Precalculus for Team-Based Inquiry Learning

2024 Edition

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TBIL Community

Editors

Steven Clontz
University of South Alabama

Drew Lewis

Contributing Authors

Tonya DeGeorge
Gwinnett County Public Schools

Abby Noble
Middle Georgia State University

Kathy Pinzon
Georgia Gwinnett College

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Website: Team-Based Inquiry Learning[1]

[1] tbil.org

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Back Matter

Chapter 1

Equations, Inequalities, and Applications (EQ)

Objectives

How do we find solutions of equations?
By the end of this chapter, you should be able to...

1. Solve linear equations in one variable. Solve linear inequalities in one variable and express the solution graphically and using interval notation.
2. Solve application problems involving linear equations.
3. Given two points, determine the distance between them and the midpoint of the line segment connecting them.
4. Solve linear equations involving an absolute value. Solve linear inequalities involving absolute values and express the answers graphically and using interval notation.
5. Solve quadratic equations using factoring, the square root property, completing the square, and the quadratic formula and express these answers in exact form.
6. Solve rational equations.
7. Solve quadratic inequalities and express the solution graphically and with interval notation. Solve rational inequalities and express the solution graphically and using interval notation.

1.1 Linear Equations and Inequalities (EQ1)

Objectives

- Solve linear equations in one variable. Solve linear inequalities in one variable and express the solution graphically and using interval notation.

Linear Equations and Inequalities (EQ1)

Remark 1.1.1 Recall that when solving a linear equation, you use addition, subtraction, multiplication and division to isolate the variable.

Activity 1.1.2 Solve the linear equations.

(a) $3x - 8 = 5x + 2$

A. $x = 2$

B. $x = 5$

C. $x = -5$

D. $x = -2$

(b) $5(3x - 4) = 2x - (x + 3)$

A. $x = \frac{17}{14}$

B. $x = \frac{14}{17}$

C. $x = \frac{23}{14}$

D. $x = \frac{14}{23}$

Linear Equations and Inequalities (EQ1)

Activity 1.1.3 Solve the linear equation.

$$\frac{2}{3}x - 8 = \frac{5x+1}{6}$$

(a) Which equation is equivalent to $\frac{2}{3}x - 8 = \frac{5x+1}{6}$ but does not contain any fractions?

A. $12x - 48 = 15x + 3$

B. $3x - 24 = 10x + 2$

C. $4x - 8 = 5x + 1$

D. $4x - 48 = 5x + 1$

(b) Use the simplified equation from part (a) to solve $\frac{2}{3}x - 8 = \frac{5x+1}{6}$.

A. $x = -17$

B. $x = -\frac{26}{7}$

C. $x = -9$

D. $x = -49$

Activity 1.1.4 It is not always the case that a linear equation has exactly one solution. Consider the following linear equations which appear similar, but their solutions are very different.

(a) Which of these equations has one unique solution?

A. $4(x-2)=4x+6$

B. $4(x-1)=4x-4$

C. $4(x-1)=x+4$

(b) Which of these equations has no solutions?

A. $4(x-2)=4x+6$

B. $4(x-1)=4x-4$

C. $4(x-1)=x+4$

(c) Which of these equations has many solutions?

A. $4(x-2)=4x+6$

B. $4(x-1)=4x-4$

C. $4(x-1)=x+4$

(d) What happens to the x variable when a linear equation has no solution or many solutions?

Definition 1.1.5 A linear equation with one unique solution is a **conditional equation.** A linear equation that is true for all values of the variable is an **identity equation**. A linear equation with no solutions is an **inconsistent equation.** ◊

Activity 1.1.6 An inequality is a relationship between two values that are not equal.

(a) What is the solution to the linear equation $3x - 1 = 5$?

(b) Which of these values is a solution of the inequality $3x - 1 \geq 5$?

A. $x = 0$

B. $x = 2$

C. $x = 4$

D. $x = 10$

(c) Express the solution of the inequality $3x - 1 \geq 5$ in interval notation.

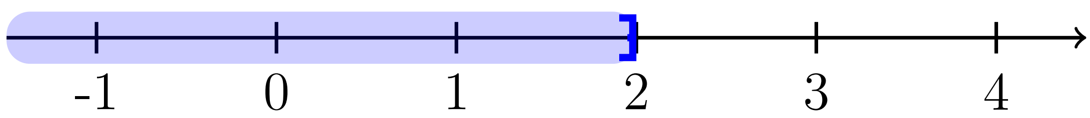
A. $(-\infty, 2]$

B. $(-\infty, 2)$

C. $(2, \infty)$

D. $[2, \infty)$

(d) Draw the solution to the inequality on a number line.



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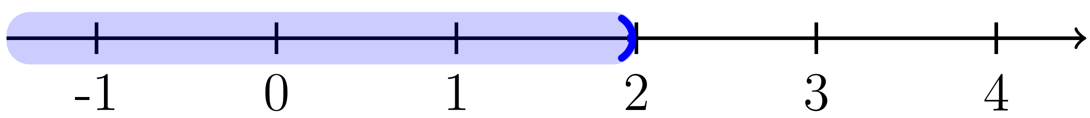

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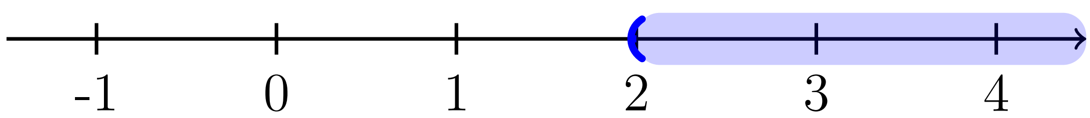

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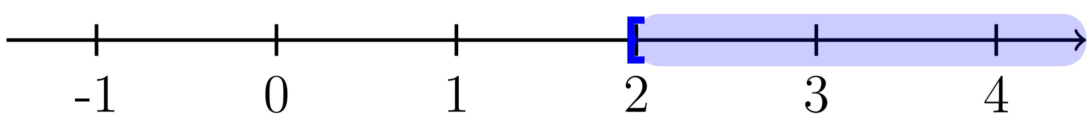

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Activity 1.1.7 Let's consider what happens to the inequality when the variable has a negative coefficient.

(a) Which of these values is a solution of the inequality $-x < 8$?

A. $x = -10$

B. $x = -8$

C. $x = 4$

D. $x = 10$

(b) Solve the linear inequality $-x < 8$. How does your solution compare to the values chosen in part (a)?

(c) Expression the solution of the inequality $-x < 8$ in interval notation.

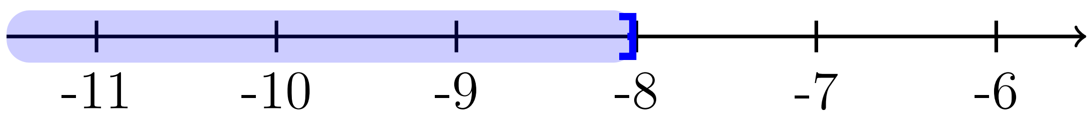
A. $(-\infty, -8]$

B. $(-\infty, -8)$

C. $(-8, \infty)$

D. $[-8, \infty)$

(d) Draw the solution to the inequality on a number line.

A.


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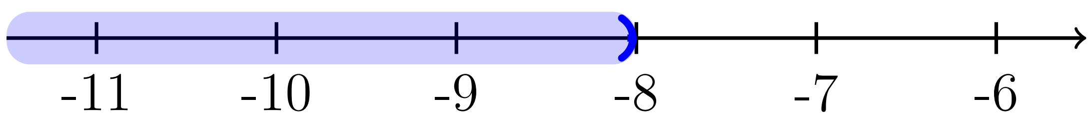

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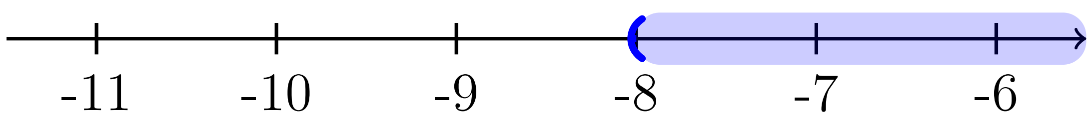

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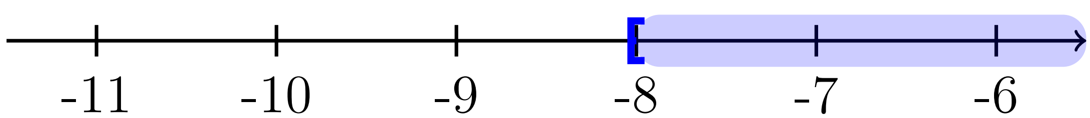


Remark 1.1.8 You can treat solving linear inequalities, just like solving an equation. The one exception is when you multiply or divide by a negative value, reverse the inequality symbol.

Activity 1.1.9 Solve the following inequalities. Express your solution in interval notation and graphically on a number line.

(a) $-3x - 1 \leq 5$

(b) $3(x + 4) > 2x - 1$

(c) $-\frac{1}{2}x \geq -\frac{3}{4} + \frac{5}{4}x$

Definition 1.1.10 A **compound inequality** includes multiple inequalities in one statement.

◊

Activity 1.1.11 Consider the statement $3 \leq x < 8$. This really means that $3 \leq x$ and $x < 8$.

(a) Which of the following inequalities are equivalent to the compound inequality $3 \leq 2x - 3 < 8$?

A. $3 \leq 2x - 3$

B. $3 \geq 2x - 3$

C. $2x - 3 < 8$

D. $2x - 3 > 8$

(b) Solve the inequality $3 \leq 2x - 3$.

A. $x \leq 0$

B. $x \geq 0$

C. $x \leq 3$

D. $x \geq 3$

(c) Solve the inequality $2x - 3 < 8$.

A. $x > \frac{11}{2}$

B. $x < \frac{11}{2}$

C. $x > \frac{5}{2}$

D. $x < \frac{5}{2}$

(d) Which compound inequality describes how the two solutions overlap?

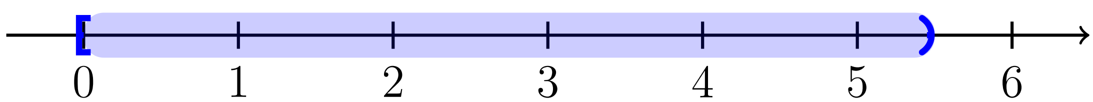
A. $0 \leq x < \frac{11}{2}$

B. $0 \leq x < \frac{5}{2}$

C. $\frac{5}{2} < x \leq 3$

D. $3 \leq x < \frac{11}{2}$

(e) Draw the solution to the inequality on a number line.



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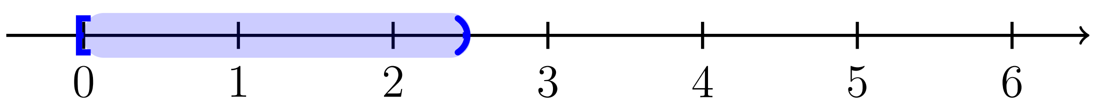

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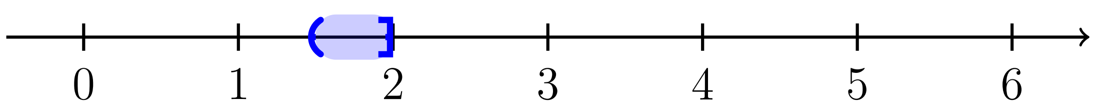

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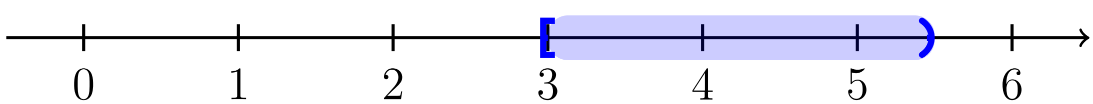


D.

Remark 1.1.12 Solving a compound linear inequality, uses the same methods as a single linear inequality ensuring that you perform the same operations on all three parts. Alternatively, you can break the compound inquality up into two and solve separately.

Activity 1.1.13 Solve the following inequalities. Express your solution in interval notation and graphically on a number line.

(a) $8 < -3x - 1 \leq 11$

(b) $-6 \leq \dfrac{x-12}{4} < -2$

1.2 Applications of Linear Equations (EQ2)

Objectives

- Solve application problems involving linear equations.

Applications of Linear Equations (EQ2)

Observation 1.2.1 Linear equations can be used to solve many types of real-world applications. We'll investigate some of those in this section.

Applications of Linear Equations (EQ2)

Remark 1.2.2 Distance, rate, and time problems are a standard example of an application of a linear equation. For these, it's important to remember that

$$d = rt$$

where d is distance, r is the rate (or speed), and t is time.

Often we will have more than one moving object, so it is helpful to denote which object's distance, rate, or time we are referring to. One way we can do this is by using a subscript. For example, if we are describing an eastbound train (as we will in the first example), it may be helpful to denote its distance, rate, and time as d_E, r_E, and t_E respectively. Notice that the subscript E is a label reminding us that we are referring to the eastbound train.

Applications of Linear Equations (EQ2)

Activity 1.2.3 Two trains leave a station at the same time. One is heading east at a speed of 75 mph, while the other is heading west at a speed of 85 mph. After how long will the trains be 400 miles apart?

(a) How fast is each train traveling?

A. $r_E = 85$ mph, $r_W = 75$ mph

B. $r_E = 75$ mph, $r_W = 85$ mph

C. $r_E = 400$ mph, $r_W = 400$ mph

D. $r_E = 75$ mph, $r_W = 400$ mph

E. $r_E = 400$ mph, $r_W = 85$ mph

(b) Which of the statements describes how the times of the eastbound and westbound train are related?

A. The eastbound train is slower than the westbound train, so $75 + t_E = 85 + t_W$.

B. The eastbound train left an hour before the westbound train, so if we let $t_E = t$, then $t_W = t - 1$.

C. Both trains have been traveling the same amount of time, so $t_E = t_W$. Since they are the same, we can just call them both t.

D. We don't know how the times relate to each other, so we must denote them separately as t_E and t_W.

E. Since the trains are traveling at different speeds, we need the proportion $\dfrac{r_E}{r_W} = \dfrac{t_E}{t_W}$.

(c) Fill in the following table using the information you've just determined about the trains' rates and times since they left the station. Some values are there to help you get started.

	rate	**time**	**distance from station**
eastbound train			$75t$
westbound train		t	

(d) At the moment in question, the trains are 400 miles apart. How does that total distance relate to the distance each train has traveled?

A. The 400 miles is irrelevant. They've been traveling the same amount of time so they must be the same distance away from the station. That tells us $d_E = d_W$.

B. The 400 miles is the difference between the distance each train traveled, so $d_E - d_W = 400$.

C. The 400 miles represents the sum of the distances that each train has traveled, so $d_E + d_W = 400$.

D. The 400 miles is the product of the distance each train traveled, so $(d_E)(d_W) = 400$.

(e) Now plug in the expressions from your table for d_E and d_w. What equation do you get?

A. $75t = 85t$

B. $75t - 85t = 400$

C. $75t + 85t = 400$

D. $(75t)(85t) = 400$

(f) Notice that we now have a linear equation in one variable, t. Solve for t, and put that answer in context of the problem.

A. The trains are 400 miles apart after 2 hours.

B. The trains are 400 miles apart after 2.5 hours.

C. The trains are 400 miles apart after 3 hours.

D. The trains are 400 miles apart after 3.5 hours.

E. The trains are 400 miles apart after 4 hours.

Remark 1.2.4 In Activity 1.2.3, we examined the motion of two objects moving at the same time in opposite directions. In Activity 1.2.5, we will examine a different perspective, but still apply $d = rt$ to solve.

Activity 1.2.5 Jalen needs groceries, so decides to ride his bike to the store. It takes him half an hour to get there. After finishing his shopping, he sees his friend Alex who offers him a ride home. He takes the same route home as he did to the store, but this time it only takes one-fifth of an hour. If his average speed was 18 mph faster on the way home, how far away does Jalen live from the grocery store?

We'll use the subscript b to refer to variables relating to Jalen's trip to the store while riding his *b*ike and the subscript c to refer to variables relating to Jalen's trip home while riding in his friend's *c*ar.

(a) How long does his bike trip from home to the store and his car trip from the store back home take?

A. $t_b = 18$ hours, $t_c = 18$ hours

B. $t_b = \frac{1}{5}$ of an hour, $t_c = \frac{1}{2}$ of an hour

C. $t_b = \frac{1}{2}$ of an hour, $t_c = \frac{1}{5}$ of an hour

D. $t_b = 2$ hours, $t_c = 5$ hours

E. $t_b = 5$ hours, $t_c = 2$ hours

(b) Which of the statements describes how the speed (rate) of the bike trip and the car trip are related?

A. Both the trip to the store and the trip home covered the same distance, so $r_b = r_c$. Since they are the same, we can just call them both r.

B. We don't know how the two rates relate to each other, so cannot write an equation comparing them and must leave them as separate variables r_b and r_c.

C. Jalen's rate on the trip home in the car was 18 mph faster than his trip to the store on his bike, so if we let $r_b = r$, then $r_c = r - 18$.

D. Jalen's rate on the trip home in the car was 18 mph faster than his trip to the store on his bike, so if we let $r_b = r$, then $r_c = r + 18$.

(c) Fill in the following table using the information you've just determined about the Jalen's rates and times on each leg of his grocery store trip. Then fill in the distance column based on how distance relates to rate and time in each case.

	rate	**time**	**distance covered**
bike trip (to the store)			
car trip (going back home)			

(d) Our goal is to figure out how far away Jalen lives from the store. To help us get there, write an equation relating d_b and d_c.

A. The distance he traveled by bike is the same as the distance he traveled by car, so $d_b = d_c$.

B. The distance he traveled by bike took longer than the distance he traveled by car, so $d_b + \frac{1}{2} = d_c + \frac{1}{5}$.

C. The distance, d, between his house and the grocery store is sum of the distance he traveled on his bike and the distance he traveled in the car, so $d_b + d_c = d$.

D. The distance, d, between his house and the grocery store is sum of the difference he traveled on his bike and the distance he traveled in the car, so $d_b - d_c = d$.

(e) Now plug in the expressions from your table for d_b and d_c into the equation you just found. Notice that it is a linear equation in one variable, r. Solve for r.

(f) Our goal was to determine the distance between Jalen's house and the grocery store. Solving for r did not tell us that distance, but it did get us one step closer. Use that value to help you determine the distance between his house and the store, and write your answer using the context of the problem.

A. The grocery store is 6 miles away from Jalen's house.

B. The grocery store is 8 miles away from Jalen's house.

C. The grocery store is 10 miles away from Jalen's house.

D. The grocery store is 12 miles away from Jalen's house.

E. The grocery store is 14 miles away from Jalen's house.

Hint. Can you find an expression involving r that we made that represents that distance?

Applications of Linear Equations (EQ2)

Remark 1.2.6 Another type of application of linear equations is called a mixture problem. In these we will mix together two things, like two types of candy in a candy store or two solutions of different concentrations of alcohol.

Applications of Linear Equations (EQ2)

Activity 1.2.7 Ammie's favorite snack to share with friends is candy salad, which is a mixture of different types of candy. Today she chooses to mix Nerds Gummy Clusters, which cost \$8.38 per pound, and Starburst Jelly Beans, which cost \$7.16 per pound. If she makes seven pounds of candy salad and spends a total of \$55.61, how many pounds of each candy did she buy?

(a) There are two "totals" in this situation: the total weight (in pounds) of candy Ammie bought and the total amount of money (in dollars) Ammie spent. Let's begin with the total weight. If we let N represent the pounds of Nerds Gummy Clusters and S represent the pounds of Starburst Jelly Beans, which of the following equations can represent the total weight?

A. $N - S = 7$

B. $NS = 7$

C. $N + S = 7$

D. $\frac{N}{S} = 7$

(b) Which expressions represent the amount she spent on each candy? Again, we will let N represent the pounds of Nerds Gummy Clusters and S represent the pounds of Starburst Jelly Beans.

A. N spent on Nerds Gummy Clusters; S spent on Starburst Jelly Beans

B. $8.38N$ spent on Nerds Gummy Clusters; $7.16S$ spent on Starburst Jelly Beans

C. $8.38 + N$ spent on Nerds Gummy Clusters; $7.16 + S$ spent on Starburst Jelly Beans

D. $8.38 - N$ spent on Nerds Gummy Clusters; $7.16 - S$ spent on Starburst Jelly Beans

(c) Now we focus on the total cost. Which of the following equations can represent the total amount she spent?

A. $N + S = 55.61$

B. $8.38N + 7.16S = 55.61$

C. $8.38 + N + 7.16 + S = 55.61$

D. $8.38 - N + 7.16 - S = 55.61$

(d) We are almost ready to solve, but we have two variables in our weight equation and our cost equation. We will get the cost equation to one variable by using the weight equation as a substitution. Which of the following is a way to express one variable in terms of the other?

A. If N is the total weight of the Nerds Gummy Clusters, then $7 - N$ could represent the weight of the Starburst Jelly Beans.

B. If N is the total weight of the Nerds Gummy Clusters, then $7+N$ could represent the weight of the Starburst Jelly Beans.

C. If S is the total weight of the Starburst Jelly Beans, then $7-S$ could represent the weight of the Nerds Gummy Clusters.

D. If S is the total weight of the Starburst Jelly Beans, then $7+S$ could represent the weight of the Nerds Gummy Clusters.

Hint. More than one answer may be correct here!

(e) Plug your expressions in to the total cost equation.

A. $8.38N + 7.16(7-N) = 55.61$

B. $8.38S + 7.16(7-S) = 55.61$

C. $8.38(7-N) + 7.16N = 55.61$

D. $8.38(7-S) + 7.16S = 55.61$

Hint. More than one of these may be correct!

(f) Now solve for N and S, and put your answer in the context of the problem.

A. Ammie bought 2.5 lbs of Nerds Gummy Clusters and 4.5 lbs of Starburst Jelly Beans.

B. Ammie bought 3.5 lbs of Nerds Gummy Clusters and 3.5 lbs of Starburst Jelly Beans.

C. Ammie bought 4.5 lbs of Nerds Gummy Clusters and 2.5 lbs of Starburst Jelly Beans.

D. Ammie bought 5.5 lbs of Nerds Gummy Clusters and 1.5 lbs of Starburst Jelly Beans.

Applications of Linear Equations (EQ2)

Activity 1.2.8 A chemist needs to mix two solutions to create a mixture consisting of 30% alcohol. She uses 20 liters of the first solution, which has a concentration of 21% alcohol. How many liters of the second solution (that is 45% alcohol) should she add to the first solution to create the mixture that is 30% alcohol?

1.3 Distance and Midpoint (EQ3)

Objectives

- Given two points, determine the distance between them and the midpoint of the line segment connecting them.

Distance and Midpoint (EQ3)

Activity 1.3.1 The points A and B are shown in the graph below. Use the graph to answer the following questions:

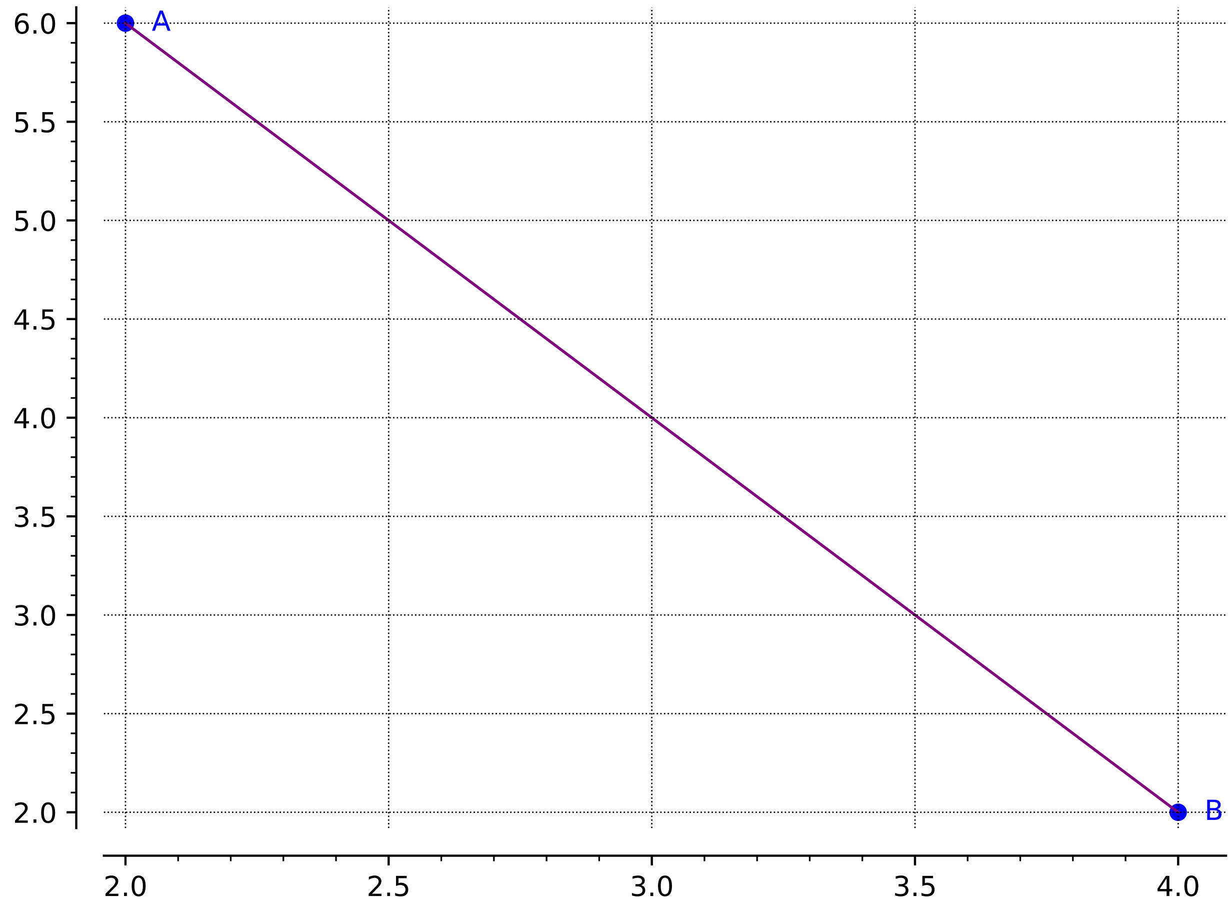


Figure 1.3.2

(a) Draw a right triangle so that the hypotenuse is the line segment between points A and B. Label the third point of the triangle C.

(b) Find the lengths of line segments AC and BC.

(c) Now that you know the lengths of AC and BC, how can you find the length of AB? Find the length of AB.

Distance and Midpoint (EQ3)

Remark 1.3.3 Using the **Pythagorean Theorem** ($a^2 + b^2 = c^2$) can be helpful in finding the distance of a line segment (as long as you create a right triangle!).

Distance and Midpoint (EQ3)

Activity 1.3.4 Suppose you are given two points (x_1, y_1) and (x_2, y_2). Let's investigate how to find the length of the line segment that connects these two points!

(a) Draw a sketch of a right triangle so that the hypotenuse is the line segment between the two points.

(b) Find the lengths of the legs of the right triangle.

(c) Find the length of the line segment that connects the two original points.

Distance and Midpoint (EQ3)

Definition 1.3.5 The distance, d, between two points, (x_1, y_1) and (x_2, y_2), can be found by using the **distance formula**:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Notice that the distance formula is an application of the Pythagorean Theorem! $\diamond$

Distance and Midpoint (EQ3)

Activity 1.3.6 Apply Definition 1.3.5 to calculate the distance between the given points.

(a) What is the distance between $(4, 6)$ and $(9, 15)$?

A. 10.2

B. 10.3

C. $\sqrt{106}$

D. $\sqrt{56}$

(b) What is the distance between $(-2, 5)$ and $(-7, -1)$?

A. $\sqrt{11}$

B. 7.8

C. 3.3

D. $\sqrt{61}$

(c) Suppose the line segment AB has one endpoint, A, at the origin. For which coordinate of B would make the line segment AB the longest?

A. $(3, 7)$

B. $(2, -8)$

C. $(-6, 4)$

D. $(-5, -5)$

Remark 1.3.7 Notice in Activity 1.3.6, you can give a distance in either exact form (leaving it with a square root) or as an approximation (as a decimal). Make sure you can give either form as sometimes one form is more useful than another!

Distance and Midpoint (EQ3)

Remark 1.3.8 A **midpoint** refers to the point that is located in the middle of a line segment. In other words, the midpoint is the point that is halfway between the two endpoints of a given line segment.

Activity 1.3.9 Two line segments are shown in the graph below. Use the graph to answer the following questions:

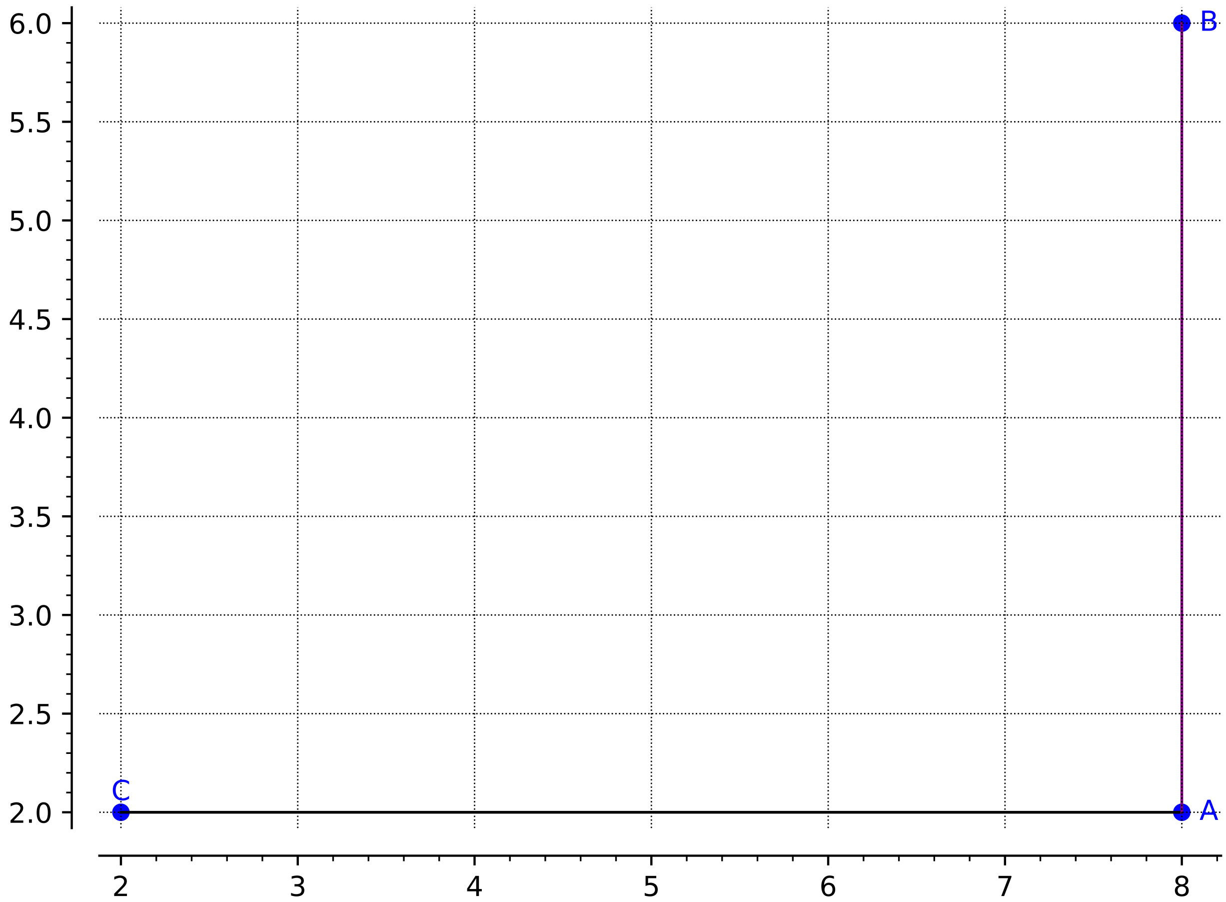


Figure 1.3.10

(a) What is the midpoint of the line segment AB?

A. $(16, 4)$

B. $(8, 4)$

C. $(8, 8)$

D. $(10, 2)$

(b) What is the midpoint of the line segment AC?

A. $(6, 0)$

B. $(4, 4)$

C. $(6, 4)$

D. $(5, 2)$

(c) Suppose we connect the two endpoints of the two line segments together, to create the new line segment, BC. Can you make an educated guess to where the midpoint of BC is?

A. $(10, 8)$

B. $(6, 4)$

C. $(5, 4)$

D. $(5, 2)$

(d) How can you test your conjecture? Is there a mathematical way to find the midpoint of any line segment?

Distance and Midpoint (EQ3)

Definition 1.3.11 The midpoint of a line segment with endpoints (x_1, y_1) and (x_2, y_2), can be found by taking the average of the x and y values. Mathematically, the **midpoint formula** states that the midpoint of a line segment can be found by:

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

$\Diamond$

Distance and Midpoint (EQ3)

Activity 1.3.12 Apply Definition 1.3.11 to calculate the midpoint of the following line segments.

(a) What is the midpoint of the line segment with endpoints $(-4, 5)$ and $(-2, -3)$?

A. $(3, 1)$
B. $(-3, 1)$
C. $(1, 1)$
D. $(1, 4)$

(b) What is the midpoint of the line segment with endpoints $(2, 6)$ and $(-6, -8)$?

A. $(-3, -1)$
B. $(-2, 0)$
C. $(-2, -1)$
D. $(4, 7)$

(c) Suppose C is the midpoint of AB and is located at $(9, 8)$. The coordinates of A are $(10, 10)$. What are the coordinates of B?

A. $(9.5, 9)$
B. $(11, 12)$
C. $(18, 16)$
D. $(8, 6)$

Distance and Midpoint (EQ3)

Activity 1.3.13 On a map, your friend Sarah's house is located at $(-2, 5)$ and your other friend Austin's house is at $(6, -2)$.

(a) How long is the direct path from Sarah's house to Austin's house?

(b) Suppose your other friend, Micah, lives in the middle between Sarah and Austin. What is the location of Micah's house on the map?

1.4 Absolute Value Equations and Inequalities (EQ4)

Objectives

- Solve linear equations involving an absolute value. Solve linear inequalities involving absolute values and express the answers graphically and using interval notation.

Absolute Value Equations and Inequalities (EQ4)

Remark 1.4.1 An absolute value, written $|x|$, is the non-negative value of x. If x is a positive number, then $|x| = x$. If x is a negative number, then $|x| = -x$.

Absolute Value Equations and Inequalities (EQ4)

Activity 1.4.2 Let's consider how to solve an equation when an absolute value is involved.

(a) Which values are solutions to the absolute value equation $|x| = 2$?

A. $x = 2$

B. $x = 0$

C. $x = -1$

D. $x = -2$

(b) Which values are solutions to the absolute value equation $|x - 7| = 2$?

A. $x = 9$

B. $x = 7$

C. $x = 5$

D. $x = -9$

(c) Which values are solutions to the absolute value equation $3|x - 7| + 5 = 11$? It may be helpful to rewrite the equation to isolate the absolute value.

A. $x = 7$

B. $x = -9$

C. $x = 5$

D. $x = 9$

Absolute Value Equations and Inequalities (EQ4)

Activity 1.4.3 Absolute value represents the distance a value is from 0 on the number line. So, $|x - 7| = 2$ means that the expression $x - 7$ is 2 units away from 0.

(a) What values on the number line could $x - 7$ equal?

A. $x = -7$

B. $x = -2$

C. $x = 0$

D. $x = 2$

E. $x = 7$

(b) This gives us two separate equations to solve. What are those two equations?

A. $x - 7 = -7$

B. $x - 7 = -2$

C. $x - 7 = 0$

D. $x - 7 = 2$

E. $x - 7 = 7$

(c) Solve each equation for x.

Absolute Value Equations and Inequalities (EQ4)

Remark 1.4.4 When solving an absolute value equation, begin by isolating the absolute value expression. Then rewrite the equation into two linear equations and solve. If $c > 0$,

$$|ax + b| = c$$

becomes the following two equations

$$ax + b = c \quad \text{and} \quad ax + b = -c$$

Absolute Value Equations and Inequalities (EQ4)

Activity 1.4.5 Solve the following absolute value equations.

(a) $|3x + 4| = 10$

A. $\{-2, 2\}$

B. $\left\{-\frac{14}{3}, 2\right\}$

C. $\{-10, 10\}$

D. No solution

(b) $3|x - 7| + 5 = 11$

A. $\{-2, 2\}$

B. $\{-9, 9\}$

C. $\{5, 9\}$

D. No solution

(c) $2|x + 1| + 8 = 4$

A. $\{-4, 4\}$

B. $\{-6, 6\}$

C. $\{5, 7\}$

D. No solution

Remark 1.4.6 Since the absolute value represents a distance, it is always a positive number. Whenever you encounter an isolated absolute value equation equal to a negative value, there will be no solution.

Absolute Value Equations and Inequalities (EQ4)

Activity 1.4.7 Just as with linear equations and inequalities, we can consider absolute value inequalities from equations.

(a) Which values are solutions to the absolute value inequality $|x - 7| \leq 2$?

A. $x = 9$

B. $x = 7$

C. $x = 5$

D. $x = -9$

(b) Rewrite the absolute value inequality $|x - 7| \leq 2$ as a compound inequality.

A. $0 \leq x - 7 \leq 2$

B. $-2 \leq x - 7 \leq 2$

C. $-2 \leq x - 7 \leq 0$

D. $2 \leq x \leq 7$

(c) Solve the compound inequality that is equivalent to $|x-7| \leq 2$ found in part (b). Write the solution in interval notation.

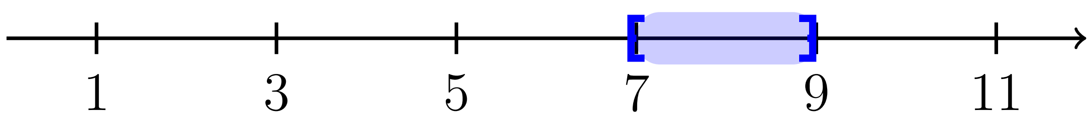
A. $[7, 9]$

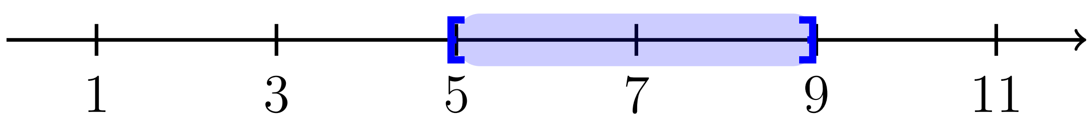
B. $[5, 9]$

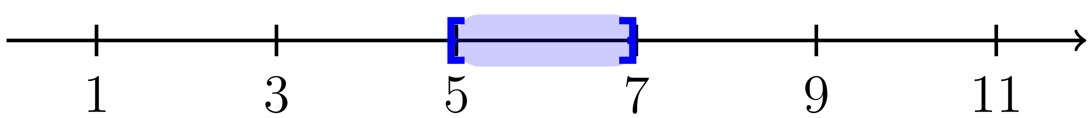
C. $[5, 7]$

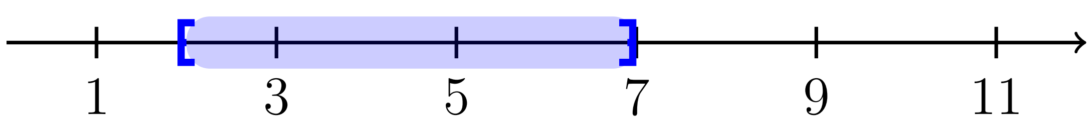
D. $[2, 7]$

(d) Draw the solution to $|x - 7| \leq 2$ on the number line.

A. 

B. 

C. 

D. 

Activity 1.4.8 Now let's consider another type of absolute value inequality.

(a) Which values are solutions to the absolute value inequality $|x-7| \geq 2$?

A. $x = 9$

B. $x = 7$

C. $x = 5$

D. $x = -9$

(b) Which two of the following inequalities are equivalent to $|x-7| \geq 2$.

A. $x - 7 \leq 2$

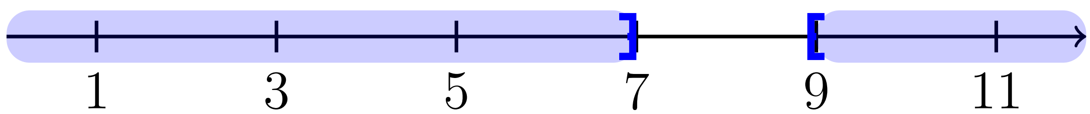
B. $x - 7 \leq -2$

C. $x - 7 \geq 2$

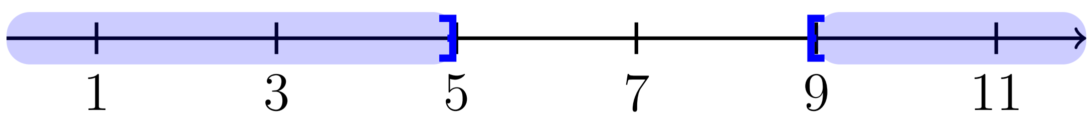
D. $x - 7 \geq -2$

(c) Solve the two inequalities found in part (b). Write the solution in interval notation and graph on the number line.

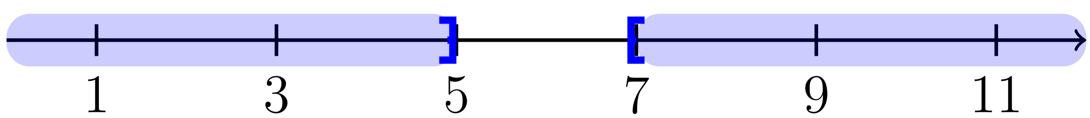
A. $(-\infty, 7] \cup [9, \infty)$



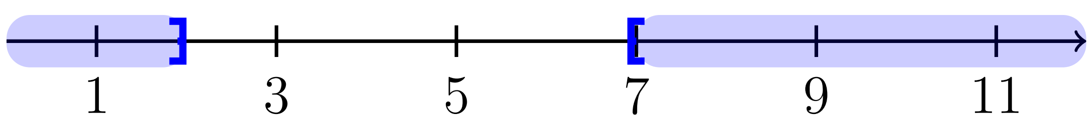
B. $(-\infty, 5] \cup [9, \infty)$



C. $(-\infty, 5] \cup [7, \infty)$



D. $(-\infty, 2] \cup [7, \infty)$



Absolute Value Equations and Inequalities (EQ4)

Definition 1.4.9 When solving an absolute value inequality, rewrite it as compound inequalities. Assume k is positive. $|x| < k$ becomes $-k < x < k$. $|x| > k$ becomes $x > k$ or $x < -k$.
$\Diamond$

Absolute Value Equations and Inequalities (EQ4)

Activity 1.4.10 Solve the following absolute value inequalities. Write your solution in interval notation and graph on a number line.

(a) $|3x + 4| < 10$

(b) $3|x - 7| + 5 > 11$

1.5 Quadratic Equations (EQ5)

Objectives

- Solve quadratic equations using factoring, the square root property, completing the square, and the quadratic formula and express these answers in exact form.

Quadratic Equations (EQ5)

Definition 1.5.1 A **quadratic equation** is of the form:

$$ax^2 + bx + c = 0$$

where a and b are coefficients (and $a \neq 0$), x is the variable, and c is the constant term. $\diamondsuit$

Quadratic Equations (EQ5)

Activity 1.5.2 Before beginning to solve quadratic equations, we need to be able to identify all various forms of quadratics.Which of the following is a quadratic equation?

A. $6 - x^2 = 3x$

B. $(2x - 1)(x + 3) = 0$

C. $4(x - 3) + 7 = 0$

D. $(x - 4)^2 + 1 = 0$

E. $5x^2 - 3x = 17 - 4x$

A, B, D, and E

Quadratic Equations (EQ5)

Definition 1.5.3 To solve a quadratic equation, we will need to apply the **zero product property**, which states that if $a \cdot b = 0$, then either $a = 0$ or $b = 0$. In other words, you can only have a product of 0 if one (or both!) of the factors is 0. $\Diamond$

Quadratic Equations (EQ5)

Activity 1.5.4 In this activity, we will look at how to apply the zero product property when solving quadratic equations.

(a) Which of the following equations can you apply Definition 1.5.3 as your first step in solving?

A. $2x^2 - 3x + 1 = 0$

B. $(2x + 1)(x + 1) = 0$

C. $3x^2 - 4 = 6x$

D. $x(3x + 5) = 0$

(b) Suppose you are given the quadratic equation, $(2x - 1)(x - 1) = 0$. Applying Definition 1.5.3 would give you:

A. $2x^2 - 3x + 1 = 0$

B. $(2x + 1) = 0$ and $(x + 1) = 0$

C. $(2x - 1) = 0$ and $(x - 1) = 0$

(c) After applying the zero product property, what are the solutions to the quadratic equation $(2x - 1)(x - 1) = 0$?

A. $x = -\frac{1}{2}$ and $x = 1$

B. $x = \frac{1}{2}$ and $x = -1$

C. $x = -\frac{1}{2}$ and $x = -1$

D. $x = \frac{1}{2}$ and $x = 1$

Remark 1.5.5 Notice in Activity 1.5.2 and Activity 1.5.4, that not all equations are set up "nicely." You will need to do some manipulation to get everything on one side (AND in factored form!) and 0 on the other ∗before∗ applying the zero product property.

Quadratic Equations (EQ5)

Activity 1.5.6 Suppose you want to solve the equation $2x^2 + 5x - 12 = 0$, which is NOT in factored form.

(a) Which of the following is the correct factored form of $2x^2 + 5x - 12 = 0$?

A. $(2x - 3)(x - 4) = 0$

B. $(2x + 3)(x - 4) = 0$

C. $(2x + 3)(x + 4) = 0$

D. $(2x - 3)(x + 4) = 0$

(b) After applying Definition 1.5.3, which of the following will be a solution to $2x^2 + 5x - 12 = 0$?

A. $x = -\frac{3}{2}$ and $x = -4$

B. $x = \frac{3}{2}$ and $x = 4$

C. $x = -\frac{3}{2}$ and $x = 4$

D. $x = \frac{3}{2}$ and $x = -4$

Quadratic Equations (EQ5)

Activity 1.5.7 Solve each of the following quadratic equations:

(a) $(2x - 5)(x + 7) = 0$

(b) $3x(4x - 1) = 0$

(c) $3x^2 - 14x - 5 = 0$

(d) $6 - x^2 = 5x$

Quadratic Equations (EQ5)

Activity 1.5.8 Suppose you are given the equation, $x^2 = 9$:

(a) How many solutions does this equation have?

A. 0

B. 1

C. 2

D. 3

(b) What are the solutions to this equation?

A. $x = 0$

B. $x = 3$

C. $x = 9, -9$

D. $x = 3, -3$

(c) How is this quadratic equation different than the equations we've solved thus far?

Quadratic Equations (EQ5)

Definition 1.5.9 The **square root property** states that a quadratic equation of the form

$$x^2 = k^2$$

(where k is a nonzero number) will give solutions $x = k$ and $x = -k$. In other words, if we have an equation with a perfect square on one side and a number on the other side, we can take the square root of both sides to solve the equation. $\Diamond$

Quadratic Equations (EQ5)

Activity 1.5.10 Suppose you are given the equation, $3x^2 - 8 = 4$:

(a) What would be the first step in solving $3x^2 - 8 = 4$?

A. Divide by 3 on both sides

B. Subtract 4 on both sides

C. Add 8 on both sides

D. Multiply by 3 on both sides

(b) Isolate the x^2 term and apply Definition 1.5.9 to solve for x.

(c) What are the solution(s) to $3x^2 - 8 = 4$?

A. $x = 6, -6$

B. $x = 2, -2$

C. $x = 0$

D. $x = 2$

Activity 1.5.11 Solve the following quadratic equations by applying the square root property (Definition 1.5.9).

(a) $3x^2 + 1 = 28$

(b) $5x^2 + 7 = 47$

(c) $2x^2 = -144$

Hint. Recall that when you have a negative number under a square root, that gives an imaginary number ($\sqrt{-1} = i$).

(d) $(x + 2)^2 + 3 = 19$

Hint. Isolate the binomial $(x + 2)$ first.

(e) $3(x - 4)^2 = 15$

Quadratic Equations (EQ5)

Remark 1.5.12 Not all quadratic equations can be factored or can be solved by using the square root property. In the next few activities, we will learn two additional methods in solving quadratics.

Quadratic Equations (EQ5)

Definition 1.5.13 Another method for solving a quadratic equation is known as **completing the square**. With this method, we add or subtract terms to both sides of an equation until we have a perfect square trinomial on one side of the equal sign and a constant on the other side. We then apply the square root property. Note: A perfect square trinomial is a trinomial that can be factored into a binomial squared. For example, $x^2 + 4x + 4$ is a perfect square trinomial because it can be factored into $(x + 2)(x + 2)$ or $(x + 2)^2$. $\diamondsuit$

Quadratic Equations (EQ5)

Activity 1.5.14 Let's work through an example together to solve $x^2 + 6x = 4$. (Notice that the methods of factoring and the square root property do not work with this equation.)

(a) In order to apply Definition 1.5.13, we first need to have a perfect square trinomial on one side of the equal sign. Which of the following number(s) could we add to the left side of the equation to create a perfect square trinomial?

A. 4

B. 9

C. -9

D. 2

(b) Add your answer from part *a* to the right side of the equation as well (i.e. whatever you do to one side of an equation you must do to the other side too!) and then factor the perfect square trinomial on the left side. Which equation best represents the equation now?

A. $(x + 3)^2 = -5$

B. $(x - 3)^2 = 13$

C. $(x + 3)^2 = 13$

D. $(x - 3)^2 = -5$

(c) Apply the square root property (Definition 1.5.9) to both sides of the equation to determine the solution(s). Which of the following is the solution(s) of $x^2 + 6x = 4$?

A. $3 + \sqrt{13}$ and $3 - \sqrt{13}$

B. $-3 + \sqrt{13}$ and $-3 - \sqrt{13}$

C. $3 + \sqrt{5}$ and $3 - \sqrt{5}$

D. $-3 + \sqrt{5}$ and $-3 - \sqrt{5}$

Remark 1.5.15 To complete the square, the leading coefficient, a (i.e., the coefficient of the x^2 term), must equal 1. If it does not, then factor the entire equation by a and then follow similar steps as in Activity 1.5.14.

Activity 1.5.16 Let's solve the equation $2x^2 + 8x - 6 = 0$ by completing the square.

(a) Rewrite the equation so that all the terms with the variable x is on one side of the equation and a constant is on the other.

(b) Notice that the coefficient of the x^2 term is not 1. What could we factor the left side of the equation by so that the coefficient of the x^2 is 1?

(c) Once you factor the left side, what equation represents the equation you now have?

A. $2(x^2 - 8x) = -6$

B. $2(x^2 - 4x) = -6$

C. $2(x^2 + 4x) = 6$

D. $2(x^2 + 8x) = 6$

(d) Just like in Activity 1.5.14, let's now try and create the perfect square trinomial (inside the parentheses) on the left side of the equation. Which of the following number(s) could we add to the left side of the equation to create a perfect square trinomial?

A. 4

B. 8

C. -8

D. 2

(e) What would we need to add to the right-hand side of the equation to keep the equation balanced?

A. 4

B. 8

C. -8

D. 2

(f) Which of the following equation represents the quadratic equation you have now?

A. $2(x + 2)^2 = 9$

B. $2(x - 2)^2 = 2$

C. $2(x + 2)^2 = 14$

D. $2(x - 2)^2 = 14$

(g) Apply the square root property and solve the quadratic equation.

Quadratic Equations (EQ5)

Activity 1.5.17 Solve the following quadratic equations by completing the square.

(a) $x^2 - 12x = -11$

(b) $x^2 + 2x - 33 = 0$

(c) $5x^2 + 29x = 6$

Quadratic Equations (EQ5)

Definition 1.5.18 The last method for solving quadratic equations is the **quadratic formula** - a formula that will solve all quadratic equations! A quadratic equation of the form $ax^2 + bx + c = 0$ can be solved by the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

where a, b, and c are real numbers and $a \neq 0$. ◊

Quadratic Equations (EQ5)

Activity 1.5.19 Use the quadratic formula (Definition 1.5.18) to solve $x^2 + 4x = -3$.

(a) When written in standard form, what are the values of a, b, and c?

(b) When applying the quadratic formula, what would you get when you substitute a, b, and c?

A. $x = \dfrac{-4 \pm \sqrt{4^2 - 4(1)(3)}}{2(1)}$

B. $x = \dfrac{4 \pm \sqrt{(-4)^2 - 4(1)(3)}}{2(1)}$

C. $x = \dfrac{-4 \pm \sqrt{(-4)^2 - 4(1)(-3)}}{2(1)}$

D. $x = \dfrac{4 \pm \sqrt{4^2 - 4(1)(-3)}}{2(1)}$

(c) What is the solution(s) to $x^2 + 4x = -3$?

A. $x = -1, 3$

B. $x = 1, 3$

C. $x = -1, -3$

D. $x = 1, -3$

Quadratic Equations (EQ5)

Activity 1.5.20 Use the quadratic formula (Definition 1.5.18) to solve $2x^2 - 13 = 7x$.

(a) When written in standard form, what are the values of a, b, and c?

(b) When applying the quadratic formula, what would you get when you substitute a, b, and c?

A. $x = \dfrac{-7 \pm \sqrt{7^2 - 4(2)(13)}}{2(1)}$

B. $x = \dfrac{7 \pm \sqrt{(-7)^2 - 4(2)(-13)}}{2(2)}$

C. $x = \dfrac{-7 \pm \sqrt{(-7)^2 - 4(1)(-13)}}{2(1)}$

D. $x = \dfrac{7 \pm \sqrt{7^2 - 4(2)(-13)}}{2(2)}$

(c) What is the solution(s) to $2x^2 - 13 = 7x$?

A. $x = \dfrac{7 + \sqrt{73}}{4}$ and $\dfrac{7 - \sqrt{73}}{4}$

B. $x = \dfrac{7 + \sqrt{153}}{4}$ and $\dfrac{-7 - \sqrt{153}}{4}$

C. $x = \dfrac{-7 + \sqrt{55}}{4}$ and $\dfrac{7 - \sqrt{55}}{4}$

D. $x = \dfrac{-7 + \sqrt{155}}{4}$ and $\dfrac{-7 - \sqrt{155}}{4}$

Quadratic Equations (EQ5)

Activity 1.5.21 Use the quadratic formula (Definition 1.5.18) to solve $x^2 = 6x - 12$.

(a) When written in standard form, what are the values of a, b, and c?

(b) When applying the quadratic formula, what would you get when you substitute a, b, and c?

A. $x = \dfrac{-6 \pm \sqrt{6^2 - 4(1)(12)}}{2(1)}$

B. $x = \dfrac{6 \pm \sqrt{(-6)^2 - 4(1)(12)}}{2(1)}$

C. $x = \dfrac{-6 \pm \sqrt{(-6)^2 - 4(1)(-12)}}{2(1)}$

D. $x = \dfrac{6 \pm \sqrt{6^2 - 4(1)(-12)}}{2(1)}$

(c) Notice that the number under the square root is a negative. Recall that when you have a negative number under a square root, that gives an imaginary number ($\sqrt{-1} = i$). What is the solution(s) to $x^2 = 6x - 12$?

A. $x = 3 + i\sqrt{3}$ and $3 - i\sqrt{3}$

B. $x = 6 + i\sqrt{12}$ and $6 - i\sqrt{12}$

C. $x = -3 + i\sqrt{3}$ and $-3 - i\sqrt{3}$

D. $x = -6 + i\sqrt{12}$ and $-6 - i\sqrt{12}$

Quadratic Equations (EQ5)

Activity 1.5.22 Solve the following quadratic equations by applying the quadratic formula (Definition 1.5.18).

(a) $2x^2 - 3x = 5$

(b) $4x^2 - 1 = -8x$

(c) $2x^2 - 7x - 13 = -10$

(d) $x^2 - 6x + 12 = 0$

Quadratic Equations (EQ5)

Activity 1.5.23 Now that you have seen all the different ways to solve a quadratic equation, you will need to know WHEN to use which method. Are some methods better than others?

(a) Which is the best method to use to solve $5x^2 = 80$?

A. Factoring and Zero Product Property
B. Square Root Property
C. Completing the Square
D. Quadratic Formula

(b) Which is the best method to use to solve $5x^2 + 9x = -4$?

A. Factoring and Zero Product Property
B. Square Root Property
C. Completing the Square
D. Quadratic Formula

(c) Which is the best method to use to solve $3x^2 + 9x = 0$?

A. Factoring and Zero Product Property
B. Square Root Property
C. Completing the Square
D. Quadratic Formula

(d) Go back to parts a, b, and c and solve each of the quadratic equations. Would you still use the same method?

1.6 Rational Equations (EQ6)

Objectives

- Solve rational equations.

Rational Equations (EQ6)

Definition 1.6.1 An algebraic expression is called a **rational expression** if it can be written as the ratio of two polynomials, p and q.

An equation is called a **rational equation** if it consists of only rational expressions and constants. $\diamondsuit$

Observation 1.6.2 Technically, linear and quadratic equations are also rational equations. They are a special case where the denominator of the rational expressions is 1. We will focus in this section on cases where the denominator is not a constant; that is, rational equations where there are variables in the denominator.

With variables in the denominator, there will often be values that cause the denominator to be zero. This is a problem because division by zero is undefined. Thus, we need to be sure to exclude any values that would make those denominators equal to zero.

Rational Equations (EQ6)

Activity 1.6.3 Which value(s) should be excluded as possible solutions to the following rational equations? Select all that apply.

(a)

$$\frac{2}{x+5} = \frac{x-3}{x-8} - 7$$

A. -7

B. -5

C. 2

D. 3

E. 8

(b)

$$\frac{x^2-6x+8}{x^2-4x+3} = 0$$

A. 0

B. 1

C. 2

D. 3

E. 4

Rational Equations (EQ6)

Activity 1.6.4 Consider the rational equation

$$5 = -\frac{6}{x-2}$$

(a) What value should be excluded as a possible solution?

A. 5
B. 6
C. -6
D. 2
E. -2

(b) To solve, we begin by clearing out the fraction involved. What can we multiply each term by that will clear the fraction?

A. $x - 5$
B. $x - 6$
C. $x + 6$
D. $x - 2$
E. $x + 2$

(c) Multiply each term by the expression you chose and simplify. Which of the following linear equations does the rational equation simplify to?

A. $5(x - 5) = -6$
B. $5(x - 6) = -6$
C. $5(x + 6) = -6$
D. $5(x - 2) = -6$
E. $5(x + 2) = -6$

(d) Solve the linear equation. Check your answer using the original rational equation.

Rational Equations (EQ6)

Activity 1.6.5 Consider the rational equation

$$\frac{4}{x+1} = -\frac{2}{x+6}$$

(a) What values should be excluded as possible solutions?

A. 2 and 4

B. 1 and 6

C. -1 and -6

D. 1 and 4

E. 2 and 6

(b) To solve, we'll once again begin by clearing out the fraction involved. Which of the following should we multiply each term by to clear out all of the fractions?

A. $x - 2$ and $x - 4$

B. $x - 1$ and $x - 6$

C. $x + 1$ and $x + 6$

D. $x - 1$ and $x - 4$

E. $x - 2$ and $x - 6$

(c) Multiply each term by the expressions you chose and simplify. Which of the following linear equations does the rational equation simplify to?

A. $4(x + 1) = -2(x + 6)$

B. $4(x + 6) = -2(x + 1)$

C. $4(x + 1)(x + 6) = -2(x + 1)(x + 6)$

D. $4(x + 1) = -2(-x - 6)$

E. $4(x + 6) = -2(-x - 1)$

(d) Solve the linear equation. Check your answer using the original rational equation.

Observation 1.6.6 In Activity 1.6.5, you may have noticed that the resulting linear equation looked like the result of cross-multiplying. This is no coincidence! Cross-multiplying is a method of clearing out fractions that works specifically when the equation is in proportional form: $\frac{a}{b} = \frac{c}{d}$.

Rational Equations (EQ6)

Activity 1.6.7 Consider the rational equation

$$\frac{x}{x+2} = -\frac{2}{x+2} - \frac{2}{5}$$

(a) What value(s) should be excluded as possible solutions?

(b) To solve, we'll once again begin by clearing out the fraction involved. Which of the following should we multiply each term by to clear out all of the fractions?

A. $x + 2$, $x + 2$, and 5

B. $x + 2$ and 5

C. $x + 2$

D. 5

(c) Multiply each term by the expressions you chose and simplify. You should end up with a linear equation.

(d) Solve the linear equation. Check your answer using the original rational equation.

Observation 1.6.8 Activity 1.6.7 demonstrates why it is so important to determine excluded values and check our answers when solving rational equations. Just because a number is a solution to the *linear* equation we found, it doesn't mean it is automatically a solution to the *rational* equation we started with.

Rational Equations (EQ6)

Activity 1.6.9 Consider the rational equation

$$\frac{2x}{x-1} - \frac{3}{x-3} = \frac{x^2 - 11x + 18}{x^2 - 4x + 3}$$

(a) What values should be excluded as possible solutions? Select all that apply.

A. 0

B. 1

C. 2

D. 3

E. 9

(b) To solve, we'll begin by clearing out any fractions involved. Which of the following should we multiply each term by to clear out all of the fractions?

A. $x - 1$

B. $x - 1$ and $x - 3$

C. $x - 1$, $x - 3$, and $x^2 - 4x + 3$

D. $x - 1$ and $x^2 - 4x + 3$

E. $x - 3$ and $x^2 - 4x + 3$

(c) Multiply each term by the expressions you chose and simplify. Notice that the result is a quadratic equation. Which of the following quadratic equations does the rational equation simplify to?

A. $x^2 + 2x - 15 = 0$

B. $x^2 - 11x + 18 = 0$

C. $x^2 - 9x - 9 = 0$

D. $x^2 - 13x + 21 = 0$

(d) Solve the quadratic equation. Check your answer using the original rational equation. What are the solutions to the rational equation?

A. $x = 3$ and $x = -5$

B. $x = -3$ and $x = 5$

C. $x = 3$

D. $x = -5$

E. $x = -3$

F. $x = 5$

Rational Equations (EQ6)

Activity 1.6.10 Consider the rational equation

$$\frac{2x}{x-2} - \frac{x^2+21x-15}{x^2+3x-10} = \frac{-6}{x+5}$$

(a) What values should be excluded as possible solutions?

(b) What expression(s) should we multiply by to clear out all of the fractions?

(c) Multiply each term by the expressions you chose and simplify. Your result should be a quadratic equation.

(d) Solve the quadratic equation. Check your answer using the original rational equation. What are the solutions to the rational equation?

Rational Equations (EQ6)

Activity 1.6.11 Solve the following rational equations.

(a) $\dfrac{4}{x} + 9 = 16$

(b) $-5 = \dfrac{2}{x-4}$

(c) $\dfrac{-3}{x-10} = \dfrac{x}{x-6}$

(d) $\dfrac{x+2}{x-3} + \dfrac{x}{2x-1} = 6$

1.7 Quadratic and Rational Inequalities (EQ7)

Objectives

- Solve quadratic inequalities and express the solution graphically and with interval notation. Solve rational inequalities and express the solution graphically and using interval notation.

Quadratic and Rational Inequalities (EQ7)

Remark 1.7.1 In Section 1.5 and Section 1.6 we learned how to solve quadratic and rational equations. In this section, we use these skills to solve quadratic and rational *inequalities*.

Quadratic and Rational Inequalities (EQ7)

Activity 1.7.2 Consider the quadratic inequality

$$x^2 - 4x - 32 > 0.$$

(a) Use a graphing utility to graph the function $f(x) = x^2 - 4x - 32$. Which part of the graph represents where $x^2 - 4x - 32 > 0$?

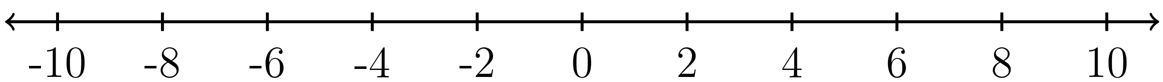
(b) Which pieces of information about $f(x) = x^2 - 4x - 32$ were needed to answer part (a)?

A. The y-intercept

B. The x-intercepts

C. The minimum value

(c) Use algebra to find the x-intercepts of $f(x) = x^2 - 4x - 32$ and mark them on a number line. Then, shade the part of the number line where $x^2 - 4x - 32 > 0$.



(d) Now use interval notation to express where $x^2 - 4x - 32 > 0$.

A. $(-4, 8)$

B. $[-4, 8]$

C. $(-\infty, -4) \cup (8, \infty)$

D. $(-\infty, -4] \cup [8, \infty)$

Definition 1.7.3 A **sign chart** is a number line representing the x-axis that shows where a function is positive or negative. Instead of shading, which can be ambiguous, it is often decorated with a '+' or a '-' to indicate which regions are positive or negative. For example, a sign chart for $f(x) = x^2 - 4x - 32$ is below.

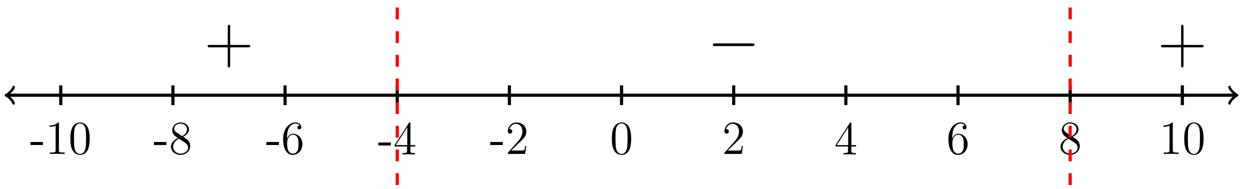


Figure 1.7.4 A sign chart for the function $f(x) = x^2 - 4x - 32$.

◊

Quadratic and Rational Inequalities (EQ7)

Activity 1.7.5 Solve the quadratic inequality algebraically

$$2x^2 - 28 < 10x.$$

(a) Write your solution using interval notation.

A. $(-\infty, -2) \cup (7, \infty)$

B. $(-\infty, -7) \cup (2, \infty)$

C. $(-7, 2)$

D. $(-2, 7)$

(b) Draw a number line representing your solution.

Quadratic and Rational Inequalities (EQ7)

Activity 1.7.6 Solve the inequality

$$-2x^2 - 10x - 10 \geq 6x + 20.$$

(a) Write your solution using interval notation.

A. $[-5, -3]$

B. $(-\infty, -5] \cup [-3, \infty)$

C. $[3, 5]$

D. $(-\infty, 3] \cup [5, \infty)$

(b) Draw a number line representing your solution.

Quadratic and Rational Inequalities (EQ7)

Activity 1.7.7 Consider the rational inequality

$$\frac{x+3}{x-2} \leq 0.$$

(a) Use a graphing utility to graph the function $f(x) = \frac{x+3}{x-2}$. Which part of the graph represents where $\frac{x+3}{x-2} \leq 0$? Write your answer in interval notation.

(b) How does the interval you found in part (a) relate to the numerator and the denominator of the function $f(x) = \frac{x+3}{x-2}$?

Quadratic and Rational Inequalities (EQ7)

Activity 1.7.8 Consider the rational inequality

$$\frac{4x+3}{x+2} > x.$$

(a) For which of the following functions will a graph help us solve the rational inequality above?

A. $f(x) = \dfrac{4x+3}{x+2}$

B. $g(x) = \dfrac{4x+3}{x+2} - x$

C. $h(x) = x - \dfrac{4x+3}{x+2}$

(b) Use a graphing utility to graph the function $g(x) = \dfrac{4x+3}{x+2} - x$. Which part of the graph represents where $\dfrac{4x+3}{x+2} - x > 0$? Write your answer in interval notation.

(c) Simplify $\dfrac{4x+3}{x+2} - x$ into a single rational expression.

A. $\dfrac{4x+3}{x+2}$

B. $\dfrac{3x+3}{x+2}$

C. $\dfrac{x^2+6x+3}{x+2}$

D. $\dfrac{-x^2+2x+3}{x+2}$

(d) How does the interval you found in part (b) relate to the numerator and the denominator of the combined rational function in part (c)?

Hint. Factor the numerator.

(e) For what values is the original inequality a true statement?

A. $x < -2$ and $-1 < x < 3$

B. $-2 < x < -1$ and $x > 3$

C. $-2 < x < -1$

D. $1 < x < 3$

(f) How can we express the answers to part (e) for the rational inequality using interval notation?

A. $(1, 3)$

B. $(-2, -1) \cup (3, \infty)$

C. $(-2, -1)$

D. $(-\infty, -2) \cup (-1, 3)$

Quadratic and Rational Inequalities (EQ7)

Definition 1.7.9 The values on the x-axis where a function is equal to zero or undefined are called **partition values.** $\Diamond$

Activity 1.7.10 Solve the rational inequality

$$\frac{x+8}{x-2} \leq \frac{x+10}{x+5}.$$

(a) Write the solution using interval notation.

A. $(-\infty, -12) \cup [-5, 2]$

B. $(-\infty, -12] \cup (-5, 2)$

C. $(-12, -5] \cup [2, \infty)$

D. $[-12, -5) \cup (2, \infty)$

(b) Draw a number line representing your solution.

(c) Compare the interval notation from Activity 1.7.8 to the interval notation for this activity. When do we include the partition values in the answer with a bracket?

Chapter 2

Functions (FN)

Objectives

How do we express relationships between two quantities?
By the end of this chapter, you should be able to...

1. Determine if a relation, equation, or graph defines a function using the definition as well as the vertical line test.
2. Use and interpret function notation to evaluate a function for a given input value and find a corresponding input value given an output value.
3. Use the graph of a function to find the domain and range in interval notation, the x- and y-intercepts, the maxima and minima, and where it is increasing and decreasing using interval notation.
4. Apply transformations including horizontal and vertical shifts, stretches, and reflections to a function. Express the result of these transformations graphically and algebraically.
5. Find the sum, difference, product, quotient, and composition of two or more functions and evaluate them.
6. Find the inverse of a one-to-one function.

2.1 Introduction to Functions (FN1)

Objectives

- Determine if a relation, equation, or graph defines a function using the definition as well as the vertical line test.

Introduction to Functions (FN1)

Definition 2.1.1 A **relation** is a relationship between sets of values. Relations in mathematics are usually represented as ordered pairs: (input, output) or (x, y). When observing relations, we often refer to the x-values as the **domain** and the y-values as the **range**. $\Diamond$

Definition 2.1.2 Mapping Notation (also known as an arrow diagram) is a way to show relationships visually between sets. For example, suppose you are given the following ordered pairs: $(3,-8),(4,6)$, and $(2,-1)$. Each of the x-values "map onto" a y-value and can be visualized in the following way:

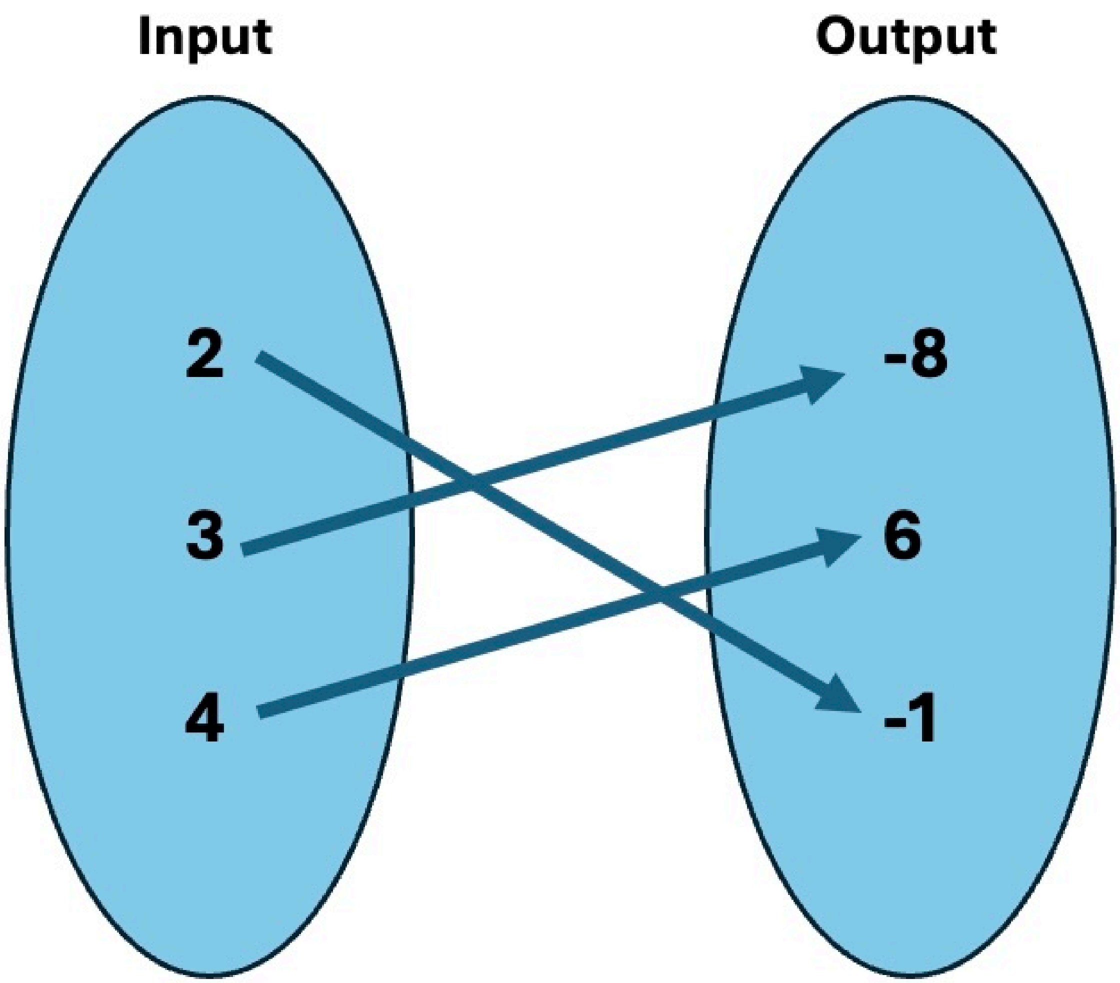


Figure 2.1.3 Every x-value from the ordered pair list is listed in the input set and every y-value is listed in the output set. An arrow is drawn from every x-value to its corresponding y-value.

Notice that an arrow is used to indicate which x-value is mapped onto its corresponding y-value. ◊

Activity 2.1.4 Use mapping notation to create a visual representation of the following relation.

$$(-1, 5), (2, 6), (4, -2)$$

(a) What is the domain?

A. $\{5, 6, -2\}$

B. $\{-1, 2, 4\}$

C. $\{-2, -1, 2, 4, 5, 6\}$

(b) What is the range?

A. $\{5, 6, -2\}$

B. $\{-1, 2, 4\}$

C. $\{-2, -1, 2, 4, 5, 6\}$

Activity 2.1.5 Use mapping notation to create a visual representation of the following relation.

$$(6,4),(3,4),(6,5)$$

(a) What is the domain?

A. $\{3,6\}$

B. $\{6,3,6\}$

C. $\{3,4,5,6\}$

D. $\{4,5\}$

(b) What is the range?

A. $\{3,6\}$

B. $\{6,3,6\}$

C. $\{3,4,5,6\}$

D. $\{4,5\}$

Activity 2.1.6 Use mapping notation to create a visual representation of the following relation.

$$(1,2),(-5,2),(-7,2)$$

(a) What is the domain?

A. $\{2,2,2\}$

B. $\{-7,-5,1,2\}$

C. $\{-7,-5,1\}$

D. $\{2\}$

(b) What is the range?

A. $\{2,2,2\}$

B. $\{-7,-5,1,2\}$

C. $\{-7,-5,1\}$

D. $\{2\}$

Remark 2.1.7 Notice that in Activity 2.1.4, Activity 2.1.5, and Activity 2.1.6, each set represents a very different relationship. Many concepts in mathematics will depend on particular relationships, so it is important to be able to visualize relationships and compare them.

Introduction to Functions (FN1)

Definition 2.1.8 A **function** is a relation where every input (or x-value) is mapped onto *exactly one* output (or y-value).

Note that all functions are relations but not all relations are functions! $\Diamond$

Introduction to Functions (FN1)

Activity 2.1.9 Relations can be expressed in multiple ways (ordered pairs, tables, and verbal descriptions).

(a) Let's revisit some of the sets of ordered pairs we've previously explored in Activity 2.1.4, Activity 2.1.5, and Activity 2.1.6. Which of the following sets of ordered pairs represent a function?

A. $(-1, 5), (2, 6), (4, -2)$

B. $(6, 4), (3, 4), (6, 5)$

C. $(1, 2), (-5, 2), (-7, 2)$

D. $(-1, 2), (-1, 9), (1, 9)$

(b) Note that relations can be expressed in a table. A table of values is shown below. Is this an example of a function? Why or why not?

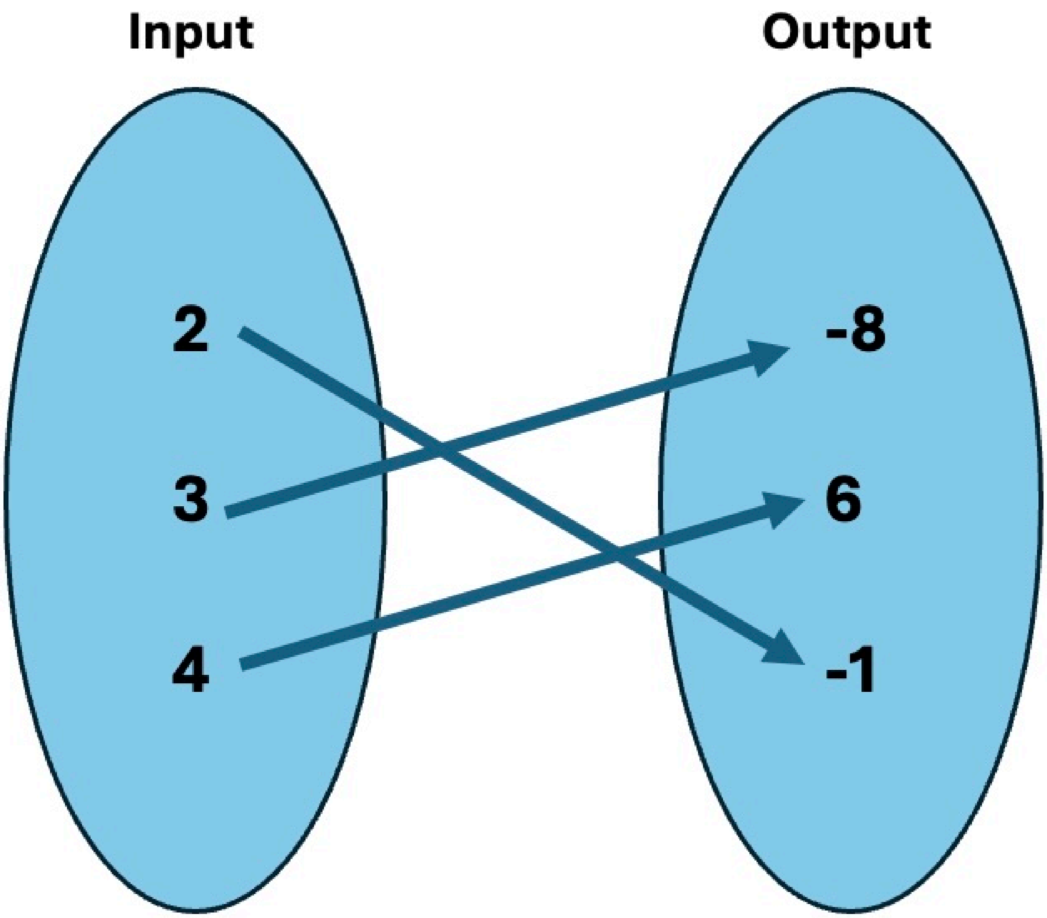
x	y
-5	-2
-4	-5
-2	8
8	-4
8	1

(c) Relations can also be expressed in words. Suppose you are looking at the amount of time you spend studying versus the grade you earn in your Algebra class. Is this an example of a function? Why or why not?

Remark 2.1.10 Notice that when trying to determine if a relation is a function, we often have to rely on looking at the domain and range values. Thus, it is important to be able to idenfity the domain and range of any relation!

Activity 2.1.11 For each of the given functions, determine the domain and range.

(a) $(-4, 3), (-1, 8), (7, 4), (1, 9)$



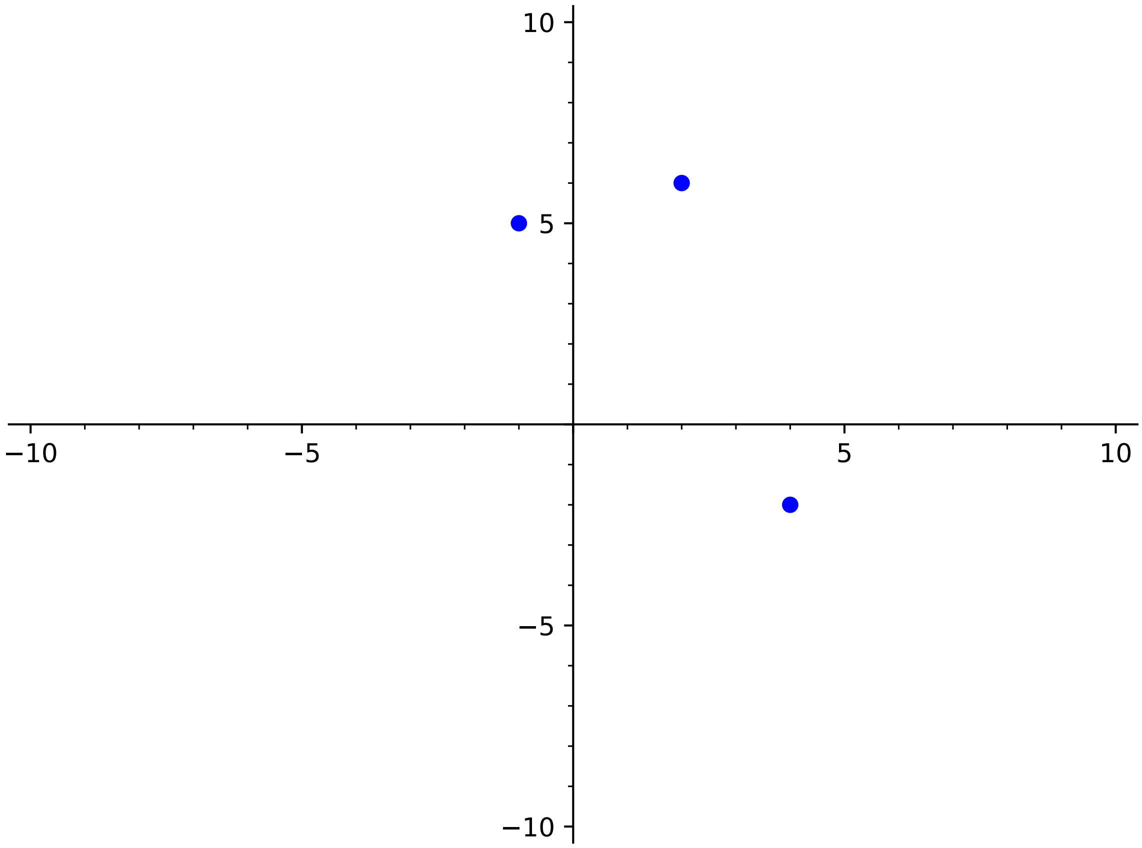
(b)

(c)

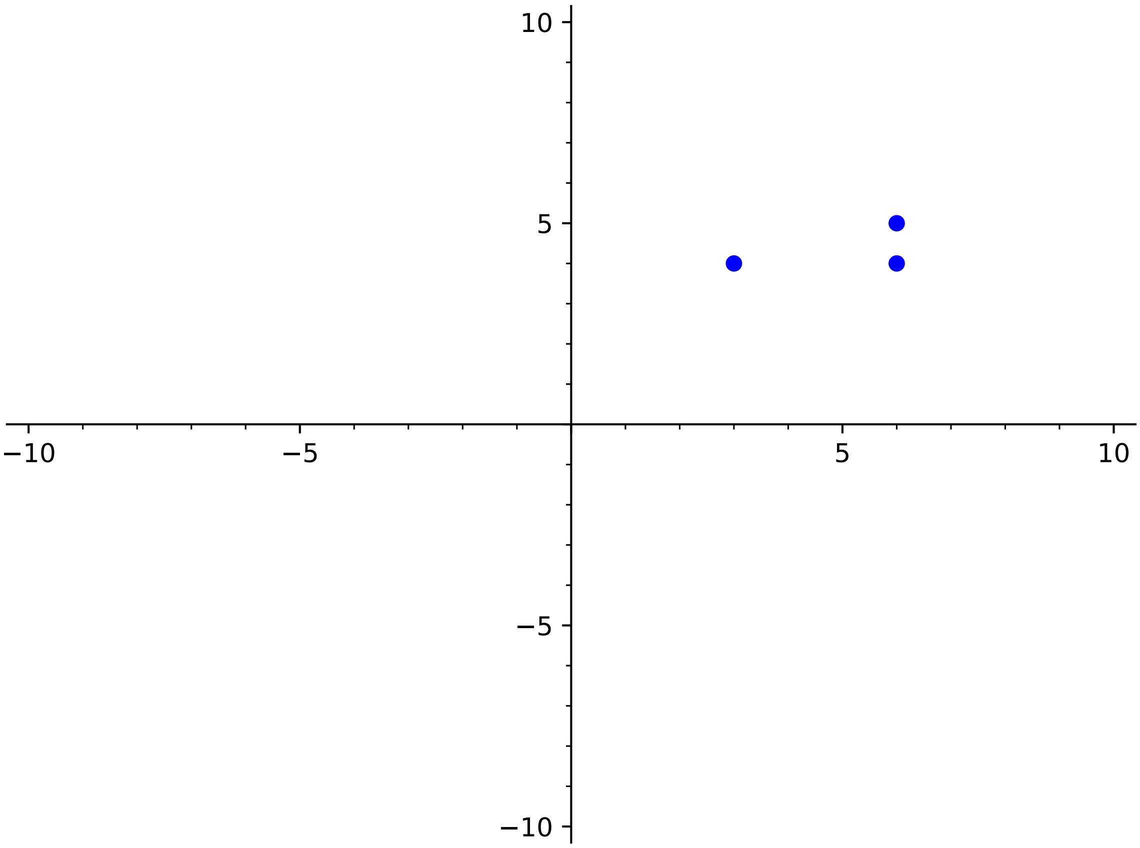
x	y
-2	5
0	4
3	6
8	1

(d) The amount of time you spend studying versus the grade you earn in your Algebra class.

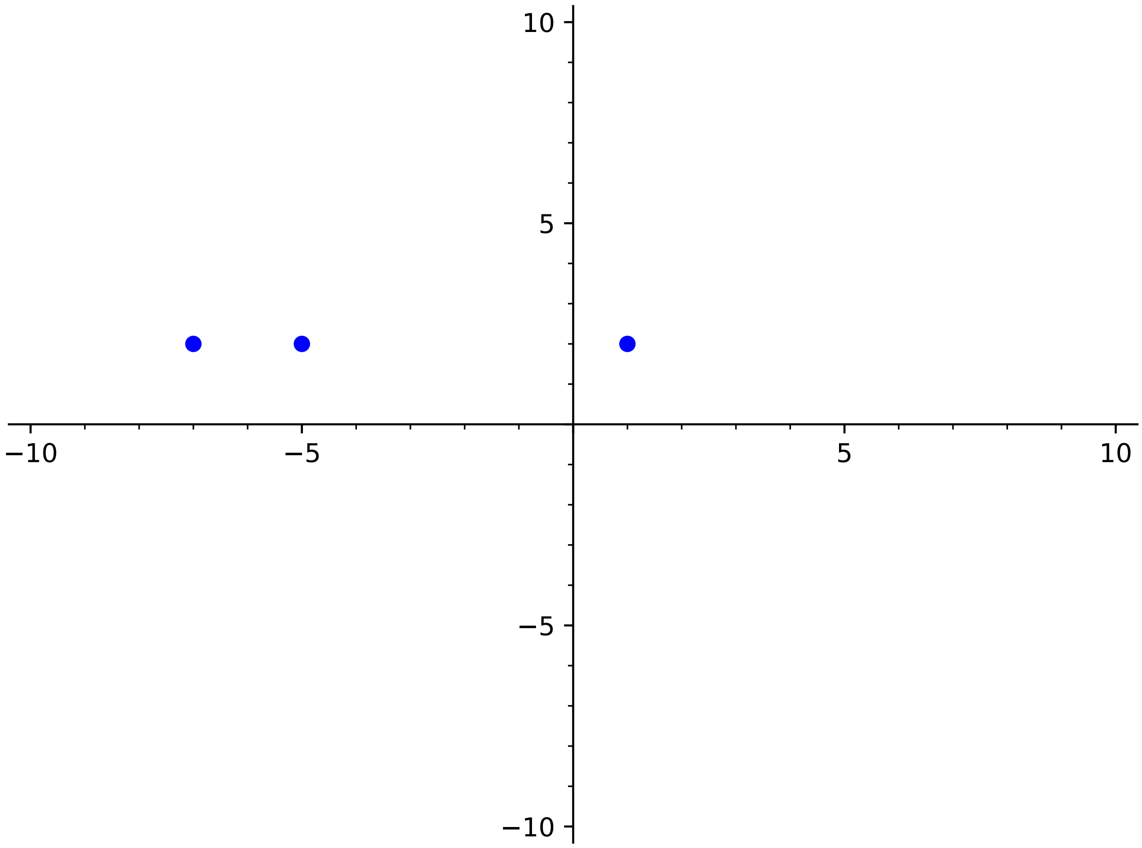
Activity 2.1.12 Determine whether each of the following relations is a function.



(a)



(b)



(c)

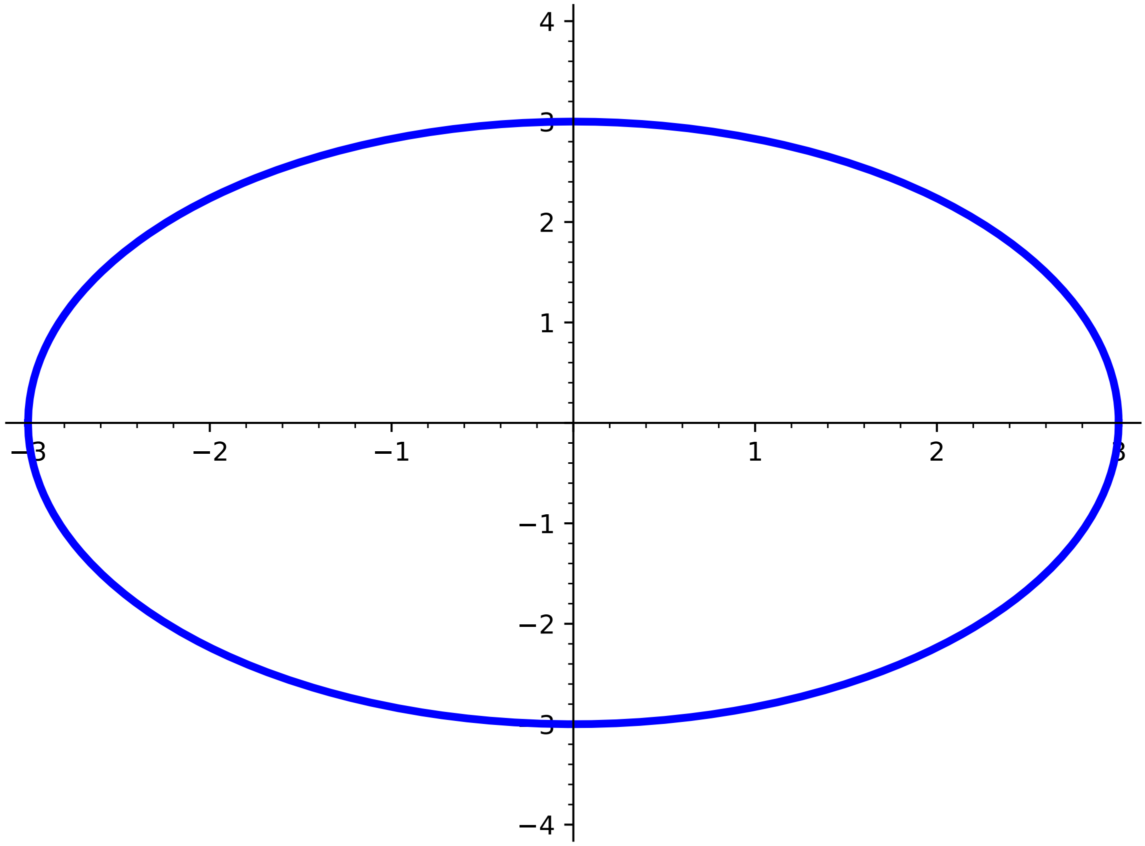
Remark 2.1.13 You probably noticed (in Activity 2.1.12) that when the graph has points that "line up" or are on top of each other, they have the same x-values. When this occurs, this shows that the same x-value has two different outputs (y-values) and that the relation is not a function.

Definition 2.1.14 The **vertical line test** is a method used to determine whether a relation on a graph is a function.

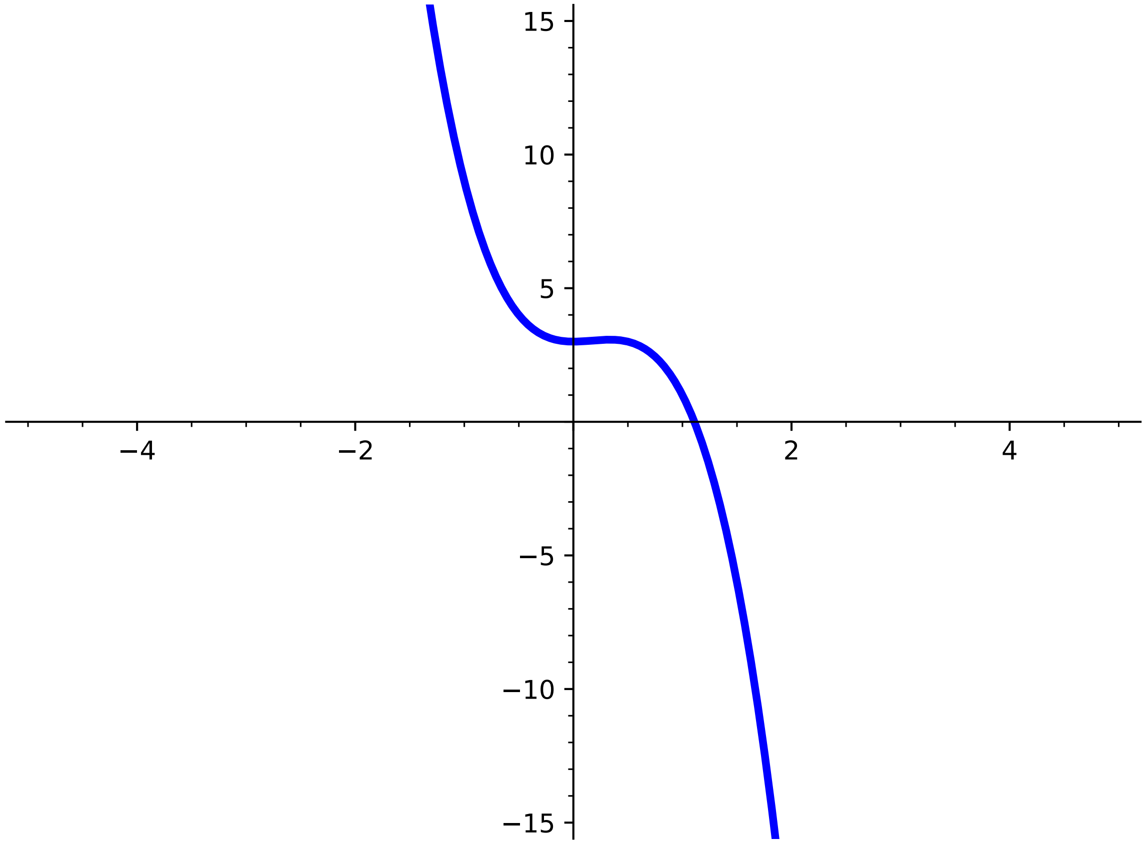
Start by drawing a vertical line anywhere on the graph and observe the number of times the relation on the graph intersects with the vertical line. If every possible vertical line intersects the graph at only one point, then the relation is a function. If, however, the graph of the relation intersects a vertical line more than once (anywhere on the graph), then the relation is not a function. $\diamondsuit$

Activity 2.1.15 Use the vertical line test (Definition 2.1.14) to determine whether each graph of a relation represents a function.

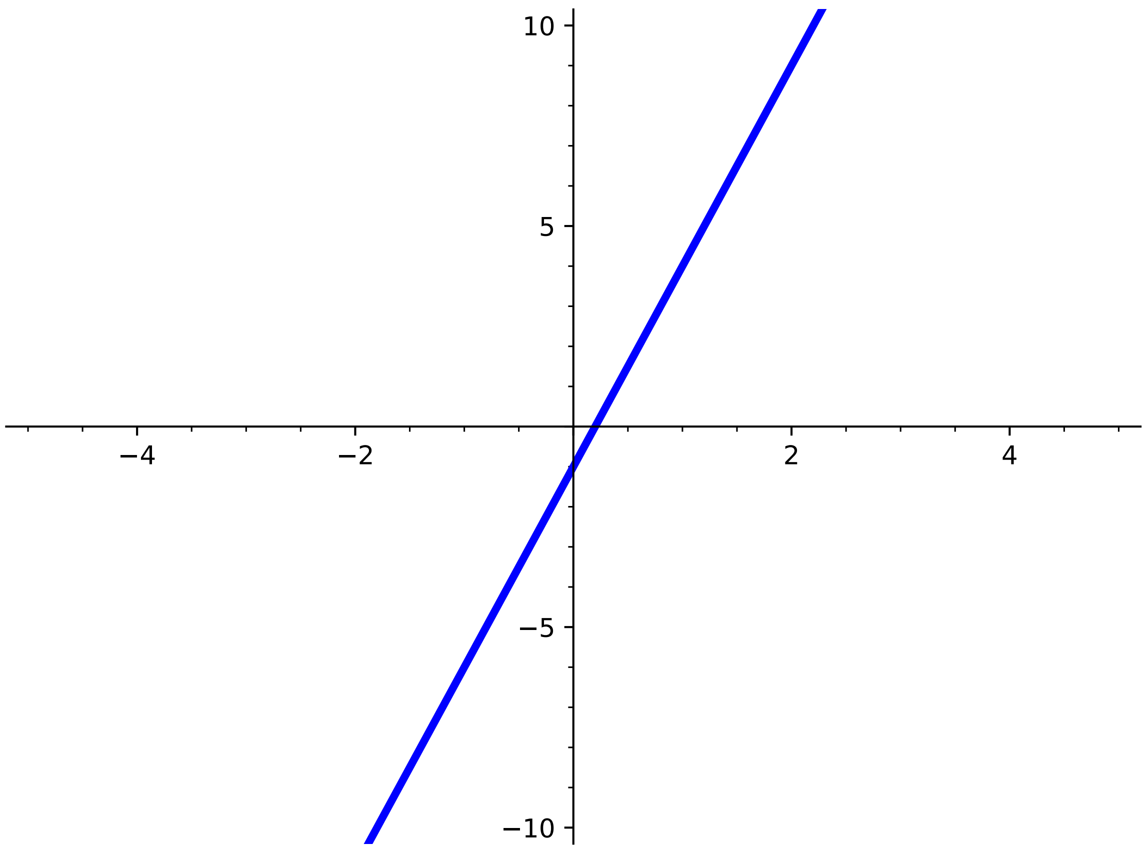
(a)



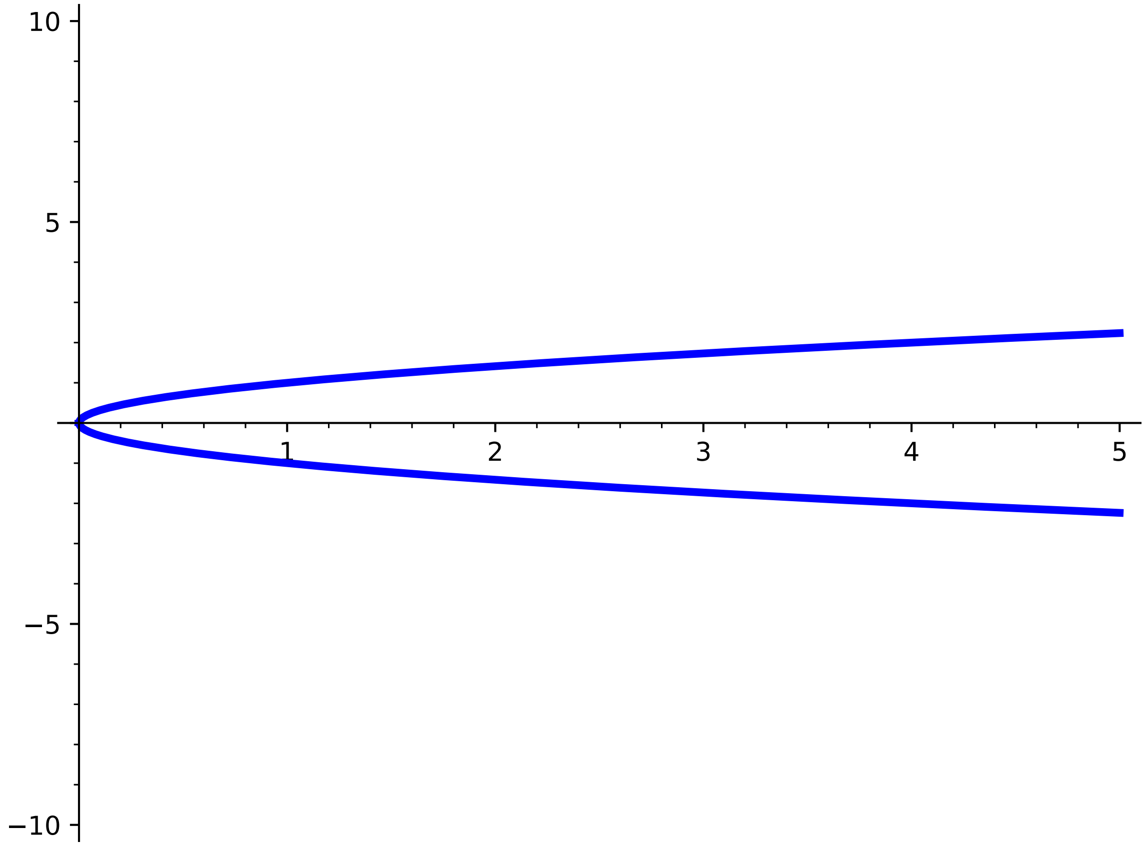
(b)



(c)



(d)



Introduction to Functions (FN1)

Activity 2.1.16 Let's explore how to determine whether an equation represents a function.

(a) Suppose you are given the equation $x = y^2$.

- If $x = 4$, what kind of y-values would you get for $x = y^2$?
- Based on this information, do you think $x = y^2$ is a function?

(b) Suppose you are given the equation $y = 3x^2 + 2$.

- If $x = 4$, what kind of y-values would you get for $y = 3x^2 + 2$?
- Based on this information, do you think $y = 3x^2 + 2$ is a function?

(c) Suppose you are given the equation $x^2 + y^2 = 25$.

- If $x = 4$, what kind of y-values would you get for $x^2 + y^2 = 25$?
- Based on this information, do you think $x^2 + y^2 = 25$ is a function?

(d) Suppose you are given the equation $y = -4x - 3$.

- If $x = 4$, what kind of y-values would you get for $y = -4x - 3$?
- Based on this information, do you think $y = -4x - 3$ is a function?

(e) How can you look at an equation to determine whether or not it is a function?

Remark 2.1.17 Notice that Activity 2.1.16 shows that equations with a y^2 term generally do not define functions. This is because to solve for a squared variable, you must consider both positive and negative inputs. For example, both $2^2 = 4$ and $(-2)^2 = 4$.

Introduction to Functions (FN1)

Activity 2.1.18 It's important to be able to determine the domain of any equation, especially when thinking about functions. Answer the following questions given the equation $y = \sqrt{x}$.

(a) What values of x would give an error (if any)?

A. -2

B. 0

C. 4

D. -5

(b) Based on this information, for what values of x would the equation exist?

A. -2

B. 0

C. 4

D. -5

(c) How can we represent the domain of this equation in interval notation?

A. $(-\infty, 0)$

B. $[0, \infty)$

C. $(0, 0)$

D. $(-\infty, \infty)$

Introduction to Functions (FN1)

Activity 2.1.19 Answer the following questions given the equation $y = -5x + 1$.

(a) What values of x would give an error (if any)?

A. -2

B. 0

C. 4

D. -5

(b) Based on this information, for what values of x would the equation exist?

A. -2

B. 0

C. 4

D. -5

(c) How can we represent the domain of this equation in interval notation?

A. $(-\infty, 0)$

B. $(0, \infty)$

C. $(-5, 1)$

D. $(-\infty, \infty)$

Activity 2.1.20 Answer the following questions given the equation $y = \frac{3}{x-5}$.

(a) What values of x would give an error (if any)?

A. -3

B. 0

C. -4

D. 5

(b) Based on this information, for what values of x would the equation exist?

A. -3

B. 0

C. -4

D. 5

(c) How can we represent the domain of this equation in interval notation?

A. $(-\infty, 5)$

B. $(5, \infty)$

C. $(-5, 5)$

D. $(-\infty, 5)\text{U}(5, \infty)$

Remark 2.1.21 When determining the domain of an equation, it is often easier to first find values of x that make the function undefined. Once you have those values, then you know that x can be any value but those.

2.2 Function Notation (FN2)

Objectives

- Use and interpret function notation to evaluate a function for a given input value and find a corresponding input value given an output value.

Function Notation (FN2)

Remark 2.2.1 As we saw in the last section, we can represent functions in many ways, like using a set of ordered pairs, a graph, a description, or an equation. When describing a function with an equation, we will often use function notation.

If y is written as a function of x, like in the equation

$$y = x + 5,$$

we can replace the y with $f(x)$ and get the function notation

$$f(x) = x + 5.$$

The x is the input variable, and $f(x)$ is the y-value or output that corresponds to x.

Generally, we use the letter f for functions. Other letters are okay as well; $g(x)$ and $h(x)$ are common. If we are using multiple functions at one time, we often denote them with different letters so we can refer to one without any confusion as to which function we mean.

Function Notation (FN2)

Activity 2.2.2 Rewrite the following equations using function notation. In each case, assume y is a function of the variable x.

(a) $y = 2x + 14$

(b) $y + x = 3x^2 - 5$

(c) $\frac{2}{x} - x^4 = y - 5$

Function Notation (FN2)

Activity 2.2.3 Let $f(x) = 3x^2 - 4x + 1$. Find the value of $f(x)$ for the given values of x.

Table 2.2.4

x	$f(x)$
-5	
$-\frac{1}{2}$	
0	
2	
10	

Function Notation (FN2)

Remark 2.2.5 If we are asked to find the value of $f(x)$ for a certain x-value, say $x = 5$, we use the notation $f(5)$ to indicate that.

Function Notation (FN2)

Activity 2.2.6 Let $f(x)$, $g(x)$, and $h(x)$ be defined as shown.

$$f(x) = 3x^2 - 4x + 1$$
$$g(x) = \sqrt{13 - x^2}$$
$$h(x) = \frac{x^2 - 6x + 8}{x^2 - 4x + 3}$$

Find the following, if they exist.

(a) $f(-4)$, $f(0)$, and $f(2)$

(b) $g(0)$, $g(2)$, and $g(8)$

(c) $h(3)$, $h(4)$, and $h(10)$

Remark 2.2.7 Sometimes functions are made up of multiple functions put together. We call these **piecewise functions**. Each piece is defined for only a certain interval, and these intervals do not overlap. When evaluating a piecewise function at a given x-value, we first need to find the interval that includes the x-value, and then plug in to the corresponding function piece.

Function Notation (FN2)

Activity 2.2.8 Let $f(x)$ be a piecewise function as shown below.

$$f(x) = \begin{cases} x^2 + 3, & x < 5 \\ 9 - 2x, & x \geq 5 \end{cases}$$

(a) On which interval from the piecewise function does the value $x = 1$ belong?

A. $x < 5$ B. $x \leq 5$ C. $x > 5$ D. $x \geq 5$

(b) Find $f(1)$.

A. 3 B. 4 C. 5 D. 6 E. 7

(c) On which interval from the piecewise function does the value $x = 5$ belong?

A. $x < 5$ B. $x \leq 5$ C. $x > 5$ D. $x \geq 5$

(d) Find $f(5)$.

A. -10 B. -5 C. -1 D. 17 E. 28

Remark 2.2.9 We've been practicing evaluating functions at specific numeric values. It's also possible to evaluate a function given an expression involving variables.

Function Notation (FN2)

Activity 2.2.10 Let $g(x) = x^2 - 3x$.

(a) Find $g(a)$.

A. $(ax)^2 - 3ax$

B. $a^2 - 3a$

C. $a(x^2 - 3x)$

D. $ax^2 - 3ax$

E. $a - 3$

(b) Find $g(x + h)$.

A. $x^2 - 3x + h$

B. $(x + h)^2 - 3x$

C. $(x + h)^2 - 3(x + h)$

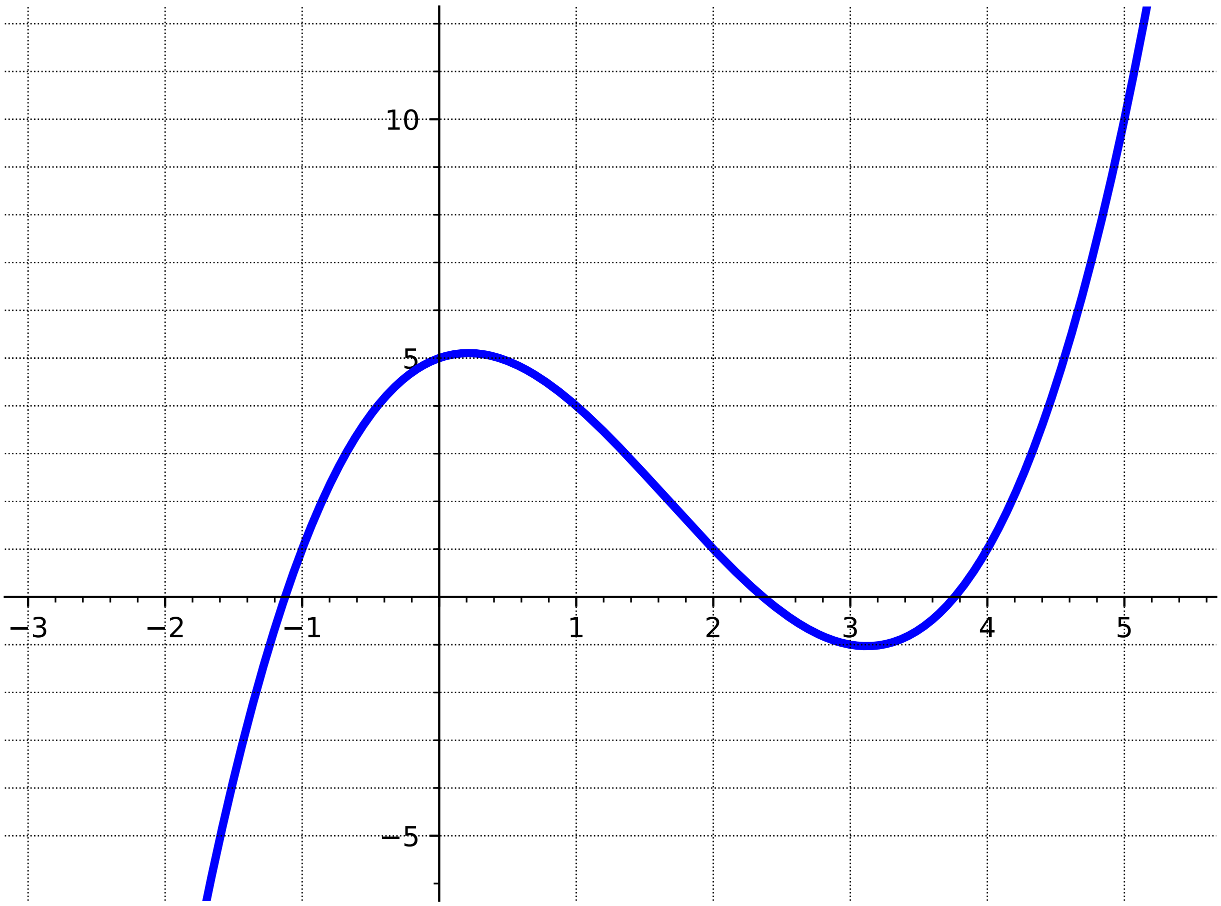
D. $x^2 - 3(x + h)$

Function Notation (FN2)

Remark 2.2.11 We should also be able to look at a graph of a function and evaluate it for different values of x. The next activity explores that.

Function Notation (FN2)

Activity 2.2.12 Let $f(x)$ be the function graphed below.



(a) Find $f(1)$.

A. -4 B. -2 C. 0 D. 2 E. 4

(b) Find $f(3)$.

A. -1 B. 0 C. 1 D. 2 E. 3 F. 4

(c) For which x-value(s) does $f(x) = 1$?

A. -1 B. 0 C. 1 D. 2 E. 3 F. 4

(d) For which x-value(s) does $f(x) = 4$. Estimate as needed!

Function Notation (FN2)

Activity 2.2.13 In these activities, we are flipping the question around. This time we know what the function equals at some x-value, and we want to recover that x-value (or values!).

(a) Let $h(x) = 5x + 7$. Find the x-value(s) such that $h(x) = -13$.

(b) Let $f(x) = x^2 - 3x - 9$. Find the x-value(s) such that $f(x) = 9$.

Function Notation (FN2)

Activity 2.2.14 Ellie has \$13 in her piggy bank, and she gets an additional \$1.50 each week for her allowance. Assuming she does not spend any money, the total amount of allowance, $A(w)$, she has after w weeks can be modeled by the function

$$A(w) = 13 + 1.50w.$$

(a) How much money will be in her piggy bank after 5 weeks?

(b) After how many weeks will she have \$40 in her piggy bank?

2.3 Characteristics of a Function's Graph (FN3)

Objectives

- Use the graph of a function to find the domain and range in interval notation, the x- and y-intercepts, the maxima and minima, and where it is increasing and decreasing using interval notation.

Characteristics of a Function's Graph (FN3)

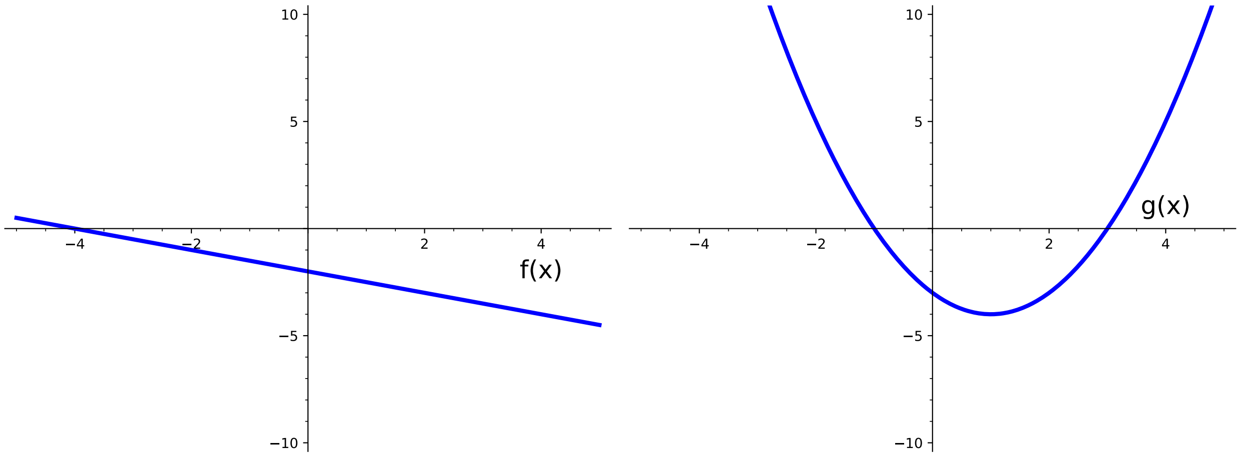
Remark 2.3.1 In this section, we will be looking at different kinds of graphs and will identify various characteristics. These ideas can span all kinds of functions, so you will see these come up multiple times!

Characteristics of a Function's Graph (FN3)

Definition 2.3.2 One of the easiest things to identify from a graph are the **intercepts**, which are points at which the graph crosses the axes. An x**-intercept** is a point at which the graph crosses the x-axis and a y**-intercept** is a point at which the graph crosses the y-axis. Because intercepts are points, they are typically written as an ordered pair: (x, y). ◊

Characteristics of a Function's Graph (FN3)

Activity 2.3.3 Use the following graphs to answer the questions.



(a) What are the x-intercept(s) of $f(x)$?

A. $(0, -4)$
B. $(-2, 0)$
C. $(-4, 0)$
D. $(0, -2)$

(b) What are the x-intercept(s) of $g(x)$?

A. $(0, -3)$
B. $(-1, 0)$
C. $(3, 0)$
D. $(-3, 0)$

(c) What are the y-intercept(s) of $f(x)$?

A. $(0, -4)$
B. $(-2, 0)$
C. $(-4, 0)$
D. $(0, -2)$

(d) What are the y-intercept(s) of $g(x)$?

A. $(0, -3)$
B. $(-1, 0)$
C. $(3, 0)$
D. $(-3, 0)$

(e) Sketch a graph of a function with the following intercepts:

- x-intercepts: $(-2, 0)$ and $(6, 0)$
- y-intercept: $(0, 4)$

(f) Sketch a graph of a function with the following intercepts:

- x-intercept: $(-1, 0)$
- y-intercept: $(0, 6)$ and $(0, -2)$

Remark 2.3.4 Notice in Activity 2.3.3, that a function can have multiple x-intercepts, but only one y-intercept. Having more than one y-intercept would create a graph that is not a function!

Characteristics of a Function's Graph (FN3)

Definition 2.3.5 The **domain** refers to the set of possible input values and the **range** refers to the set of possible output values. If given a graph, however, it would be impossible to list out all the values for the domain and range so we use interval notation to represent the set of values.

Recall that the terms **domain** and **range** were first introduced in Definition 2.1.1.

$\Diamond$

Activity 2.3.6 Use the following graph to answer the questions below.

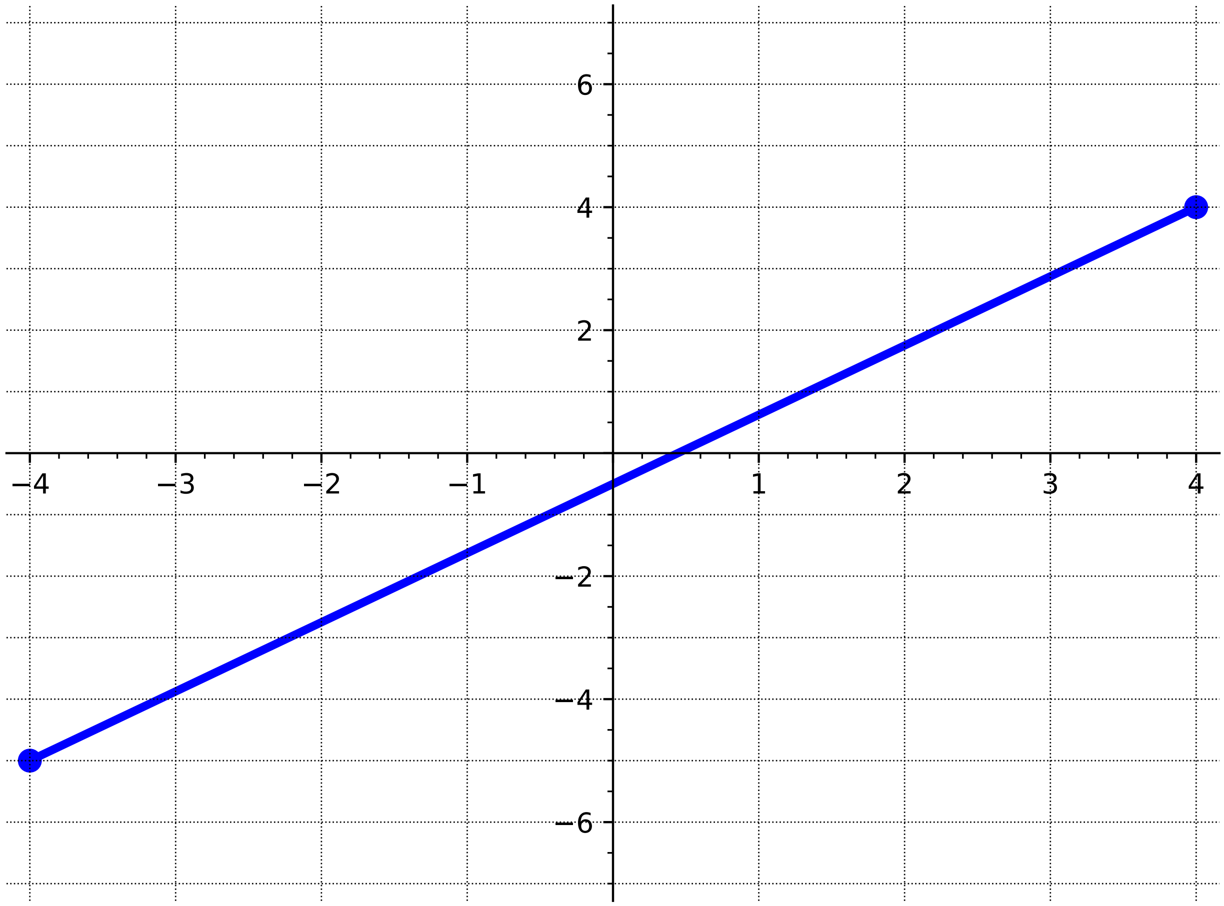


Figure 2.3.7

(a) Draw on the x-axis all the values in the domain.

(b) What interval represents the domain you drew in part (a)?

A. $[4, -4]$
B. $[-4, 4]$
C. $(-4, 4)$
D. $(4, -4)$

(c) Draw on the y-axis all the values in the range.

(d) What interval represents the range you drew in part (c)?

A. $(-5, 4)$
B. $[-4, 4]$
C. $[-5, 4]$
D. $(4, -5)$

Activity 2.3.8 Use the following graph to answer the questions below.

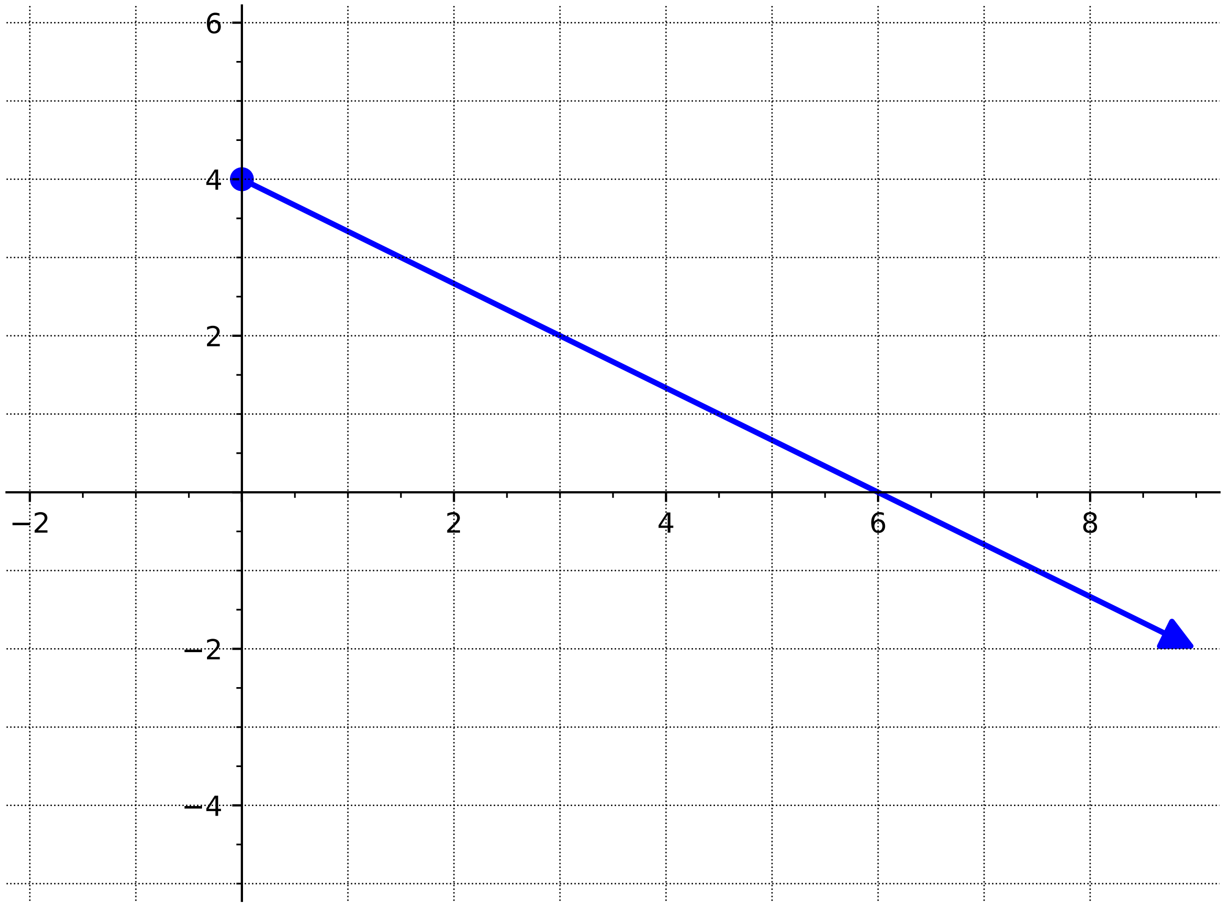


Figure 2.3.9

(a) What is the domain of this graph?

A. $[4, \infty)$

B. $(-\infty, 0]$

C. $(-\infty, 4]$

D. $[0, \infty)$

(b) What is the range of this graph?

A. $[4, \infty)$

B. $(-\infty, 0]$

C. $(-\infty, 4]$

D. $[0, \infty)$

Remark 2.3.10 When writing your intervals for domain and range, notice that you will need to write them from the smallest values to the highest values. For example, we wouldn't write $[4, -\infty)$ as an interval because $-\infty$ is smaller than 4. For domain, read the graph from left to right. For range, read the graph from bottom to top.

Activity 2.3.11 Use the following graph to answer the questions below.

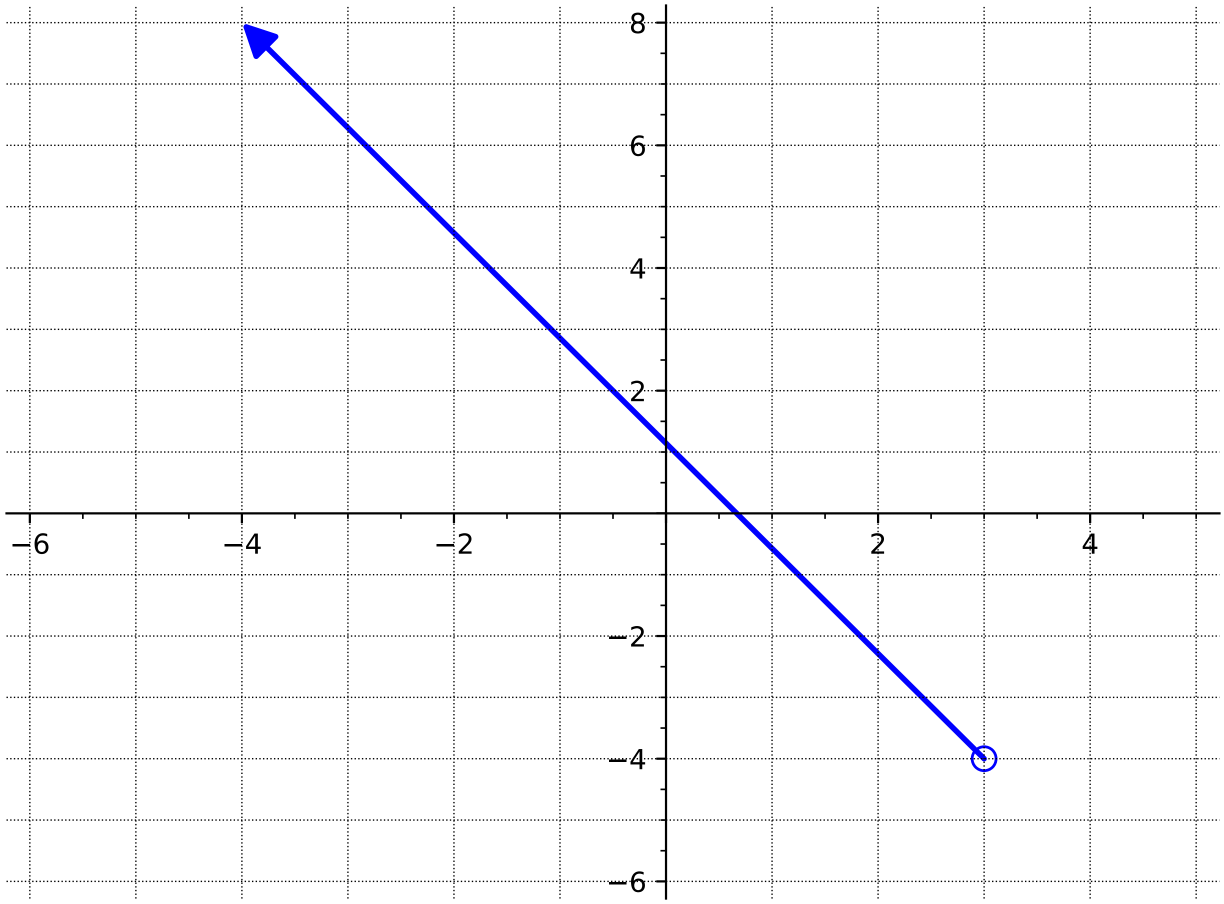


Figure 2.3.12

(a) What is the domain of this graph?

A. $(-\infty, 3)$

B. $(\infty, -4]$

C. $(-4, \infty)$

D. $(-\infty, 3]$

(b) What is the range of this graph?

A. $(-\infty, 3)$

B. $(\infty, -4]$

C. $(-4, \infty)$

D. $(-\infty, 3]$

Activity 2.3.13 Use the following graph to answer the questions below.

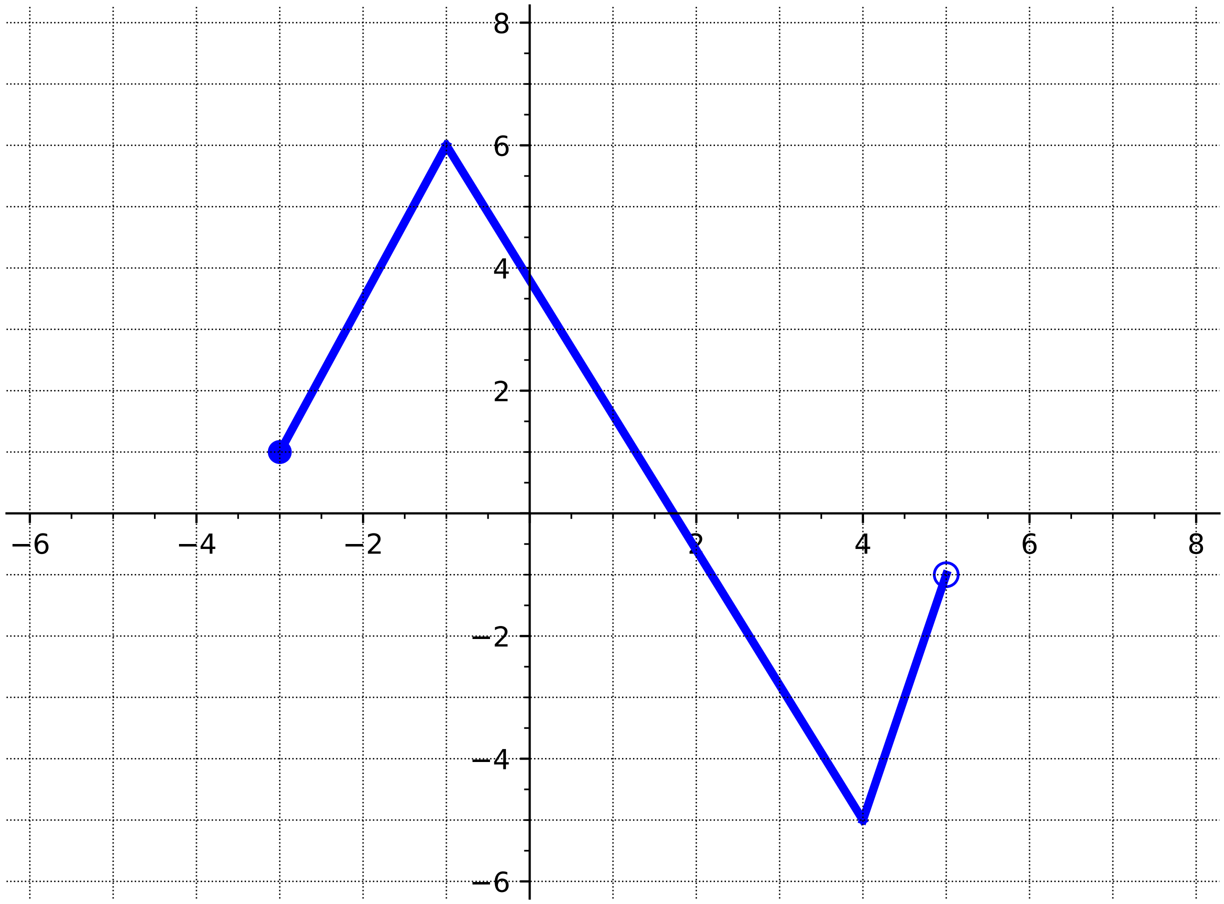


Figure 2.3.14

(a) What is the domain of this graph?

A. $(-3, 5)$

B. $(-5, 7)$

C. $[-5, 7]$

D. $[-3, 5)$

(b) What is the range of this graph?

A. $(-3, 5)$

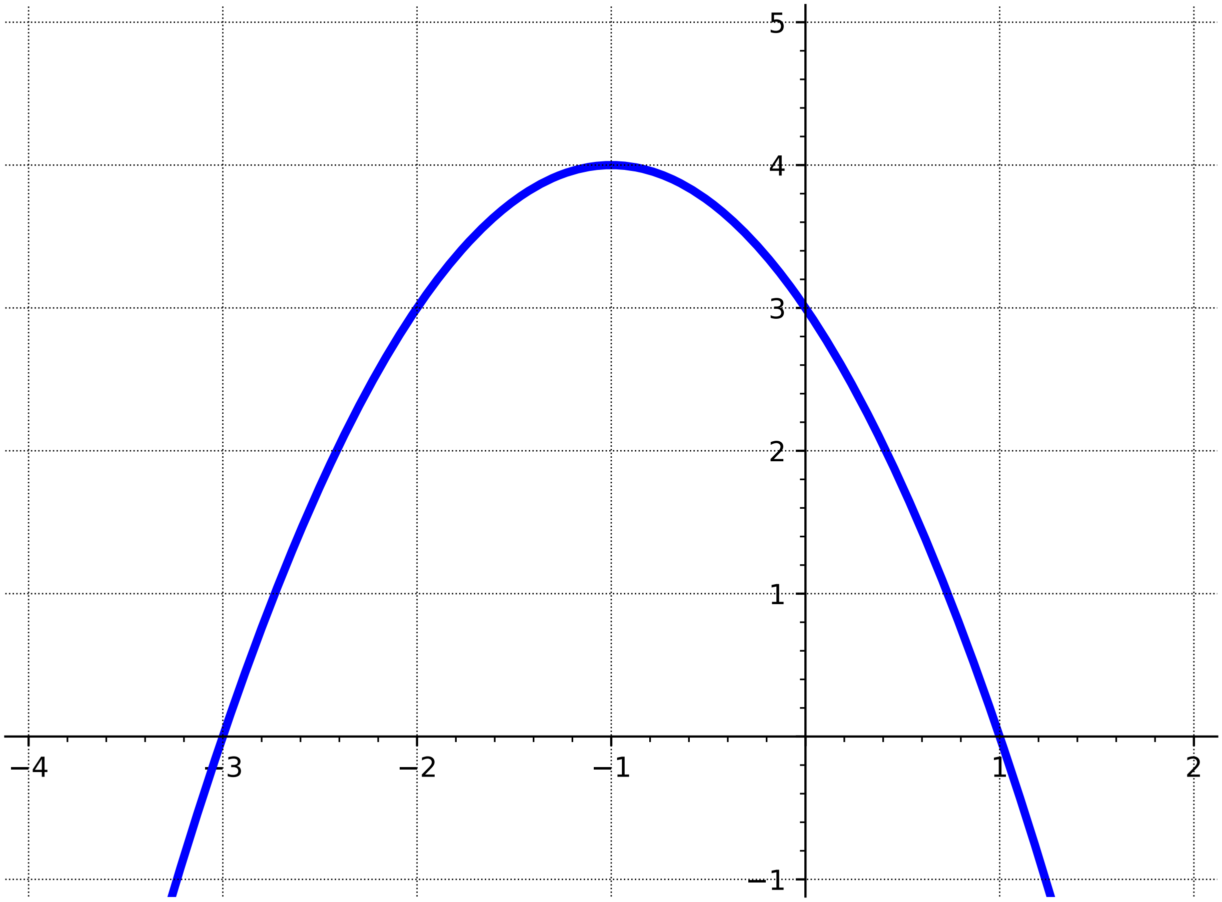
B. $(-5, 6)$

C. $[-5, 6]$

D. $[-3, 5)$

Remark 2.3.15 Notice that finding the domain and range can be tricky! Be sure to pay attention to the x- and y-values of the entire graph - not just the endpoints!

Activity 2.3.16 In this activity, we will look at where the function is increasing and decreasing. Use the following graph to answer the questions below.



(a) Where do you think the graph is increasing?

(b) Which interval best represents where the function is increasing?

A. $(-\infty, -1]$

B. $(-\infty, -1)$

C. $(-1, \infty)$

D. $[-1, \infty)$

(c) Where do you think the graph is decreasing?

(d) Which interval best represents where the function is decreasing?

A. $(-\infty, -1]$

B. $(-\infty, -1)$

C. $(-1, \infty)$

D. $[-1, \infty)$

(e) Based on what you see on the graph, do you think this graph has any maxima or minima?

Characteristics of a Function's Graph (FN3)

Definition 2.3.17 As you noticed in Activity 2.3.16, functions can increase or decrease (or even remain constant!) for a period of time. The **interval of increase** is when the y-values of the function increase as the x-values increase. The **interval of decrease** is when the y-values of the function decrease as the x-values increase. The function is constant when the y-values remain constant as x-values increase (also known as the **constant interval**).

The easiest way to identify these intervals is to read the graph from left to right and look at what is happening to the y-values. ◊

Characteristics of a Function's Graph (FN3)

Definition 2.3.18 The **maximum**, or **global maximum**, of a graph is the point where the y-coordinate has the largest value. The **minimum**, or **global minimum** is the point on the graph where the y-coordinate has the smallest value.

Graphs can also have **local maximums** and **local minimums**. A local maximum point is a point where the function value (i.e, y-value) is larger than all others in some neighborhood around the point. Similarly, a local minimum point is a point where the function value (i.e, y-value) is smaller than all others in some neighborhood around the point. $\Diamond$

Remark 2.3.19 Global extrema are sometimes referred to as absolute extrema, while local extrema are sometimes referred to as relative extrema.

Activity 2.3.20 Use the following graph to answer the questions below.

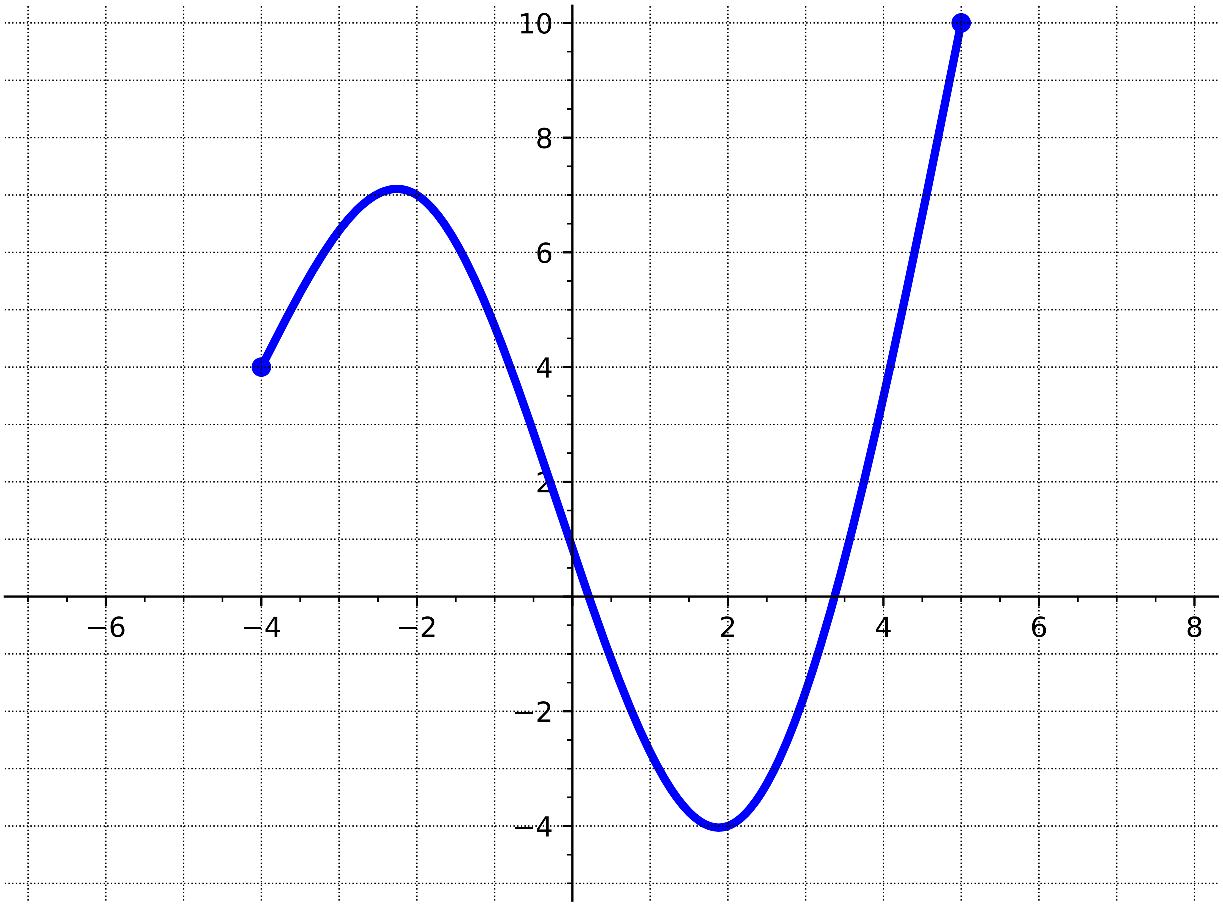


Figure 2.3.21

(a) At what value of x is there a global maximum?

A. $x = -4$

B. $x = -2$

C. $x = 2$

D. $x = 5$

(b) What is the global maximum value?

A. 10

B. 7

C. 4

D. -4

(c) At what value of x is there a global minimum?

A. $x = -4$

B. $x = -2$

C. $x = 2$

D. $x = 5$

(d) What is the global minimum value?

A. 10

B. 7

C. 4

D. -4

(e) At approximately what value of x is there a local maximum?

A. $x \approx -4$

B. $x \approx -2$

C. $x \approx 2$

D. $x \approx 5$

(f) What is the local maximum value?

A. 10

B. 7

C. 4

D. -4

(g) At approximately what value of x is there a local minimum?

A. $x \approx -4$

B. $x \approx -2$

C. $x \approx 2$

D. $x \approx 5$

(h) What is the local minimum value?

A. 10

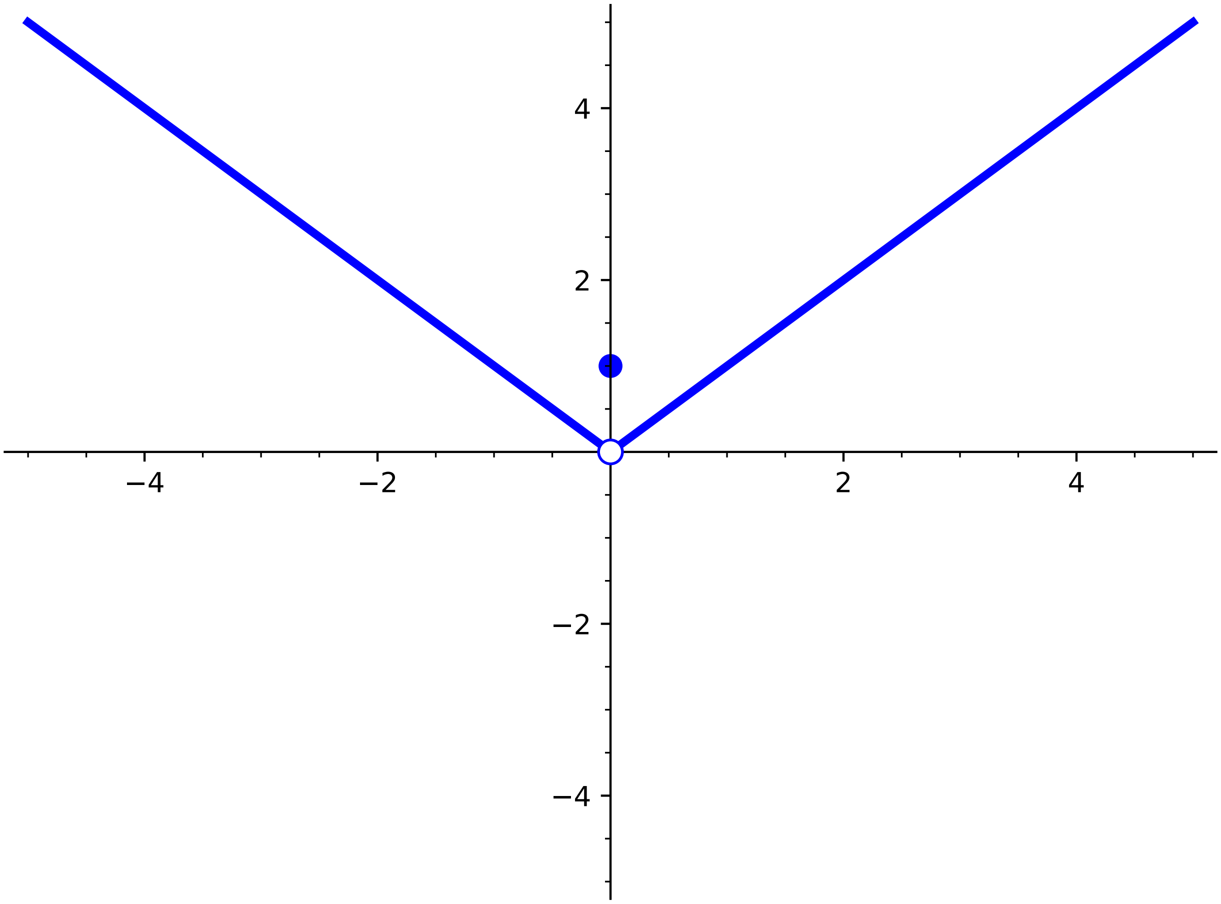
B. 7

C. 4

D. -4

Remark 2.3.22 Notice that in Activity 2.3.20, there are two ways we talk about max and min. We might want to know the location of where the max or min are (i.e., determining at which x-value the max or min occurs at) or we might want to know what the max or min values are (i.e., the y-value). Also, note that in Activity 2.3.20, a local minimum is also a global minimum.

Activity 2.3.23 Sometimes, it is not always clear what the maxima or minima are or if they exist. Consider the following graph of $f(x)$:



(a) What is the value of $f(0)$?

A. 1

B. 0

C. There is no local minimum

(b) What is the local minimum of $f(x)$?

A. 1

B. 0

C. There is no local minimum

(c) What is the global minimum of $f(x)$?

A. 1

B. 0

C. There is no global minimum

Activity 2.3.24 Use the following graph to answer the questions below.

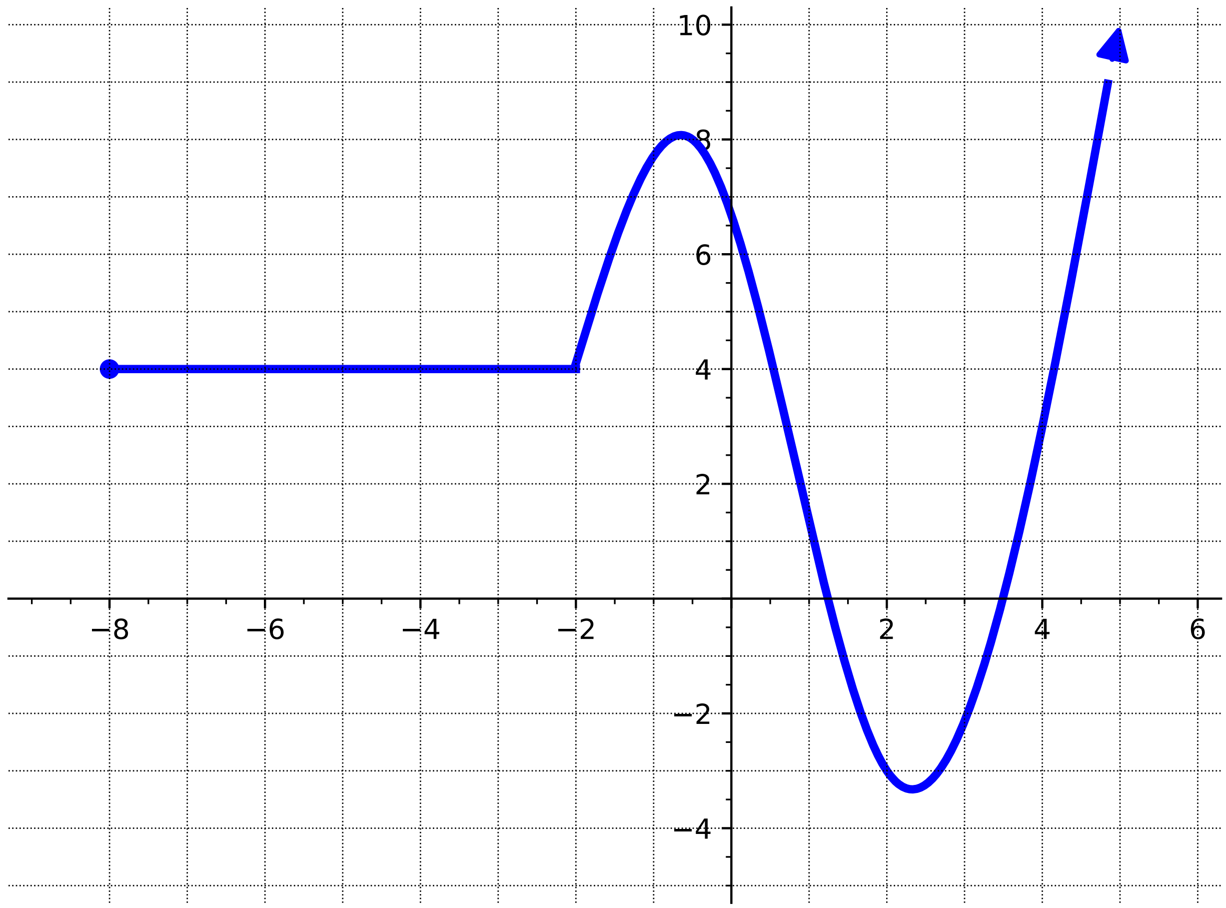


Figure 2.3.25

(a) What is the domain?

(b) What is the range?

(c) What is the x-intercept(s)?

(d) What is the y-intercept?

(e) Where is the function increasing?

(f) Where is the function decreasing?

(g) Where is the constant interval?

(h) At what x-values do the local maxima occur?

(i) At what x-values do the local minima occur?

(j) What are the global max and min?

2.4 Transformation of Functions (FN4)

Objectives

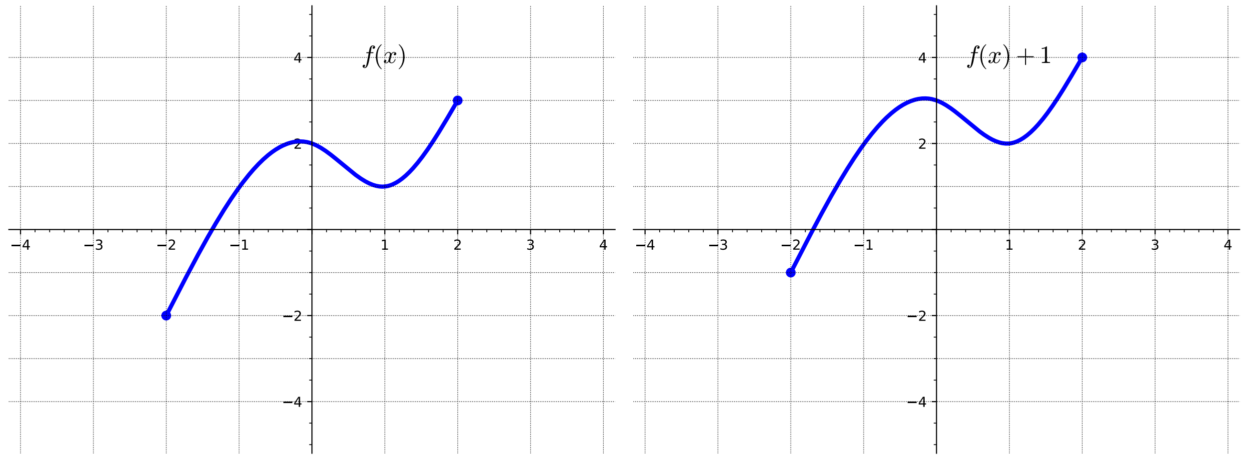
- Apply transformations including horizontal and vertical shifts, stretches, and reflections to a function. Express the result of these transformations graphically and algebraically.

Transformation of Functions (FN4)

Remark 2.4.1 Informally, a transformation of a given function is an algebraic process by which we change the function to a related function that has the same fundamental shape, but may be shifted, reflected, and/or stretched in a systematic way.

Transformation of Functions (FN4)

Activity 2.4.2 Consider the following two graphs.



(a) How is the graph of $f(x) + 1$ related to that of $f(x)$?

A. Shifted up 1 unit

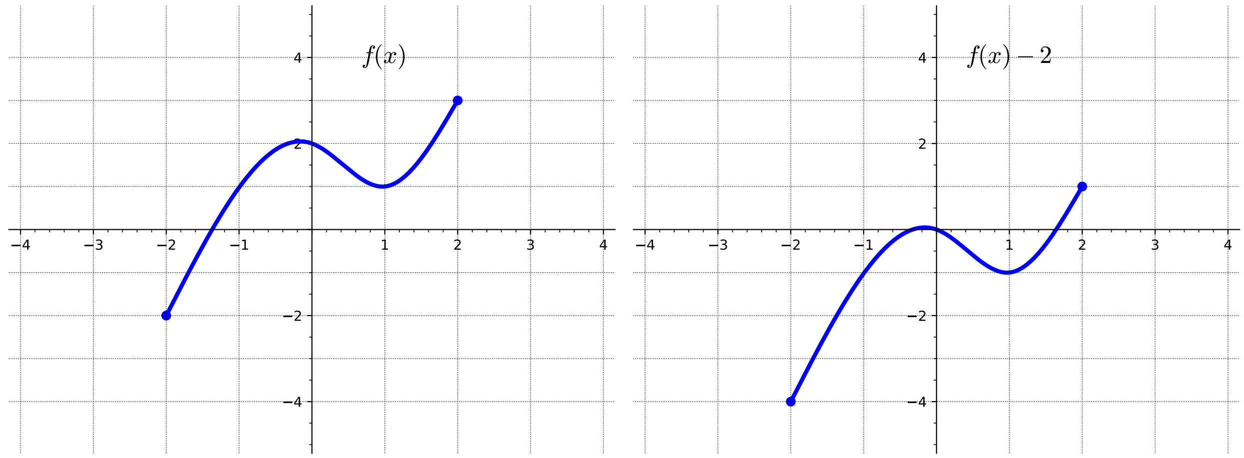
B. Shifted left 1 unit

C. Shifted down 1 unit

D. Shifted right 1 unit

Transformation of Functions (FN4)

Activity 2.4.3 Consider the following two graphs.



(a) How is the graph of $f(x) - 2$ related to that of $f(x)$?

A. Shifted up 2 units

B. Shifted left 2 units

C. Shifted down 2 units

D. Shifted right 2 units

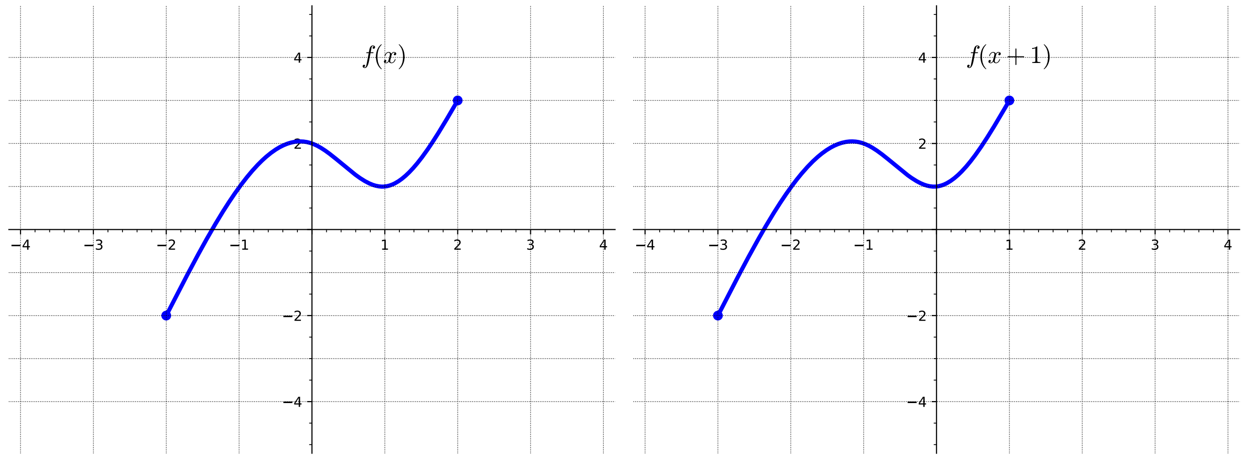
Remark 2.4.4 Notice that in Activity 2.4.2 and Activity 2.4.3, the y-values of the transformed graph are changed while the x-values remain the same.

Transformation of Functions (FN4)

Definition 2.4.5 Given a function $f(x)$ and a constant c, the transformed function $g(x) = f(x) + c$ is a **vertical translation** of the graph of $f(x)$. That is, all the outputs change by c units. If c is positive, the graph will shift up. If c is negative, the graph will shift down. $\Diamond$

Transformation of Functions (FN4)

Activity 2.4.6 Consider the following two graphs.



(a) How is the graph of $f(x+1)$ related to that of $f(x)$?

A. Shifted up by 1 unit

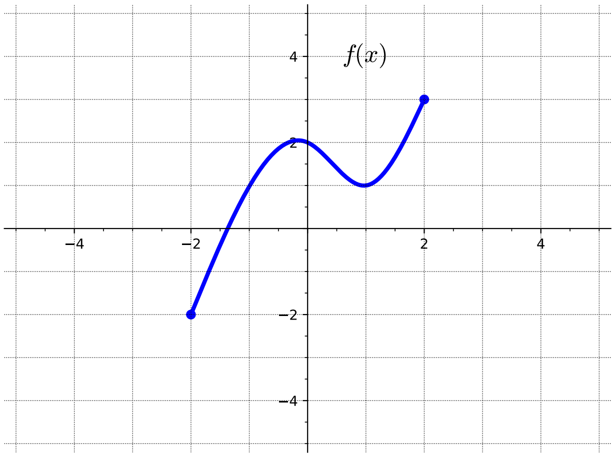
B. Shifted left 1 unit

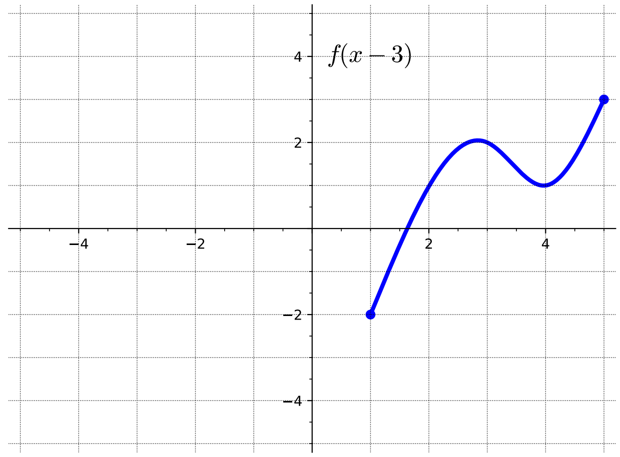
C. Shifted down 1 unit

D. Shifted right 1 unit

Transformation of Functions (FN4)

Activity 2.4.7 Consider the following two graphs.





(a) How is the graph of $f(x-3)$ related to that of $f(x)$?

A. Shifted up by 3 units

B. Shifted left 3 units

C. Shifted down 3 units

D. Shifted right 3 units

Remark 2.4.8 Notice that in Activity 2.4.6 and Activity 2.4.7, the x-values of the transformed graph are changed while the y-values remain the same.

Transformation of Functions (FN4)

Definition 2.4.9 Given a function $f(x)$ and a constant c, the transformed function $g(x) = f(x+c)$ is a **horizontal translation** of the graph of $f(x)$. If c is positive, the graph will shift left. If c is negative, the graph will shift right. $\Diamond$

Transformation of Functions (FN4)

Activity 2.4.10 Describe how the graph of the function is a transformation of the graph of the original function f.

(a) $f(x-4)+1$

A. Shifted down 4 units

B. Shifted left 4 units

C. Shifted down 1 unit

D. Shifted right 4 units

E. Shifted up 1 unit

(b) $f(x+3)-2$

A. Shifted down 2 units

B. Shifted left 3 units

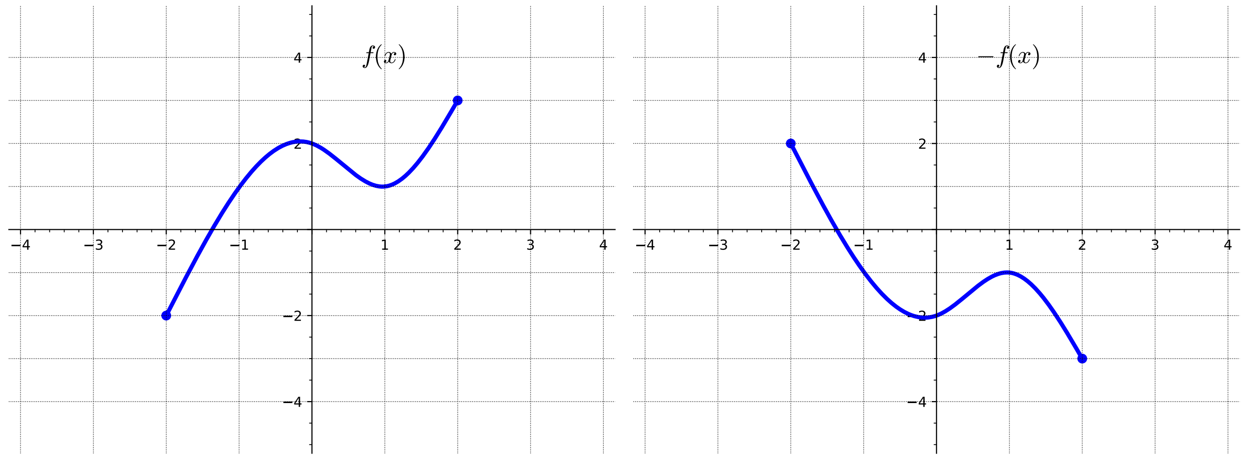
C. Shifted up 3 unit

D. Shifted right 3 units

E. Shifted up 2 unit

Transformation of Functions (FN4)

Activity 2.4.11 Consider the following two graphs.



(a) How is the graph of $-f(x)$ related to that of $f(x)$?

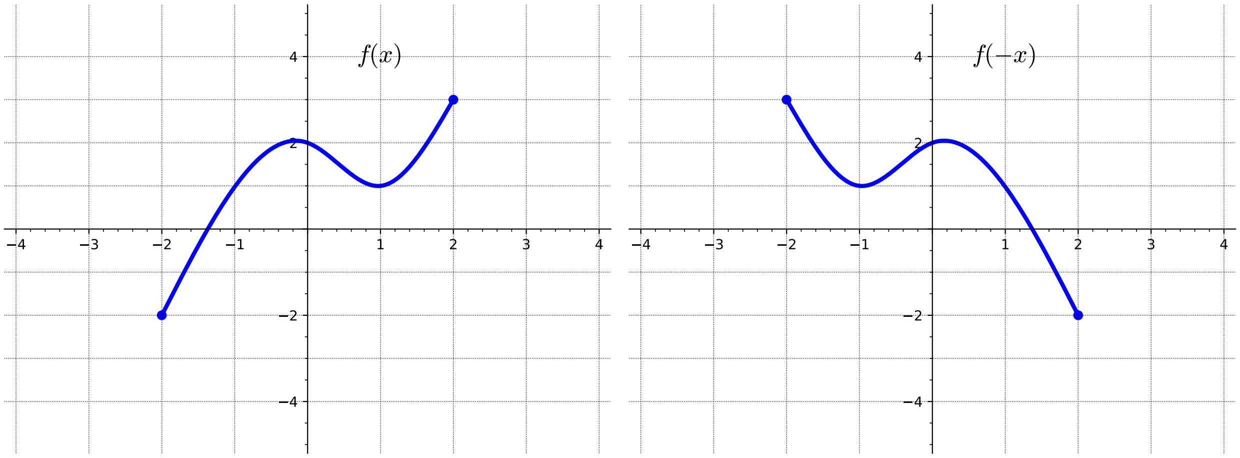
A. Shifted down 2 units

B. Reflected over the x-axis

C. Reflected over the y-axis

D. Shifted right 2 units

Activity 2.4.12 Consider the following two graphs.



(a) How is the graph of $f(-x)$ related to that of $f(x)$?

A. Shifted down 2 units

B. Reflected over the x-axis

C. Reflected over the y-axis

D. Shifted left 2 units

Remark 2.4.13 Notice that in Activity 2.4.11, the y-values of the transformed graph are changed while the x-values remain the same. While in Activity 2.4.12, the x-values of the transformed graph are changed while the y-values remain the same.

Transformation of Functions (FN4)

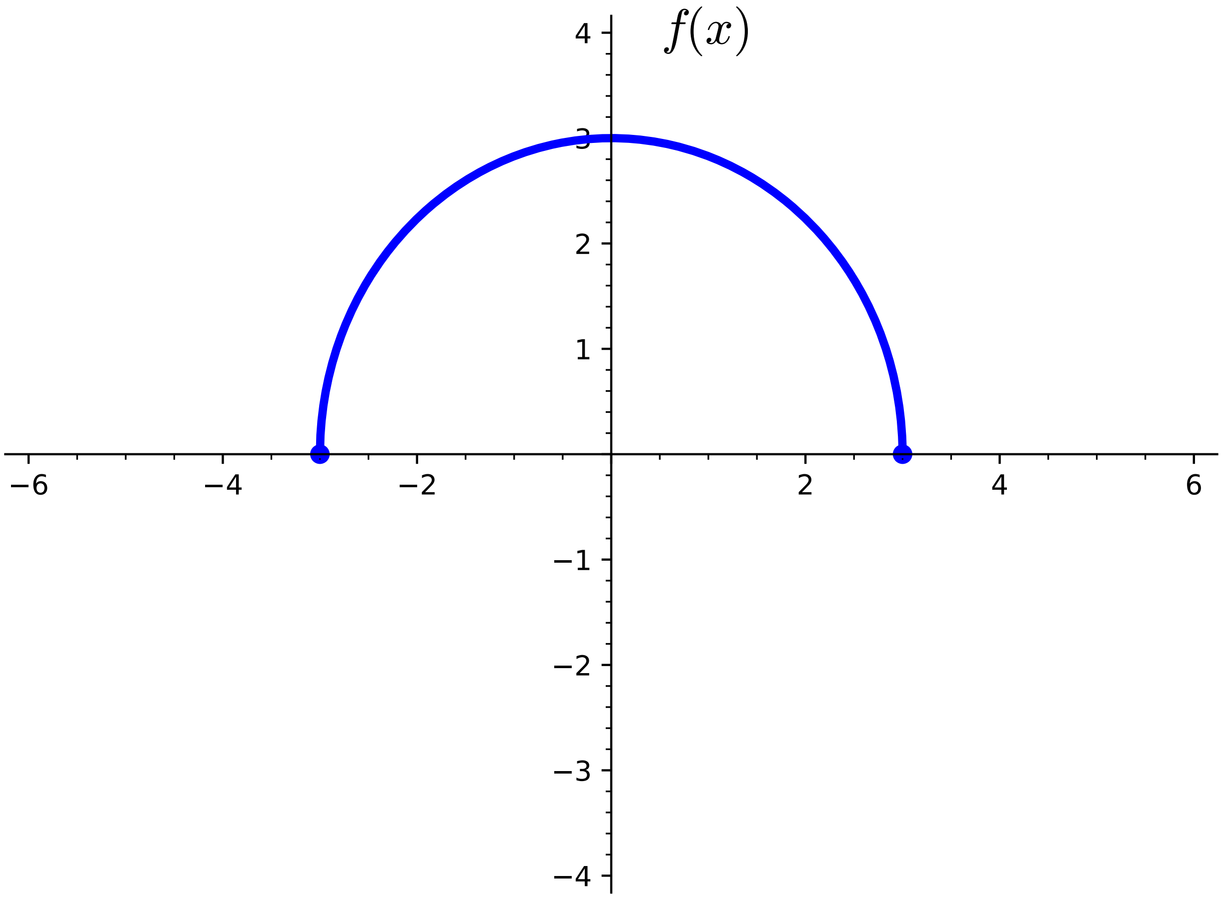
Definition 2.4.14 Given a function $f(x)$, the transformed function $g(x) = -f(x)$ is a **vertical reflection** of the graph of $f(x)$. That is, all the outputs are multiplied by -1. The new graph is a reflection of the old graph about the x-axis. $\diamond$

Transformation of Functions (FN4)

Definition 2.4.15 Given a function $f(x)$, the transformed function $y = g(x) = f(-x)$ is a **horizontal reflection** of the graph of $f(x)$. That is, all the inputs are multiplied by -1. The new graph is a reflection of the old graph about the y-axis. $\Diamond$

Transformation of Functions (FN4)

Activity 2.4.16 Consider the following graph of the function $f(x)$.



(a) How is the graph of $-f(x+2)+3$ related to that of $f(x)$?

A. Shifted up 2 units

B. Shifted up 3 units

C. Reflected over the x-axis

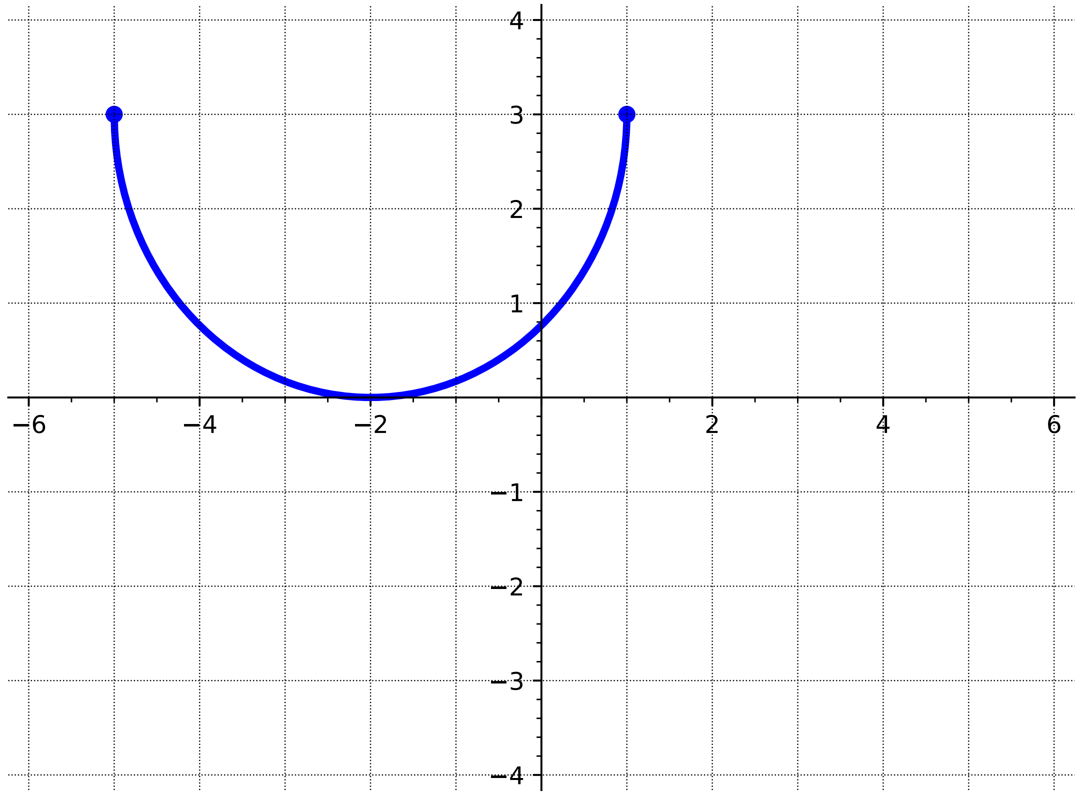
D. Reflected over the y-axis

E. Shifted left 3 units

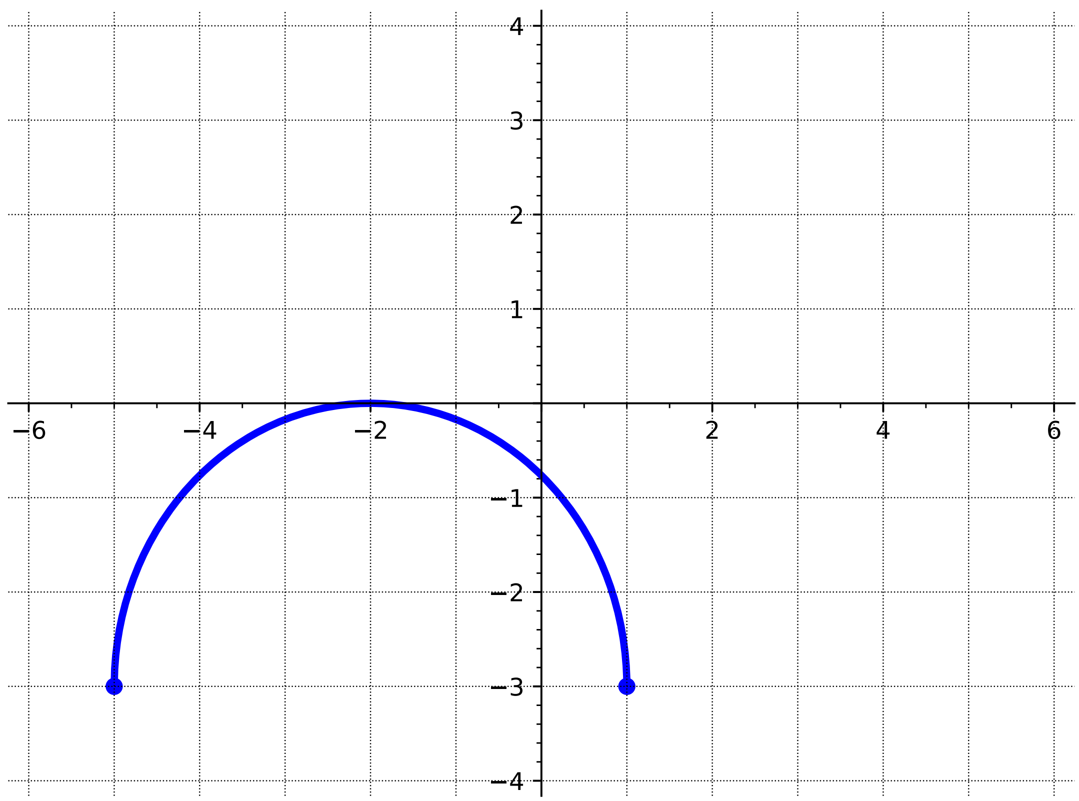
F. Shifted left 2 units

(b) Which of the following represents the graph of the transformed function $g(x) = -f(x+2)+3$?

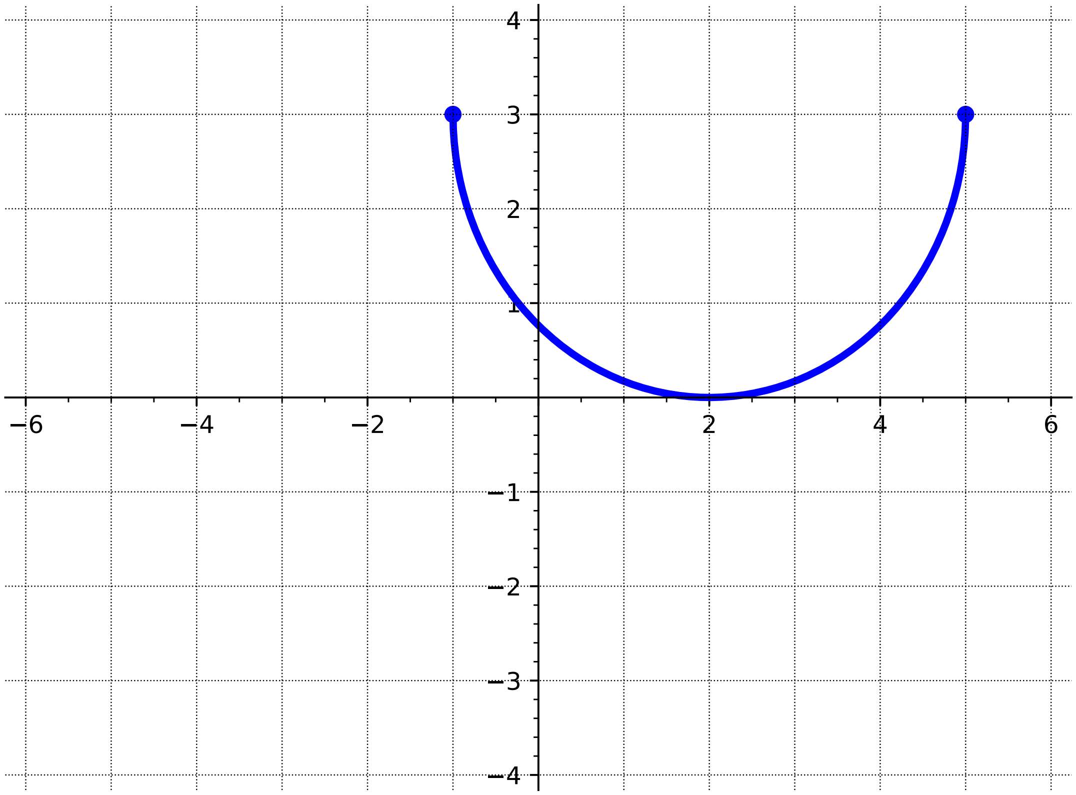
Transformation of Functions (FN4)



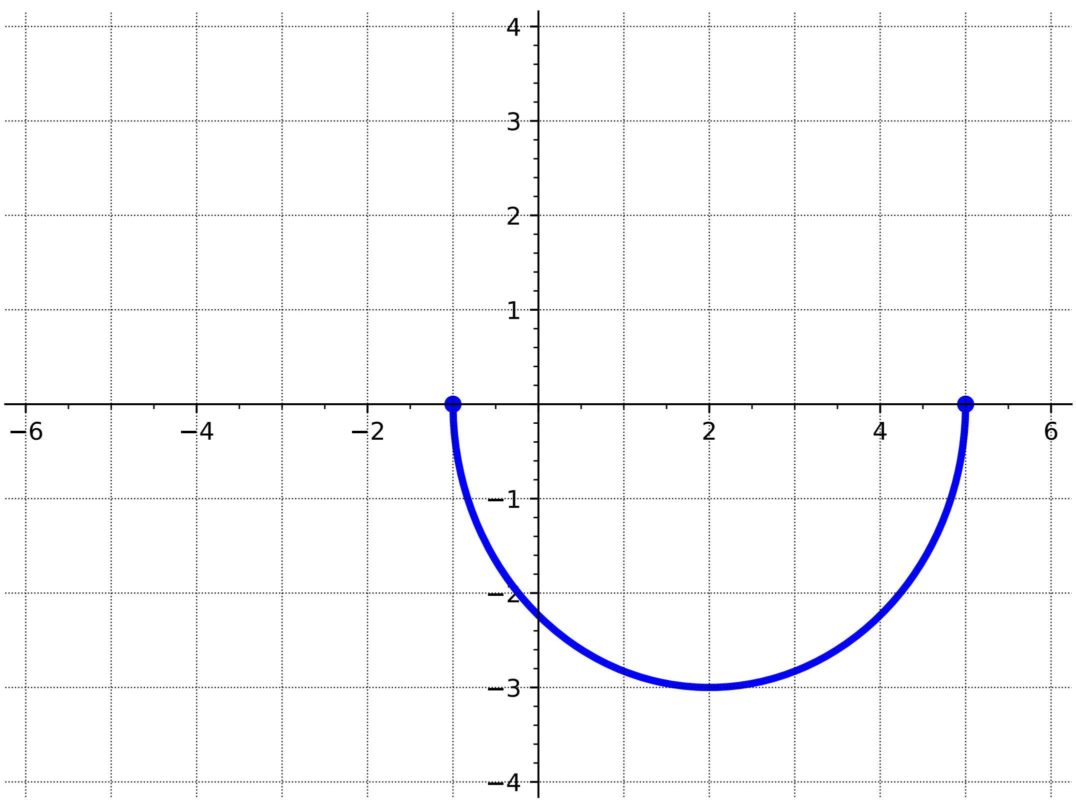
A.



B.



C.

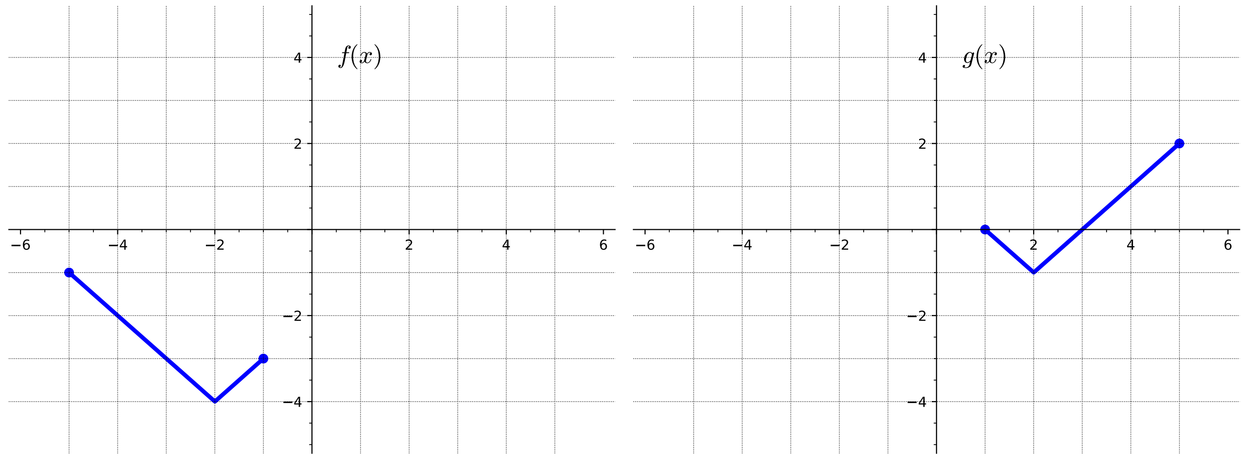


D.

Remark 2.4.17 Notice that in Activity 2.4.16 the resulting graph is different if you perform the reflection first and then the vertical shift, versus the other order. When combining transformations, it is very important to consider the order of the transformations. Be sure to follow the order of operations.

Transformation of Functions (FN4)

Activity 2.4.18 Consider the following two graphs.



(a) How is the graph of $g(x)$ related to that of $f(x)$?

A. Shifted up 3 units

B. Shifted up 1 unit

C. Reflected over the x-axis

D. Reflected over the y-axis

E. Shifted left 1 unit

F. Shifted right 4 units

(b) List the order the transformations must be applied.

(c) Write an equation for the graphed function $g(x)$ using transformations of the graph $f(x)$.

A. $g(x) = -f(x) + 3$

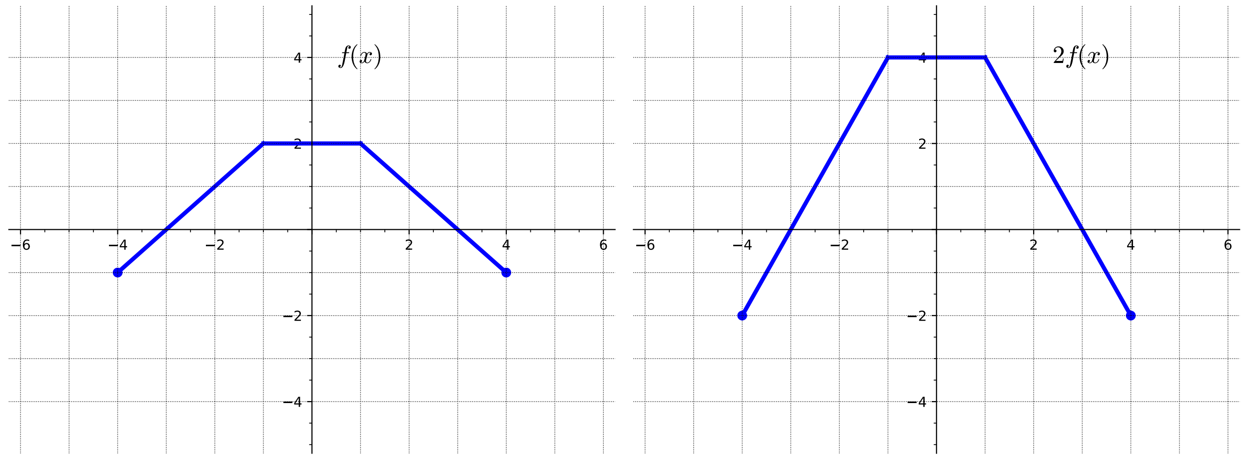
B. $g(x) = f(-x) + 3$

C. $g(x) = f(-x + 3)$

D. $g(x) = -f(x + 3)$

Transformation of Functions (FN4)

Activity 2.4.19 Consider the following two graphs.



(a) Consider the y-value of the two graphs at $x = 1$. How do they compare?

A. The y-value of $2f(x)$ is twice that of $f(x)$.

B. The y-value of $2f(x)$ is half that of $f(x)$.

C. The y-value of $2f(x)$ and $f(x)$ are the same.

D. The y-value of $2f(x)$ is negative that of $f(x)$.

(b) How is the graph of $2f(x)$ related to that of $f(x)$?

A. Vertically stretched by a factor of 2

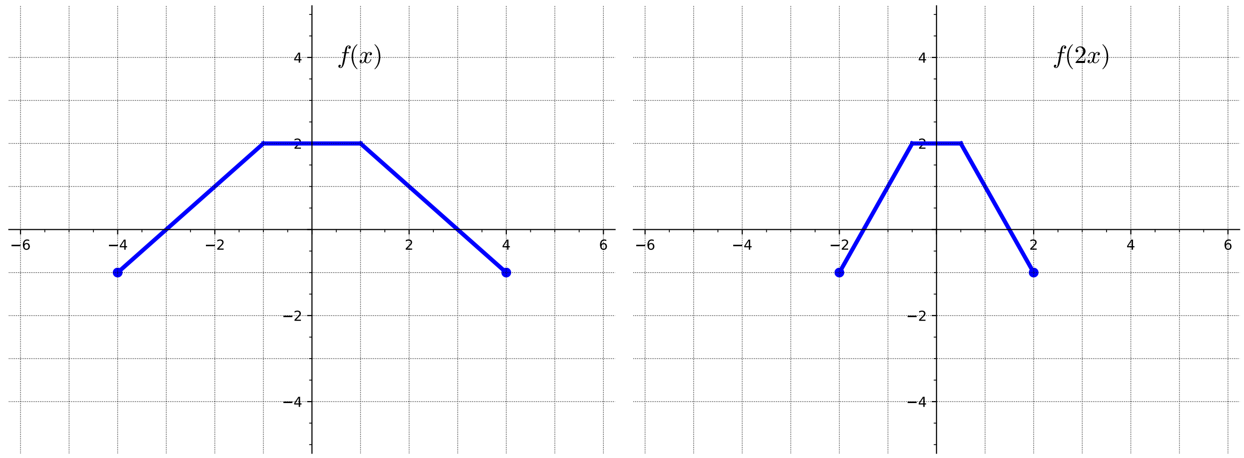
B. Vertically compressed by a factor of 2

C. Horizontally stretched by a factor of 2

D. Horizontally compressed by a factor of 2

Transformation of Functions (FN4)

Activity 2.4.20 Consider the following two graphs.



(a) Consider a x-value of the two graphs at $y = 1$. How do they compare?

A. The x-value of $2f(x)$ is twice that of $f(x)$.

B. The x-value of $2f(x)$ is half that of $f(x)$.

C. The x-value of $2f(x)$ and $f(x)$ are the same.

D. The x-value of $2f(x)$ is negative that of $f(x)$.

(b) How is the graph of $f(2x)$ related to that of $f(x)$?

A. Vertically stretched by a factor of 2

B. Vertically compressed by a factor of 2

C. Horizontally stretched by a factor of 2

D. Horizontally compressed by a factor of 2

Remark 2.4.21 Notice that in Activity 2.4.19 the y-values are doubled while the x-values remain the same. While, in Activity 2.4.20 the x-values are cut in half while the y-values remain the same.

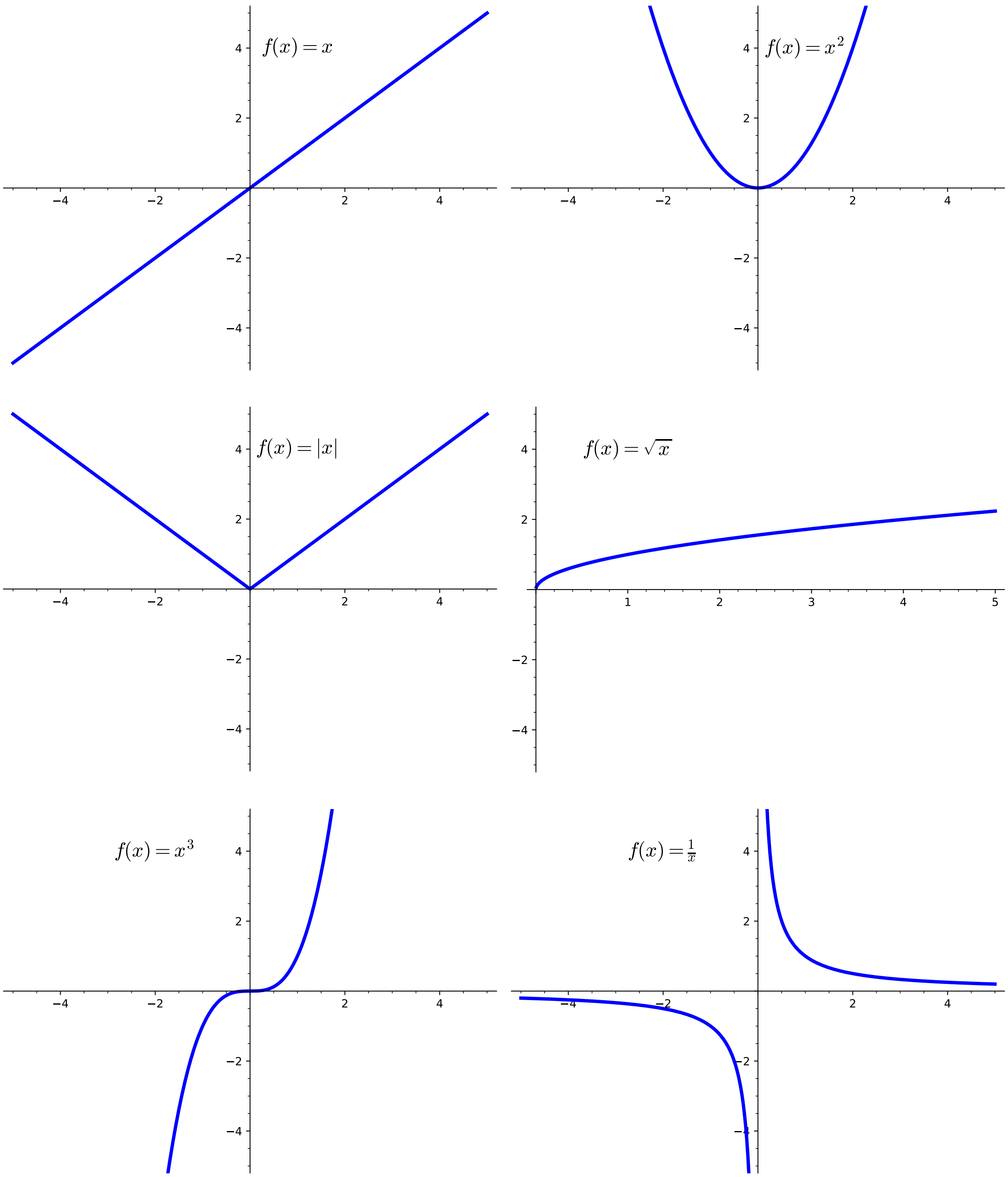
Transformation of Functions (FN4)

Definition 2.4.22 Given a function $f(x)$, the transformed function $g(x) = af(x)$ is a **vertical stretch** or **vertical compression** of the graph of $f(x)$. That is, all the outputs are multiplied by a. If $a > 1$, the new graph is a vertical stretch of the old graph away from the x-axis. If $0 < a < 1$, the new graph is a vertical compression of the old graph towards the x-axis. Points on the x-axis are unchanged. $\Diamond$

Transformation of Functions (FN4)

Definition 2.4.23 Given a function $f(x)$, the transformed function $g(x) = f(ax)$ is a **horizontal stretch** or **horizontal compression** of the graph of $f(x)$. That is, all the inputs are divided by a. If $a > 1$, the new graph is a horizontal compression of the old graph toward the y-axis. If $0 < a < 1$, the new graph is a horizontal stretch of the old graph away from the y-axis. Points on the y-axis are unchanged. $\Diamond$

Remark 2.4.24 We often use a set of basic functions with which to begin transformations. We call these parent functions.



Transformation of Functions (FN4)

Activity 2.4.25 Consider the function $g(x) = 3\sqrt{-x} + 2$

(a) Identify the parent function $f(x)$.

A. $f(x) = x^2$

B. $f(x) = |x|$

C. $f(x) = \sqrt{x}$

D. $f(x) = x$

(b) Graph the parent function $f(x)$.

(c) How is the graph of $g(x)$ related to that of the parent function$f(x)$?

A. Reflected over the x-axis

B. Reflected over the y-axis

C. Shifted down 2 units

D. Shifted up 2 units

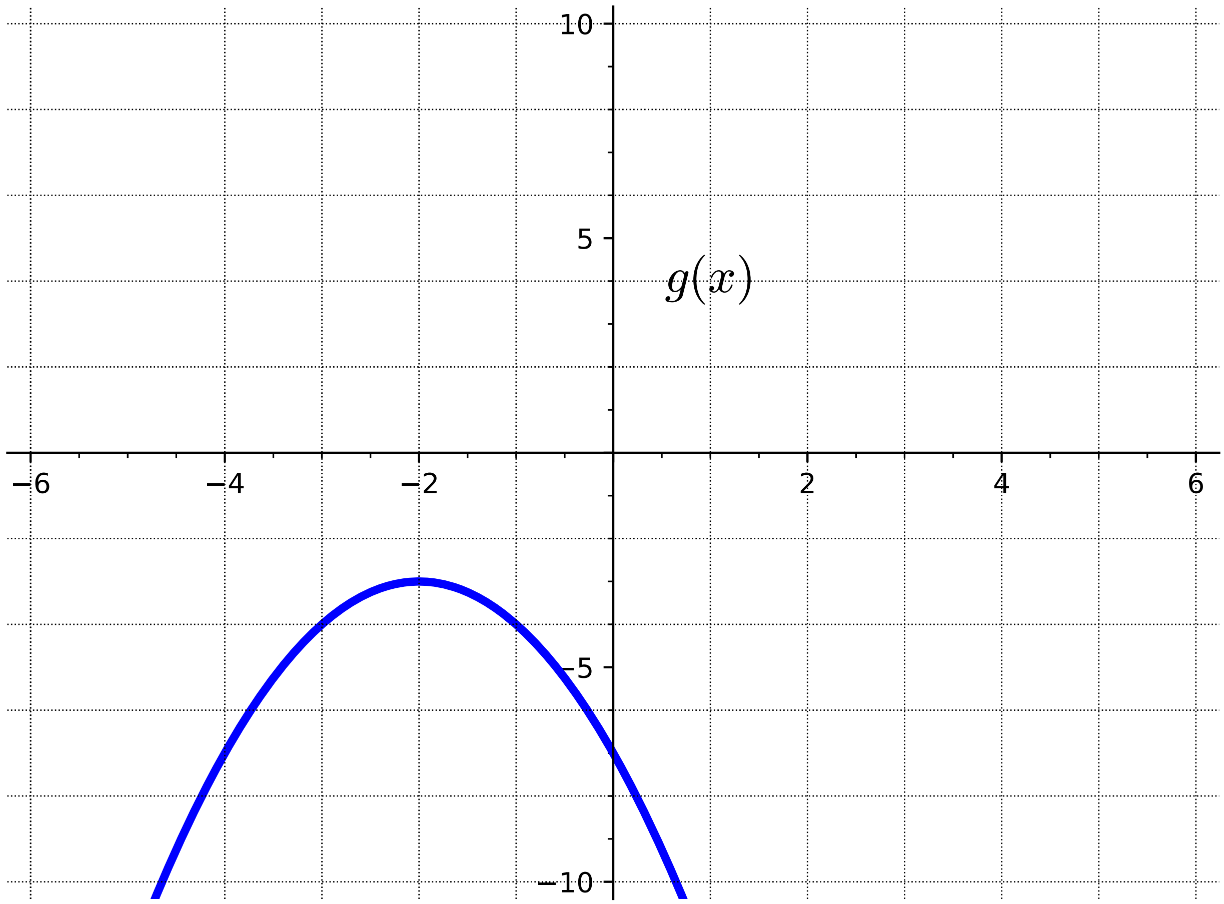
E. Vertically stretched by a factor of 3

F. Horizontally compressed by a factor of 3

(d) Graph the transformed function $g(x)$.

Transformation of Functions (FN4)

Activity 2.4.26 Consider the following graph of the function $g(x)$.



(a) Identify the parent function.

A. $f(x) = x^2$

B. $f(x) = |x|$

C. $f(x) = \sqrt{x}$

D. $f(x) = x$

(b) How is the graph of $g(x)$ related to that of the parent function$f(x)$?

A. Reflected over the x-axis

B. Reflected over the y-axis

C. Shifted down 3 units

D. Shifted up 3 units

E. Shifted left 2 units

F. Shifted right 2 units

(c) Write an equation to represent the transformed function $g(x)$.

A. $g(x) = -(x-2)^2 - 3$

B. $g(x) = -(x+2)^2 + 3$

C. $g(x) = (-x+2)^2 - 3$

D. $g(x) = -(x+2)^2 - 3$

2.5 Combining and Composing Functions (FN5)

Objectives

- Find the sum, difference, product, quotient, and composition of two or more functions and evaluate them.

Combining and Composing Functions (FN5)

Activity 2.5.1 Let $f(x) = x^2 - 3x$ and $g(x) = x^3 - 4x^2 + 7$.

(a) Which of the following seems likely to be the most simplified form of $f(x) + g(x)$?

A. $x^2 - 3x + x^3 - 4x^2 + 7$

B. $x^5 - 7x^3 + 7$

C. $-x^3 + 5x^2 - 3x - 7$

D. $x^3 - 3x^2 - 3x + 7$

(b) Which of the following seems likely to be the most simplified form of $f(x) - g(x)$?

A. $x^3 - 3x^2 - 3x + 7$

B. $-x^3 + 5x^2 - 3x - 7$

C. $-x^3 - 3x^2 - 3x + 7$

D. $x^2 - 3x - x^3 + 4x^2 - 7$

Activity 2.5.2 Let $f(x) = \sqrt{x+1}$ and $g(x) = 5x$.

(a) Which of the following seems likely to be the most simplified form of $f(x) \cdot g(x)$?

A. $\sqrt{5x+1}$

B. $5\sqrt{x+1}$

C. $\sqrt{5x^2+5x}$

D. $5x\sqrt{x+1}$

(b) Which of the following seems likely to be the most simplified form of $\dfrac{f(x)}{g(x)}$?

A. $\dfrac{5x}{\sqrt{x+1}}$

B. $\dfrac{\sqrt{x+1}}{5x}$

C. $\sqrt{\dfrac{x}{5x}+\dfrac{1}{5x}}$

D. $\sqrt{\dfrac{5x}{x}+\dfrac{5x}{1}}$

Remark 2.5.3 In Activity 2.5.1 and Activity 2.5.2, we have found the sum, difference, product, and quotient of two functions. We can use the following notation for these newly created functions:

$$
\begin{aligned}
(f+g)(x) &= f(x)+g(x)\\
(f-g)(x) &= f(x)-g(x)\\
(f\cdot g)(x) &= f(x)\cdot g(x)\\
\left(\frac{f}{g}\right)(x) &= \frac{f(x)}{g(x)}
\end{aligned}
$$

With $\left(\frac{f}{g}\right)(x)$, we note that the quotient is only defined when $g(x) \neq 0$.

Activity 2.5.4 Let $f(x) = \frac{1}{3x-5}$.

(a) Find $f(4)$.

A. $\frac{4}{3x-5}$

B. $\frac{1}{4(3x-5)}$

C. $\frac{1}{7}$

D. 7

Hint. See Remark 2.2.5 for a reminder of what this notation means!

(b) If you were asked to find $f(x^3-2)$, how do you think you would proceed?

A. Multiply the original function $\frac{1}{3x-5}$ by x^3-2.

B. Plug the expression x^3-2 in for all the x-values in $\frac{1}{3x-5}$.

C. Plug the original function $\frac{1}{3x-5}$ in for all the x-values in x^3-2.

D. Multiply $3x-5$ by x^3-2.

(c) Find $f(x^3-2)$.

A. $\frac{1}{3x-5} \cdot (x^3-2)$

B. $\frac{1}{3(x^3-2)-5}$

C. $\left(\frac{1}{3x-5}\right)^3 - 2$

D. $(3x-5)(x^3-2)$

(d) What if we gave the expression x^3-2 a name? Let's define $g(x) = x^3-2$. What's another way we could denote $f(x^3-2)$?

A. $f(x) \cdot g(x)$

B. $g(f(x))$

C. $f(g(x))$

D. $\frac{f(x)}{g(x)}$

Combining and Composing Functions (FN5)

Definition 2.5.5 Given the functions $f(x)$ and $g(x)$, we define the **composition of** f **and** g to be the new function $h(x)$ given by

$$h(x) = f(g(x)).$$

We also sometimes use the notation

$$f \circ g$$

or

$$(f \circ g)(x)$$

to refer to $f(g(x))$. $\Diamond$

Remark 2.5.6 When discussing the composite function $f(g(x))$, also written as $(f \circ g)(x)$, we often call $g(x)$ the "inner function" and $f(x)$ the "outer function". It is important to note that the inner function is actually the first function that gets applied to a given input, and then the outer function is applied to the output of the inner function.

Activity 2.5.7 Let $f(x) = \dfrac{1}{3x-5}$ and $g(x) = x^3 - 2$.

(a) Find $f(g(x))$.

A. $\dfrac{x^3-2}{3x-5}$

B. $\dfrac{1}{(3x-5)(x^3-2)}$

C. $\dfrac{1}{3(x^3-2)-5}$

D. $\left(\dfrac{1}{3x-5}\right)^3 - 2$

(b) Find $g(f(x))$.

A. $\dfrac{x^3-2}{3x-5}$

B. $\dfrac{1}{(3x-5)(x^3-2)}$

C. $\dfrac{1}{3(x^3-2)-5}$

D. $\left(\dfrac{1}{3x-5}\right)^3 - 2$

Remark 2.5.8 We can also evaluate the composition of two functions at a particular value just as we did with one function. For example, we may be asked to find something like $f(g(2))$ or $(g \circ f)(-3)$.

Combining and Composing Functions (FN5)

Activity 2.5.9 Let $f(x) = 2x^3$ and $g(x) = \sqrt{6-x}$.

(a) Find $f(g(2))$.

A. 14

B. 16

C. 18

D. 20

E. undefined

(b) Find $(g \circ f)(-3)$.

A. 50

B. 54

C. $\sqrt{60}$

D. $\sqrt{-48}$

E. undefined

(c) Find $(f \circ g)(10)$.

A. $2(\sqrt{-4})^3$

B. 16

C. $\sqrt{-1994}$

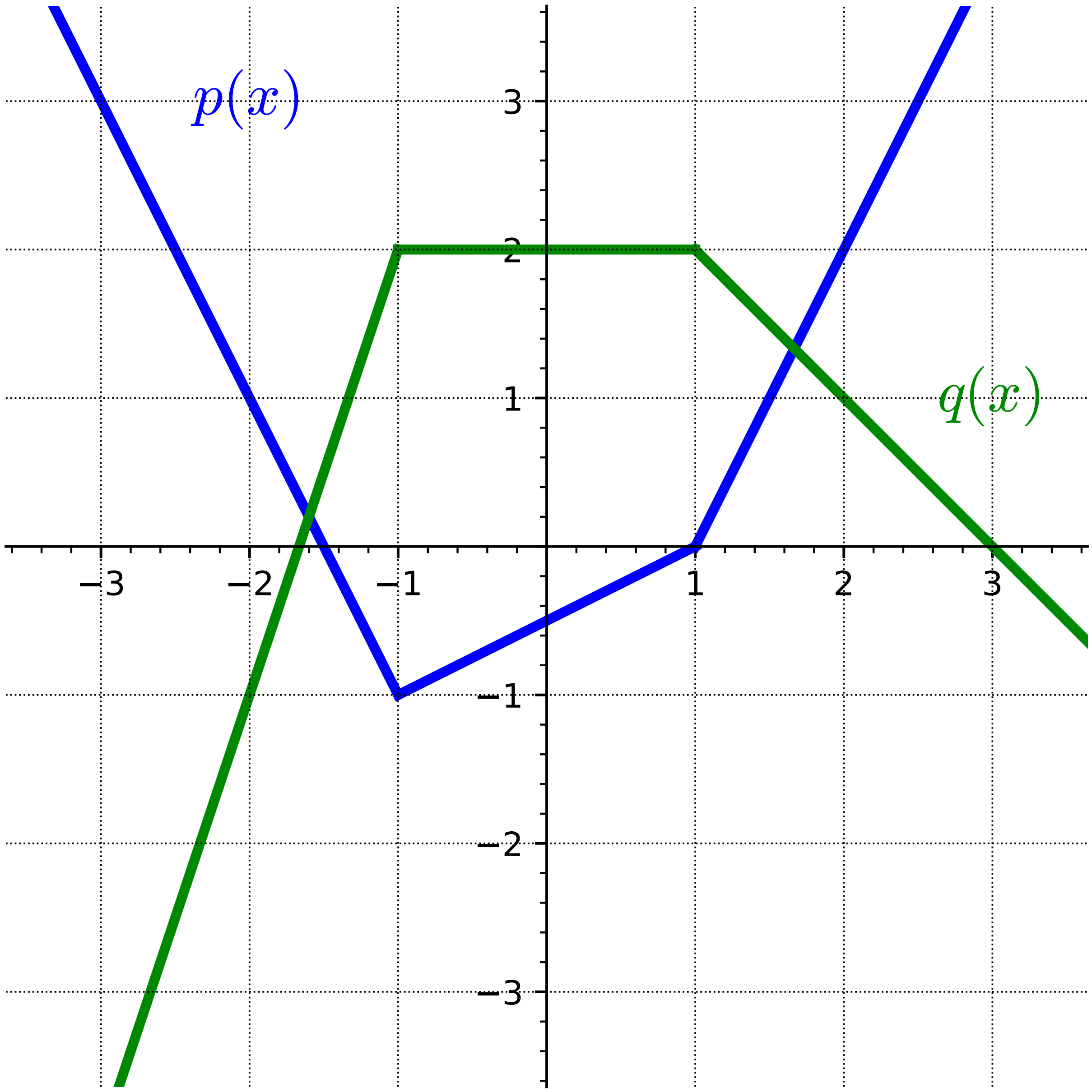
D. -16

E. undefined

Remark 2.5.10 As we saw in Activity 2.5.9, in order for a composite function to make sense, we need to ensure that the range of the inner function lies within the domain of the outer function so that the resulting composite function is defined at every possible input.

Remark 2.5.11 In addition to the possibility that functions are given by formulas, functions can be given by tables or graphs. We can think about composite functions in these settings as well, and the following activities prompt us to consider functions given in this way.

Activity 2.5.12 Let functions p and q be given by the graphs below.



Find each of the following. If something is not defined, explain why.

(a) $(p \circ q)(0)$

(b) $q(p(0))$

(c) $p(p(1))$

(d) $(q \circ p)(-3)$

(e) Find two values of x such that $q(p(x)) = 2$.

Activity 2.5.13 Let functions f and g be given by the tables below.

x	$f(x)$
0	6
1	4
2	3
3	4
4	7

Table 2.5.14

x	$g(x)$
0	1
1	3
2	0
3	5
4	2

Table 2.5.15

Find each of the following. If something is not defined, explain why.

(a) $(f \circ g)(2)$

(b) $(g \circ f)(3)$

(c) $g(f(4))$

(d) For what value(s) of x is $f(g(x)) = 4$?

(e) What are the domain and range of $(f \circ g)(x)$?

2.6 Finding the Inverse Function (FN6)

Objectives

- Find the inverse of a one-to-one function.

Finding the Inverse Function (FN6)

Remark 2.6.1 A function is a process that converts a collection of inputs to a corresponding collection of outputs. One question we can ask is: for a particular function, can we reverse the process and think of the original function's outputs as the inputs?

Finding the Inverse Function (FN6)

Activity 2.6.2 Temperature can be measured using many different units such as Fahrenheit, Celsius, and Kelvin. Fahrenheit is what is usually reported on the news each night in the United States, while Celsius is commonly used for scientific work. We will begin by converting between these two units. To convert from degrees Fahrenheit to Celsius use the following formula.

$$C = \frac{5}{9}(F - 32)$$

(a) Room temperature is around 68 degrees Fahrenheit. Use the above equation to convert this temperature to Celsius.

A. 5.8
B. 20
C. 155.4
D. 293

(b) Solve the equation $C = \frac{5}{9}(F - 32)$ for F in terms of C.

A. $F = \frac{5}{9}C + 32$
B. $F = \frac{5}{9}C - 32$
C. $F = \frac{9}{5}(C + 32)$
D. $F = \frac{9}{5}C + 32$

(c) Alternatively, 20 degrees Celsius is a fairly comfortable temperature. Use your solution for F in terms of C to convert this temperature to Fahrenheit.

A. 43.1
B. -20.9
C. 93.6
D. 68

Finding the Inverse Function (FN6)

Remark 2.6.3 Notice that when you converted 68 degrees Fahrenheit, you got a value of 20 degrees Celsius. Alternatively, when you converted 20 degrees Celsius, you got 68 degrees Fahrenheit. This indicates that the equation you were given for C and the equation you found for F are inverses.

Finding the Inverse Function (FN6)

Definition 2.6.4 Let f be a function. If there exists a function g such that

$$f(g(x)) = x \quad \text{and} \quad g(f(x)) = x$$

for all x, then we say f has an **inverse function**, or that g is the **inverse of** f. When a given function f has an inverse function, we usually denote it as f^{-1}, which is read as "f inverse". ◊

Finding the Inverse Function (FN6)

Remark 2.6.5 An inverse is a function that "undoes" another function. For any input in the domain, the function g will reverse the process of f.

Finding the Inverse Function (FN6)

Activity 2.6.6 It is important to note that in Definition 2.6.4 we say "if there exists a function," but we don't guarantee that this is always the case. How can we determine whether a function has a corresponding inverse or not? Consider the following two functions f and g represented by the tables.

Table 2.6.7

x	$f(x)$
0	6
1	4
2	3
3	4
4	6

Table 2.6.8

x	$g(x)$
0	3
1	1
2	4
3	2
4	0

(a) Use the definition of $g(x)$ in Table 2.6.8 to find an x such that $g(x) = 4$.

A. $x = 0$

B. $x = 1$

C. $x = 2$

D. $x = 3$

E. $x = 4$

(b) Is it possible to reverse the input and output rows of the function $g(x)$ and have the new table result in a function?

(c) Use the definition of $f(x)$ in Table 2.6.7 to find an x such that $f(x) = 4$.

A. $x = 0$

B. $x = 1$

C. $x = 2$

D. $x = 3$

E. $x = 4$

(d) Is it possible to reverse the input and output rows of the function $f(x)$ and have the new table result in a function?

Finding the Inverse Function (FN6)

Remark 2.6.9 Some functions, like $f(x)$ in Table 2.6.7, have a given output value that corresponds to two or more input values: $f(0) = 6$ and $f(4) = 6$. If we attempt to reverse the process of this function, we have a situation where the new input 6 would correspond to two potential outputs.

Finding the Inverse Function (FN6)

Definition 2.6.10 A **one-to-one function** is a function in which each output value corresponds to exactly one input. ◊

Finding the Inverse Function (FN6)

Remark 2.6.11 A function must be one-to-one in order to have an inverse.

Finding the Inverse Function (FN6)

Activity 2.6.12 Consider the function $f(x) = \dfrac{x-5}{3}$.

(a) When you evaluate this expression for a given input value of x, what operations do you perform and in what order?

A. divide by 3, subtract 5

B. subtract 5, divide by 3

C. add 5, multiply by 3

D. multiply by 3, add 5

(b) When you "undo" this expression to solve for a given ouput value of y, what operations do you perform and in what order?

A. divide by 3, subtract 5

B. subtract 5, divide by 3

C. add 5, multiply by 3

D. multiply by 3, add 5

(c) This set of operations reverses the process for the original function, so can be considered the inverse function. Write an equation to express the inverse function f^{-1}.

A. $f^{-1}(x) = \dfrac{x}{3} - 5$

B. $f^{-1}(x) = \dfrac{x-5}{3}$

C. $f^{-1}(x) = 5(x+3)$

D. $f^{-1}(x) = 3x + 5$

(d) Check your answer to the previous question by finding $f(f^{-1}(x))$ and $f^{-1}(f(x))$.

Finding the Inverse Function (FN6)

Observation 2.6.13 To find the inverse of a one-to-one function, perform the reverse operations in the opposite order.

Finding the Inverse Function (FN6)

Activity 2.6.14 Let's look at an alternate method for finding an inverse by solving the function for x and then interchanging the x and y.

$$h(x) = \frac{x}{x+1}$$

(a) Interchange the variables x and y.

A. $y = \dfrac{x}{x+1}$

B. $x = \dfrac{y}{x+1}$

C. $x = \dfrac{y}{y+1}$

D. $x = \dfrac{x}{y+1}$

(b) Eliminate the denominator.

A. $y(x+1) = x$

B. $x(x+1) = y$

C. $x(y+1) = x$

D. $x(y+1) = y$

(c) Distribute and gather the y terms together.

A. $yx + y = x$

B. $x^2 + x = y$

C. $xy - y = -x$

D. $xy = 0$

(d) Write the inverse function, by factoring and solving for y.

A. $h^{-1}(x) = \dfrac{x}{x-1}$

B. $h^{-1}(x) = \dfrac{x}{1-x}$

C. $h^{-1}(x) = \dfrac{-x}{1-x}$

D. $h^{-1}(x) = \dfrac{x+1}{x}$

Finding the Inverse Function (FN6)

Activity 2.6.15 Find the inverse of each function, using either method. Check your answer using function composition.

(a) $g(x) = \dfrac{4x-1}{7}$

A. $g^{-1}(x) = \dfrac{7x+1}{4}$

B. $g^{-1}(x) = \dfrac{7x}{4} + 1$

C. $g^{-1}(x) = \dfrac{4x+1}{7}$

D. $g^{-1}(x) = \dfrac{7}{4x-1}$

(b) $f(x) = 3 - \sqrt{x+5}$

A. $f^{-1}(x) = 3 + \sqrt{x-5}$

B. $f^{-1}(x) = (x-3)^2 + 5$

C. $f^{-1}(x) = \dfrac{1}{3 - \sqrt{x+5}}$

D. $f^{-1}(x) = (3-x)^2 - 5$

Chapter 3

Linear Functions (LF)

Objectives

How do we model scenarios that have a constant rate of change?
By the end of this chapter, you should be able to...

1. Determine the average rate of change of a given function over a given interval. Find the slope of a line.

2. Determine an equation for a line when given two points on the line and when given the slope and one point on the line. Express these equations in slope-intercept or point-slope form and determine the slope and y-intercept of a line given an equation.

3. Graph a line given its equation or some combination of characteristics, such as points on the graph, a table of values, the slope, or the intercepts.

4. Use slope relationships to determine whether two lines are parallel or perpendicular, and find the equation of lines parallel or perpendicular to a given line through a given point.

5. Build linear models from verbal descriptions, and use the models to establish conclusions, including by contextualizing the meaning of slope and intercept parameters.

6. Solve a system of two linear equations in two variables.

7. Solve questions involving applications of systems of equations.

3.1 Slope and Average Rate of Change (LF1)

Objectives

- Determine the average rate of change of a given function over a given interval. Find the slope of a line.

Slope and Average Rate of Change (LF1)

Remark 3.1.1 This section will explore ideas around average rate of change and slope. To help us get started, let's take a look at a context in which these ideas can be helpful.

Slope and Average Rate of Change (LF1)

Activity 3.1.2 Robert came home one day after school to a very hot house! When he got home, the temperature on the thermostat indicated that it was 85 degrees! Robert decided that was too hot for him, so he turned on the air conditioner. The table of values below indicate the temperature of his house after turning on the air conditioner.

Table 3.1.3

Time (minutes)	**Temperature (degrees Fahrenheit)**
0	85
1	84.3
2	83.6
3	82.9
4	82.2
5	81.5
6	80.8

(a) How much did the temperature change from 0 to 2 minutes?

A. The temperature decreased by 0.7 degrees

B. The temperature decreased by 1.4 degrees

C. The temperature increased by 0.7 degrees

D. The temperature increased by 1.4 degrees>

(b) How much did the temperature change from 4 to 6 minutes?

A. The temperature decreased by 0.7 degrees

B. The temperature decreased by 1.4 degrees

C. The temperature increased by 0.7 degrees

D. The temperature increased by 1.4 degrees>

(c) If Robert wanted to know how much the temperature was decreasing each minute, how could he figure that out?

(d) How would you describe the overall behavior of the temperature of Robert's house?

Remark 3.1.4 Notice in Activity 3.1.2 that the temperature appears to be decreasing at a constant rate (i.e., the temperature decreased 1.4 degrees for every 2-minute interval). Upon further investigation, you might have also noticed that the temperature decreased by 0.7 degrees every minute.

Activity 3.1.5 Refer back to the data Robert collected of the temperature of his house after turning on the air conditioner (Table 3.1.3).

(a) If this pattern continues, what will the temperature be after 8 minutes?

A. 80.1

B. 78.7

C. 80.8

D. 79.4

(b) If this pattern continues, how long will it take for Robert's house to reach 78 degrees?

A. 12 minutes

B. 9 minutes

C. 10 minutes

D. 11 minutes

Remark 3.1.6 An average rate of change helps us to see and understand how a function is generally behaving. For example, in Activity 3.1.2 and Activity 3.1.5, we began to see how the temperature of Robert's house was decreasing every minute the air conditioner was on. In other words, when looking at average rate of change, we are comparing how one quantity is changing with respect to something else changing.

Slope and Average Rate of Change (LF1)

Definition 3.1.7 An **average rate of change** of a function calculates the amount of change in one item divided by the corresponding amount of change in another.

To calculate the average rate of change for any function $f(x)$, we pick two points, a and b, and evaluate the function at those two points. We then find the difference between the y-values and x-values to calcuate the average rate of change:

Recall that we can use function notation to describe x- and y-values. $f(b)$, for instance represents the y-value when plugging in a value for x (or b).

$$\frac{f(b) - f(a)}{b - a}.$$

◊

Slope and Average Rate of Change (LF1)

Activity 3.1.8 Use the table below to answer the questions.

Table 3.1.9

x	$f(x)$
-5	28
-4	19
-3	12
-2	7
-1	4

(a) Applying Definition 3.1.7, what is the average rate of change when $x = -5$ to $x = -2$?

A. $\frac{1}{7}$

B. -3

C. -7

D. 7

(b) What is the average rate of change on the interval $[-4, -1]$?

A. -5

B. -3

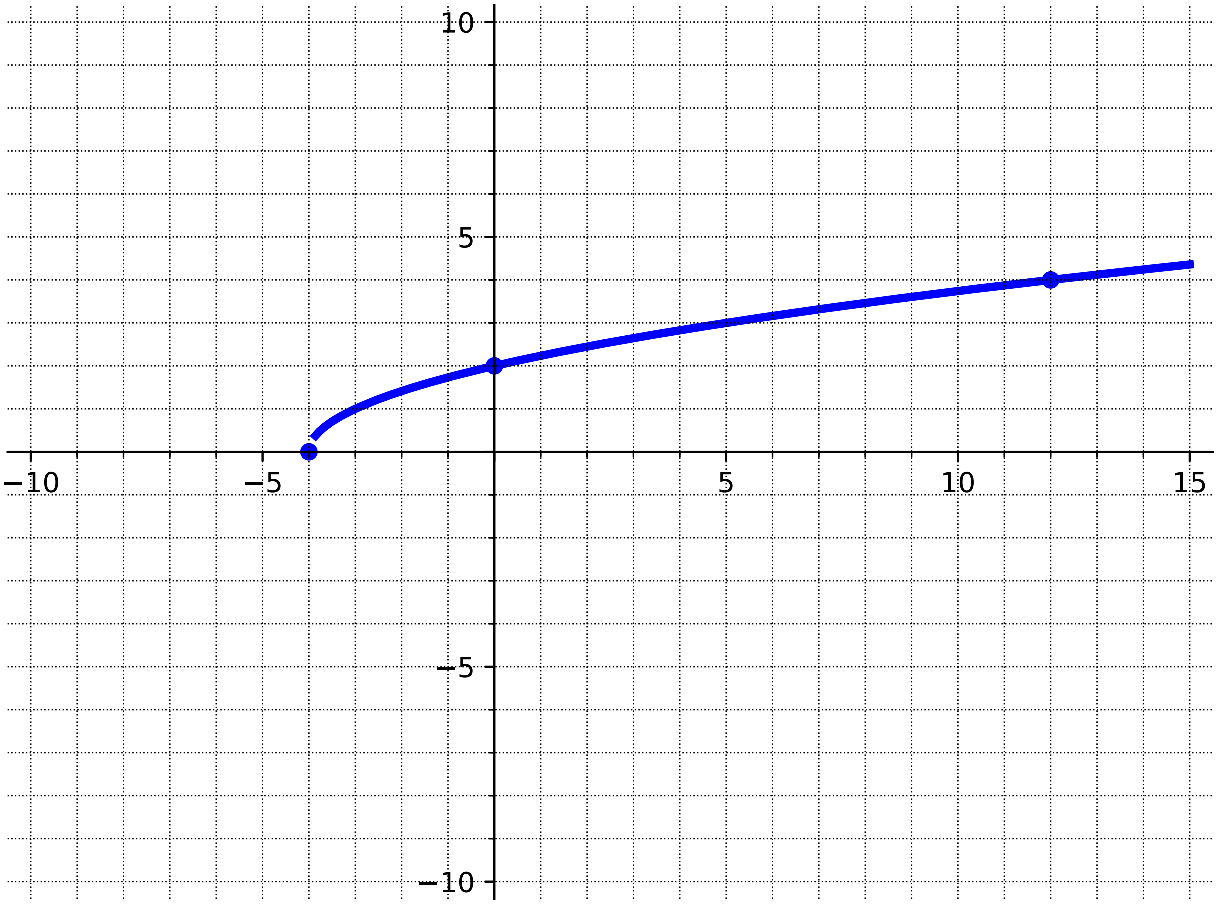
C. 5

D. 3

(c) Does this function have a constant average rate of change?

Slope and Average Rate of Change (LF1)

Activity 3.1.10 Use the graph to calculate the average rate of change on the given intervals.



(a) Applying Definition 3.1.7, what is the average rate of change on the interval $[-4, 0]$?

A. $-\frac{1}{2}$

B. $\frac{1}{2}$

C. -2

D. 2

(b) What is the average rate of change on the interval $[0, 12]$?

A. $\frac{1}{6}$

B. -6

C. $-\frac{1}{6}$

D. 6

Slope and Average Rate of Change (LF1)

Activity 3.1.11 Just like with tables and graphs, you should be able to find the average rate of change when given a function. For this activity, use the function

$$f(x) = -3x^2 - 1$$

to answer the following questions.

(a) Applying Definition 3.1.7, what is the average rate of change on the interval $[-2, 3]$?

A. $\frac{41}{5}$

B. -3

C. $-\frac{1}{3}$

D. $\frac{5}{41}$

(b) What is the average rate of change on the interval $[0, 4]$?

A. $\frac{2}{25}$

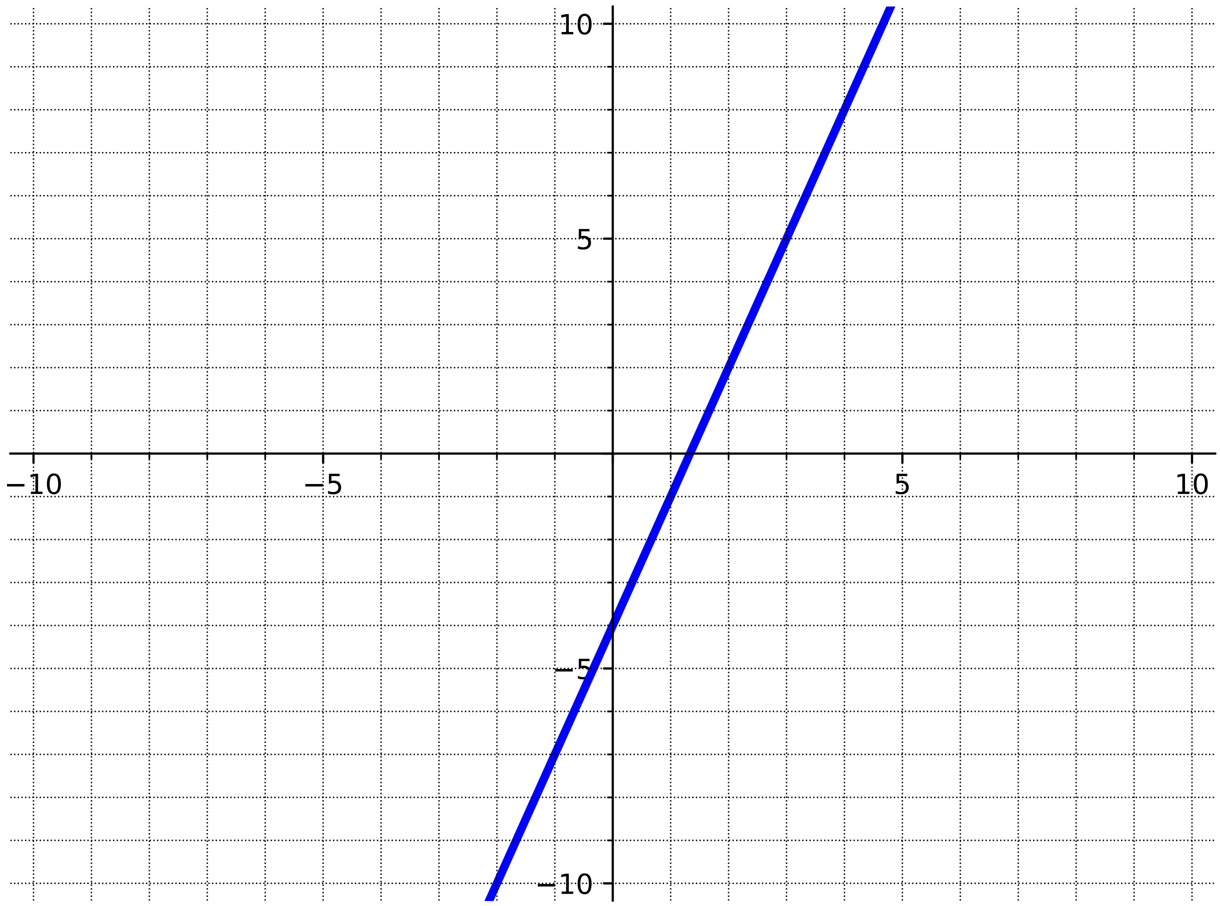
B. -50

C. $-\frac{1}{12}$

D. -12

Slope and Average Rate of Change (LF1)

Activity 3.1.12 Use the given graph of the function, $f(x) = 3x - 4$, to investigate the average rate of change of a linear function.



(a) What is the average rate of change on the interval $[-2, 0]$?

A. $\frac{1}{3}$

B. 3

C. $-\frac{1}{7}$

D. -7

(b) What is the average rate of change on the interval $[-1, 5]$? Notice that you cannot see the point at $x = 5$. How could you use the equation of the line to determine the y-value when $x = 5$?

A. 3

B. $\frac{1}{3}$

C. $\frac{2}{3}$

D. -3

(c) Based on your observations in parts (a) and (b), what do you think will be the average rate of change on the interval $[5, 25]$?

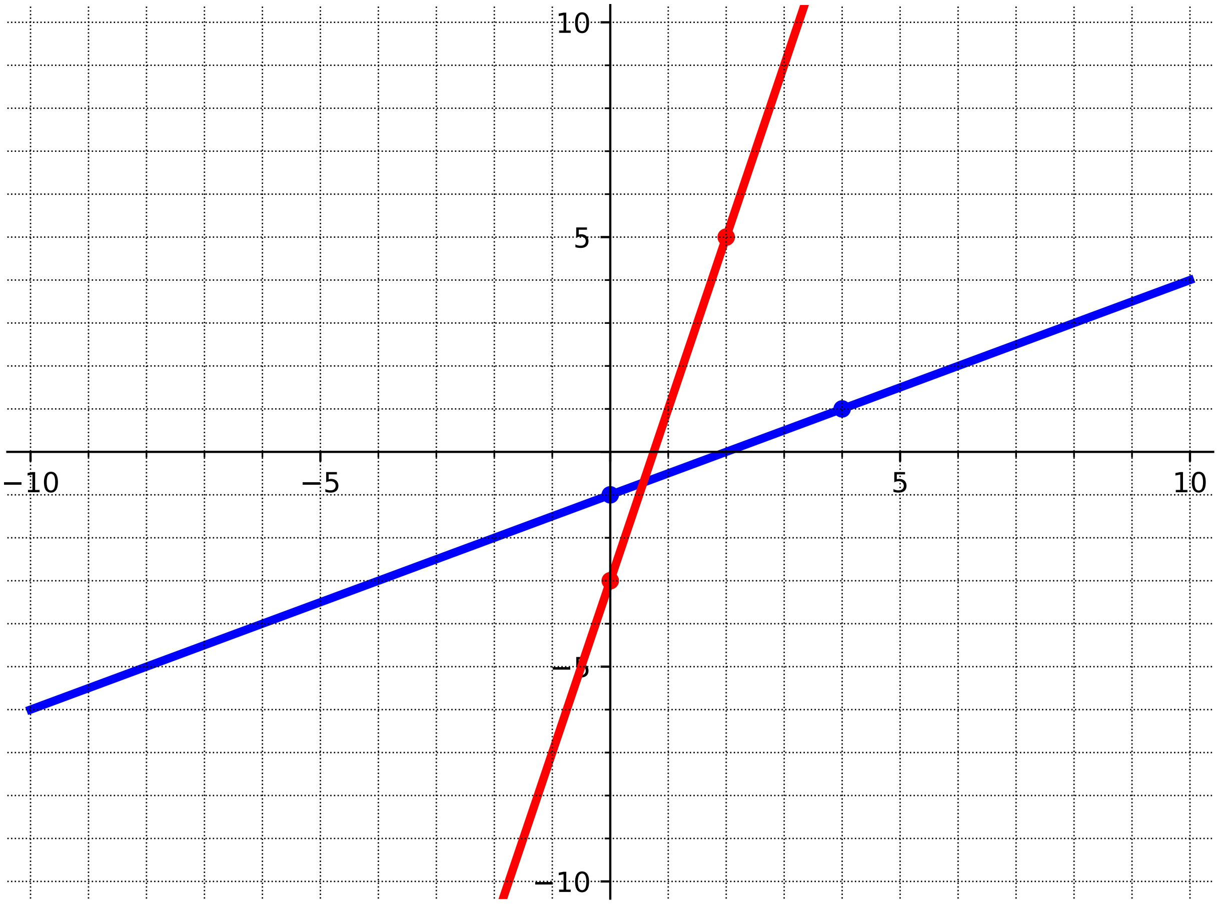
Remark 3.1.13 Notice in Activity 3.1.12, the average rate of change was the same regardless of which interval you were given. But in Activity 3.1.10, the average rate of change was not the same across different intervals.

Slope and Average Rate of Change (LF1)

Definition 3.1.14 The **slope** of a line has a constant that represents the direction and steepness of the line. For a linear function, the slope never changes - meaning it has a constant average rate of change. ◊

Slope and Average Rate of Change (LF1)

Activity 3.1.15 The steepness of a line depends on the vertical and horizontal distances between two points on the line. Use the graph below to compare the steepness, or slope, of the two lines.



(a) What is the vertical distance between the two points on the red line?

A. 2

B. 4

C. 8

D. $\frac{1}{2}$

(b) What is the horizontal distance between the two points on the red line?

A. 2

B. 4

C. 8

D. $\frac{1}{2}$

(c) Using information from parts (a) and (b), what value could we use to describe the steepness of the red line?

A. 2

B. 4

C. 8

D. $\frac{1}{2}$

(d) What is the vertical distance between the two points on the blue line?

A. 2

B. 4

C. 8

D. $\frac{1}{2}$

(e) What is the horizontal distance between the two points on the blue line?

A. 2

B. 4

C. 8

D. $\frac{1}{2}$

(f) Using information from parts (d) and (e), what value could we use to describe the steepness of the blue line?

A. 2

B. 4

C. 8

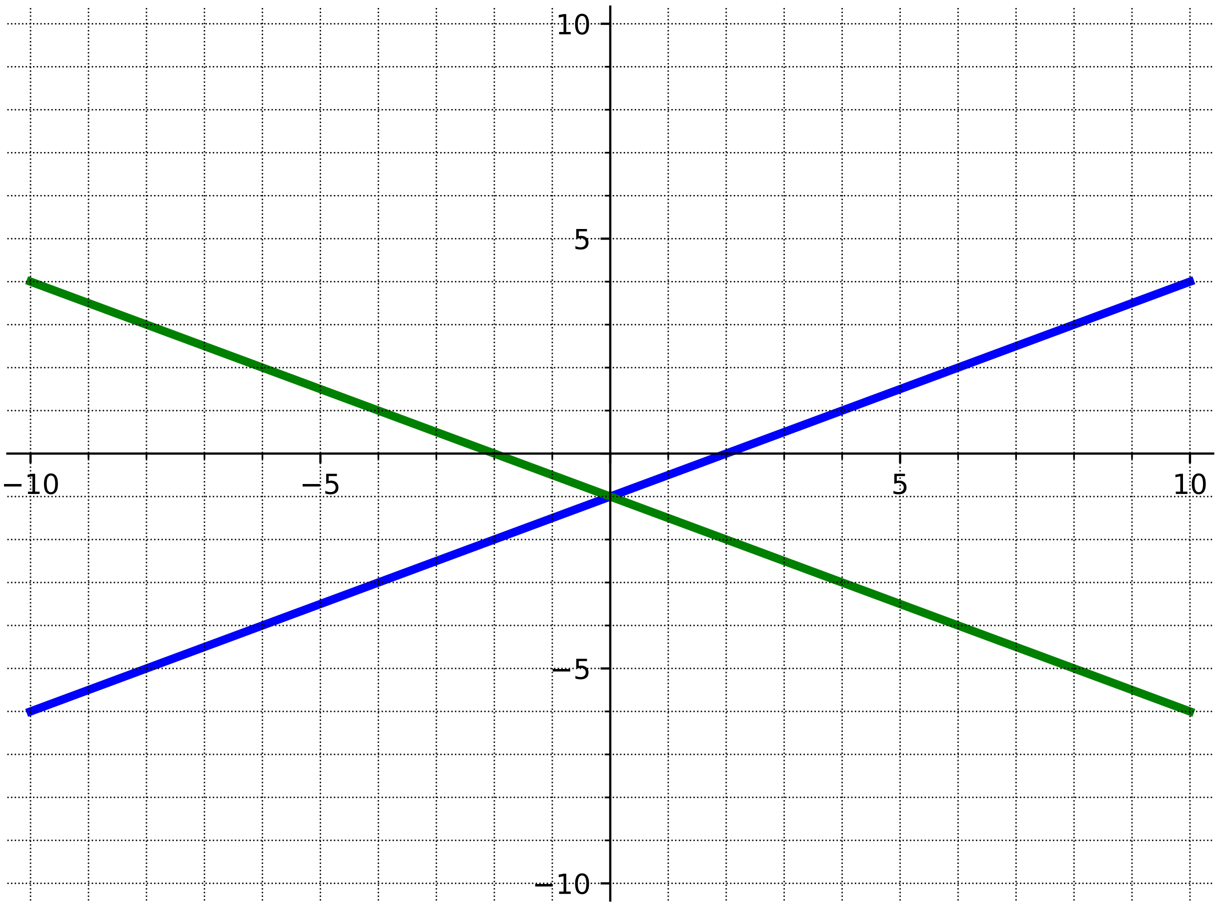
D. $\frac{1}{2}$

(g) Which line is the steepest?

Remark 3.1.16 The steepness, or slope, of a line can be found by the change in y (the vertical distance between two points on the line) divided by the change in x (the horizontal distance between two points on the line). Slope can be calculated as "rise over run." Slope is a way to describe the steepness of a line. The red line in Activity 3.1.15 has a larger value for it's slope than the blue line. Thus, the red line is steeper than the blue line.

Slope and Average Rate of Change (LF1)

Activity 3.1.17 Now that we know how to find the slope (or steepness) of a line, let's look at other properties of slope. Use the graph below to answer the following questions.



(a) What is the slope of the blue line?

A. $\frac{1}{2}$

B. 2

C. $-\frac{1}{2}$

D. -2

(b) What is the slope of the green line?

A. $\frac{1}{2}$

B. 2

C. $-\frac{1}{2}$

D. -2

(c) How are the slopes of the lines similar?

(d) How are the slopes of the lines different?

Remark 3.1.18 Notice in Activity 3.1.17 that the slope does not just indicate how steep a line is, but also it's direction. A negative slope indicates that the line is decreasing (from left to right) and a positive slope indicates that the line is increasing (from left to right).

Slope and Average Rate of Change (LF1)

Activity 3.1.19 Suppose $(-3, 7)$ and $(7, 2)$ are two points on a line.

(a) Plot these points on a graph and find the slope by using "rise over run."

A. $\frac{1}{2}$

B. 2

C. $-\frac{1}{2}$

D. -2

(b) Now calculate the slope by using the change in y over the change in x.

A. $\frac{1}{2}$

B. 2

C. $-\frac{1}{2}$

D. -2

(c) What do you notice about the slopes you got in parts (a) and (b)?

Slope and Average Rate of Change (LF1)

Remark 3.1.20 We can calculate slope (m) by finding the change in y and dividing by the change in x. Mathematically, this means that when given (x_1, y_1) and (x_2, y_2),

$$m = \frac{y_2 - y_1}{x_2 - x_1}.$$

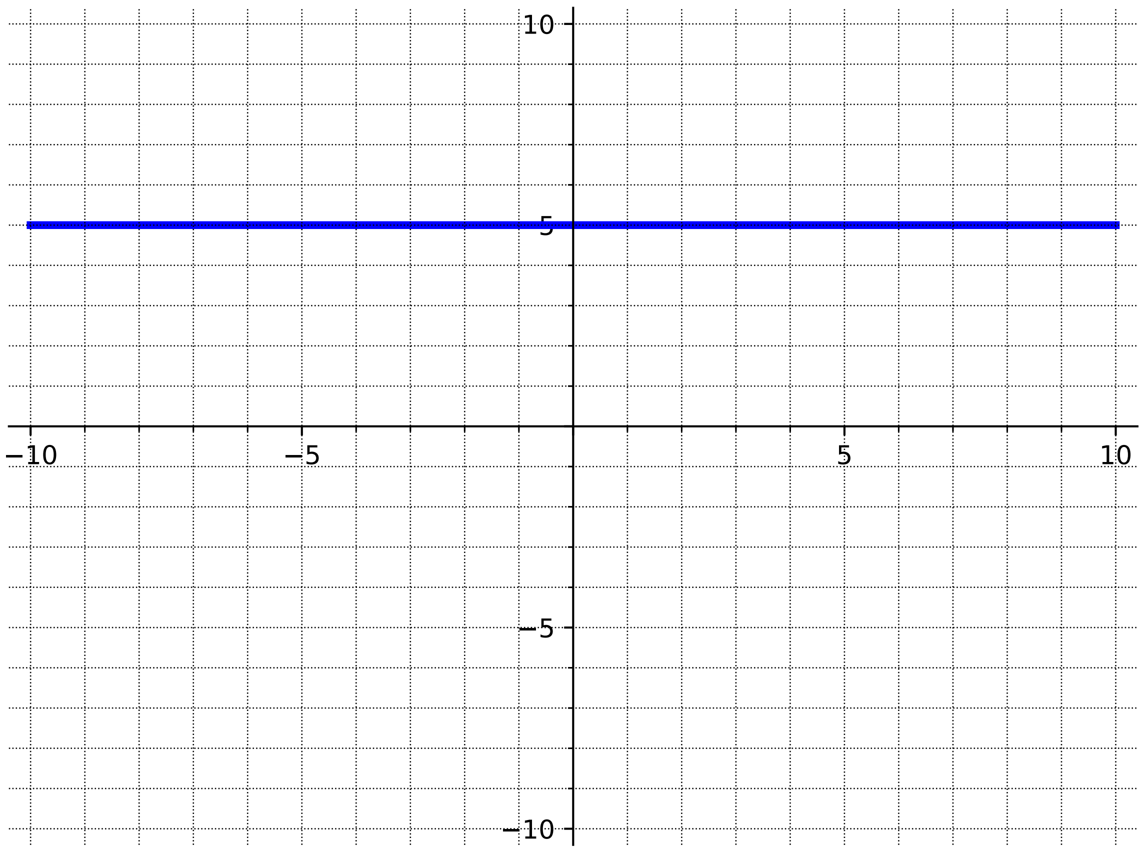
Slope and Average Rate of Change (LF1)

Activity 3.1.21 Calculate the slope of each representation of a line using the slope formula.

(a)

x	$f(x)$
-2	-7
-1	-4
0	-1
1	2
2	5

(b)



$m = 0$

(c) $(-4, 7)$ and $(-4, 1)$

Remark 3.1.22 In Activity 3.1.21, there were slopes that were 0 and undefined. When a line is vertical, the slope is undefined. This means that there is only a vertical distance between two points and there is no horizontal distance. When a line is horizontal, the slope is 0. This means that the line never rises vertically, giving a vertical distance of zero.

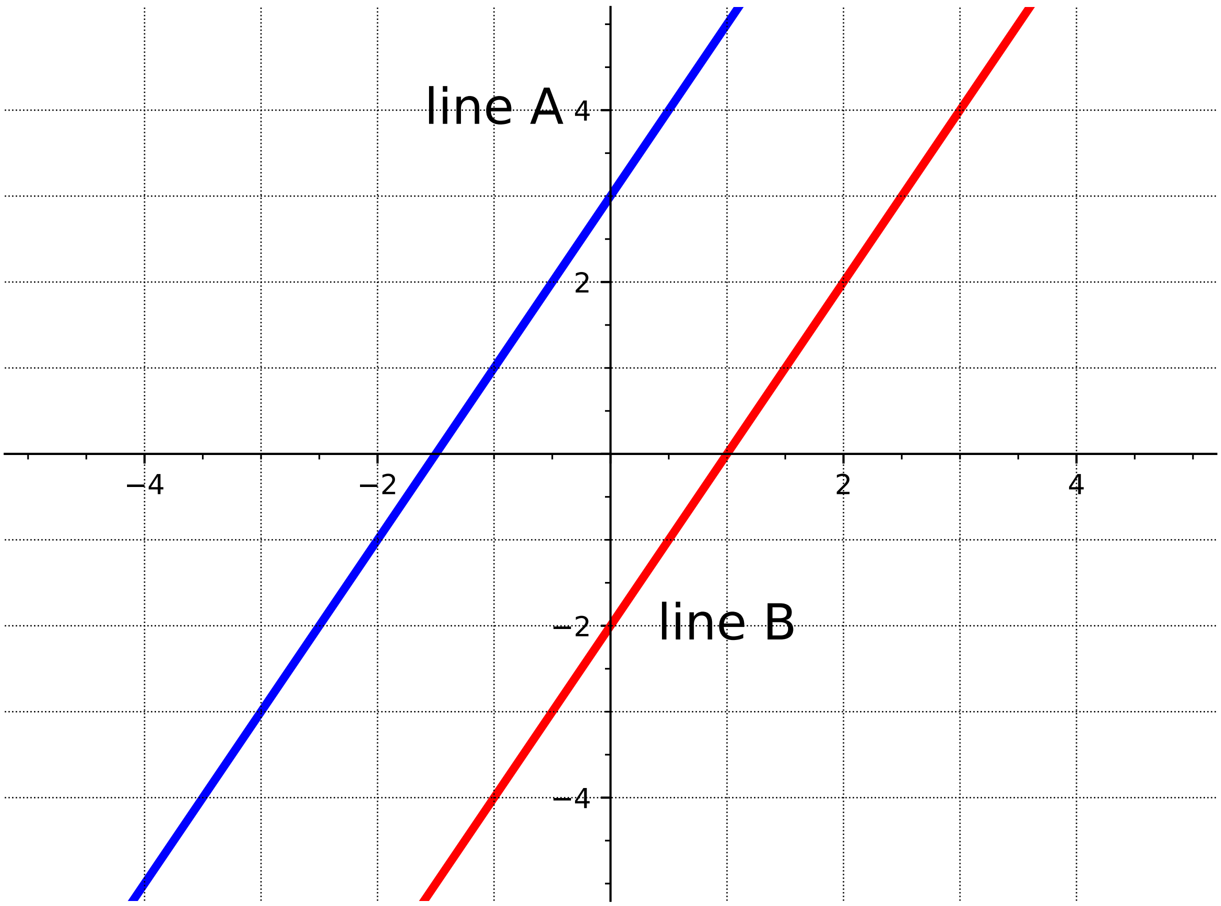
3.2 Equations of Lines (LF2)

Objectives

- Determine an equation for a line when given two points on the line and when given the slope and one point on the line. Express these equations in slope-intercept or point-slope form and determine the slope and y-intercept of a line given an equation.

Equations of Lines (LF2)

Activity 3.2.1 Consider the graph of two lines.



(a) Find the slope of line A.

A. 1

B. 2

C. $\frac{1}{2}$

D. -2

(b) Find the slope of line B.

A. 1

B. 2

C. $\frac{1}{2}$

D. -2

(c) Find the y-intercept of line A.

A. -2

B. -1.5

C. 1

D. 3

(d) Find the y-intercept of line B.

A. -2

B. -1.5

C. 1

D. 3

(e) What is the same about the two lines?

(f) What is different about the two lines?

Remark 3.2.2 Notice that in Activity 3.2.1 the lines have the same slope but different y-intercepts. It is not enough to just know one piece of information to determine a line, you need both a slope and a point.

Equations of Lines (LF2)

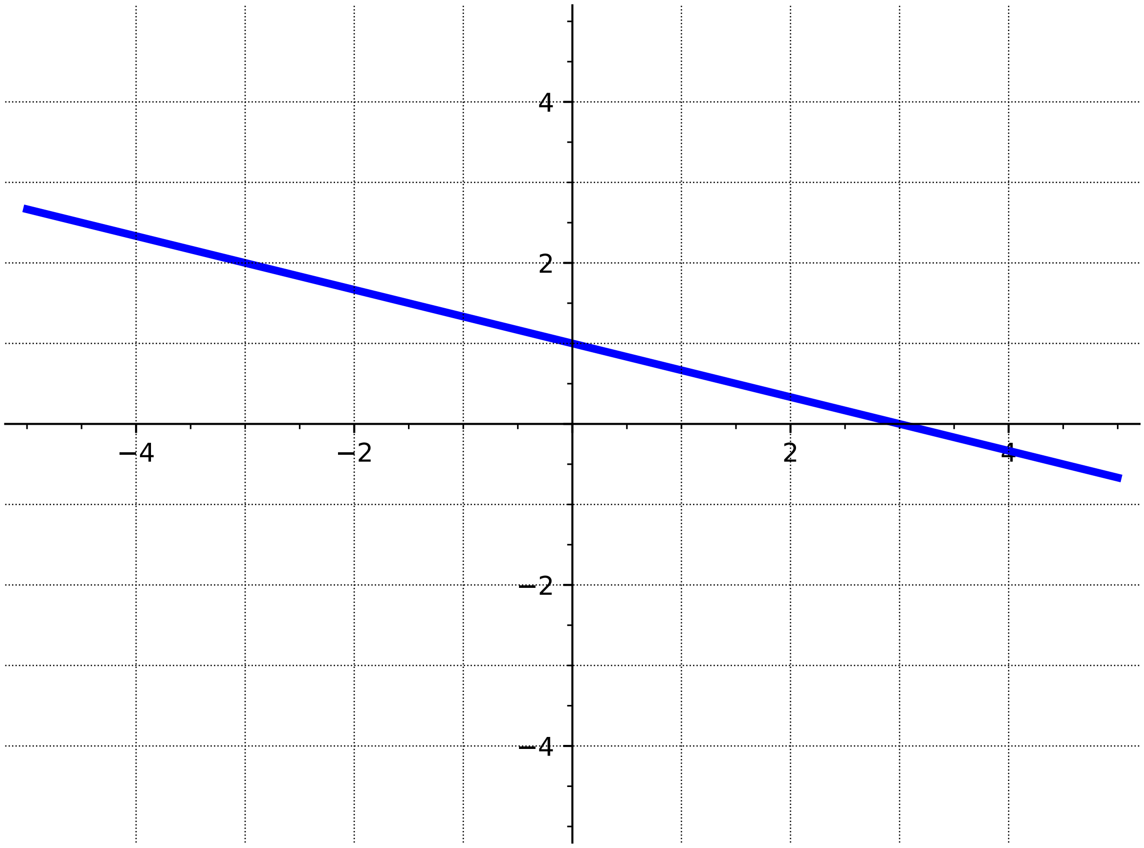
Definition 3.2.3 Linear functions can be written in **slope-intercept form**

$$f(x) = mx + b$$

where b is the y-intercept (or starting value) and m is the slope (or constant rate of change).

$\Diamond$

Activity 3.2.4 Write the equation of each line in slope-intercept form.



(a)

A. $y = -3x + 1$

B. $y = -x + 3$

C. $y = -\frac{1}{3}x + 1$

D. $y = -\frac{1}{3}x + 3$

(b) The slope is 4 and the y-intercept is $(0, -3)$.

A. $f(x) = 4x - 3$

B. $f(x) = 3x - 4$

C. $f(x) = -4x + 3$

D. $f(x) = 4x + 3$

(c) Two points on the line are $(0, 1)$ and $(2, 4)$.

A. $y = 2x + 1$

B. $y = -\frac{3}{2}x + 4$

C. $y = \frac{3}{2}x + 1$

D. $y = \frac{3}{2}x + 4$

(d)

x	$f(x)$
-2	-8
0	-2
1	1
4	10

A. $f(x) = -3x - 2$

B. $f(x) = -\frac{1}{3}x - 2$

C. $f(x) = 3x + 1$

D. $f(x) = 3x - 2$

Equations of Lines (LF2)

Activity 3.2.5 Let's try to write the equation of a line given two points that don't include the y-intercept.

(a) Plot the points $(2, 1)$ and $(-3, 4)$.

(b) Find the slope of the line joining the points.

A. $-\frac{5}{3}$

B. $-\frac{3}{5}$

C. $\frac{3}{5}$

D. -3

(c) When you draw a line connecting the two points, it's often hard to draw an accurate enough graph to determine the y-intercept of the line exactly. Let's use the slope-intercept form and one of the given points to solve for the y-intercept. Try using the slope and one of the points on the line to solve the equation $y = mx + b$ for b.

A. 2

B. $\frac{11}{5}$

C. $\frac{5}{2}$

D. 3

(d) Write the equation of the line in slope-intercept form.

Equations of Lines (LF2)

Remark 3.2.6 It would be nice if there was another form of the equation of a line that works for any points and does not require the y-intercept.

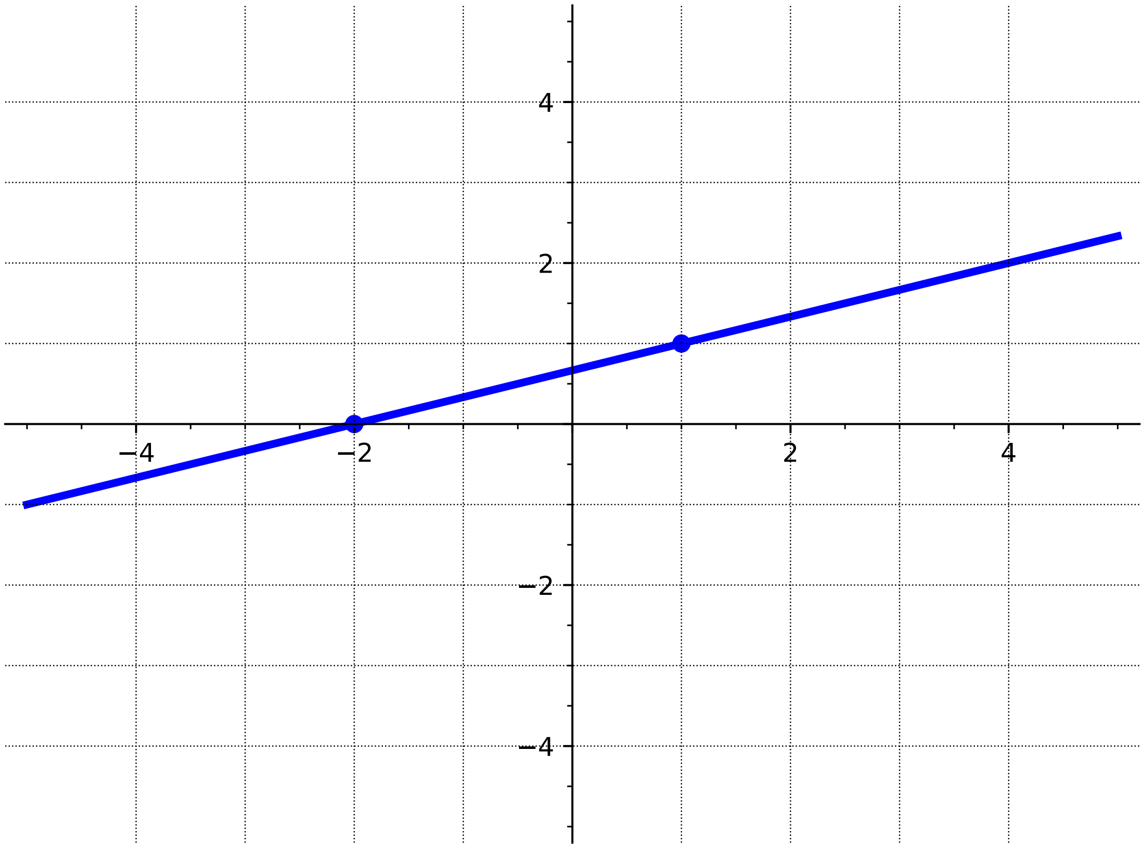
Equations of Lines (LF2)

Definition 3.2.7 Linear functions can be written in **point-slope form**

$$y - y_0 = m(x - x_0)$$

where (x_0, y_0) is any point on the line and m is the slope. ◊

Activity 3.2.8 Write an equation of each line in point-slope form.



(a)

A. $y = \frac{1}{3}x + \frac{2}{3}$

B. $y - 1 = 3(x - 1)$

C. $y - 1 = \frac{1}{3}(x - 1)$

D. $y + 2 = \frac{1}{3}(x + 2)$

E. $y = \frac{1}{3}(x + 2)$

(b) The slope is 4 and $(-1, -7)$ is a point on the line.

A. $y + 7 = 4(x + 1)$

B. $y - 7 = 4(x - 1)$

C. $y + 1 = 4(x + 7)$

D. $y - 4 = 7(x - 1)$

(c) Two points on the line are $(1, 0)$ and $(2, -4)$.

Equations of Lines (LF2)

A. $y = -4x + 1$

B. $y - 0 = -2(x - 1)$

C. $y + 4 = -4(x - 2)$

D. $y + 4 = -3(x - 2)$

(d)

x	$f(x)$
-2	-8
1	1
4	10

A. $y + 8 = 3(x - 2)$

B. $y - 1 = -\frac{1}{3}(x - 1)$

C. $y + 8 = -\frac{1}{3}(x + 2)$

D. $y - 10 = 3(x - 4)$

Equations of Lines (LF2)

Activity 3.2.9 Consider again the two points from Activity 3.2.5, $(2, 1)$ and $(-3, 4)$.

(a) Use point-slope form to find an equation of the line.

A. $y = -\frac{3}{5}x + \frac{11}{5}$

B. $y - 1 = -\frac{3}{5}(x - 2)$

C. $y - 4 = -\frac{3}{5}(x + 3)$

D. $y - 2 = -\frac{3}{5}(x - 1)$

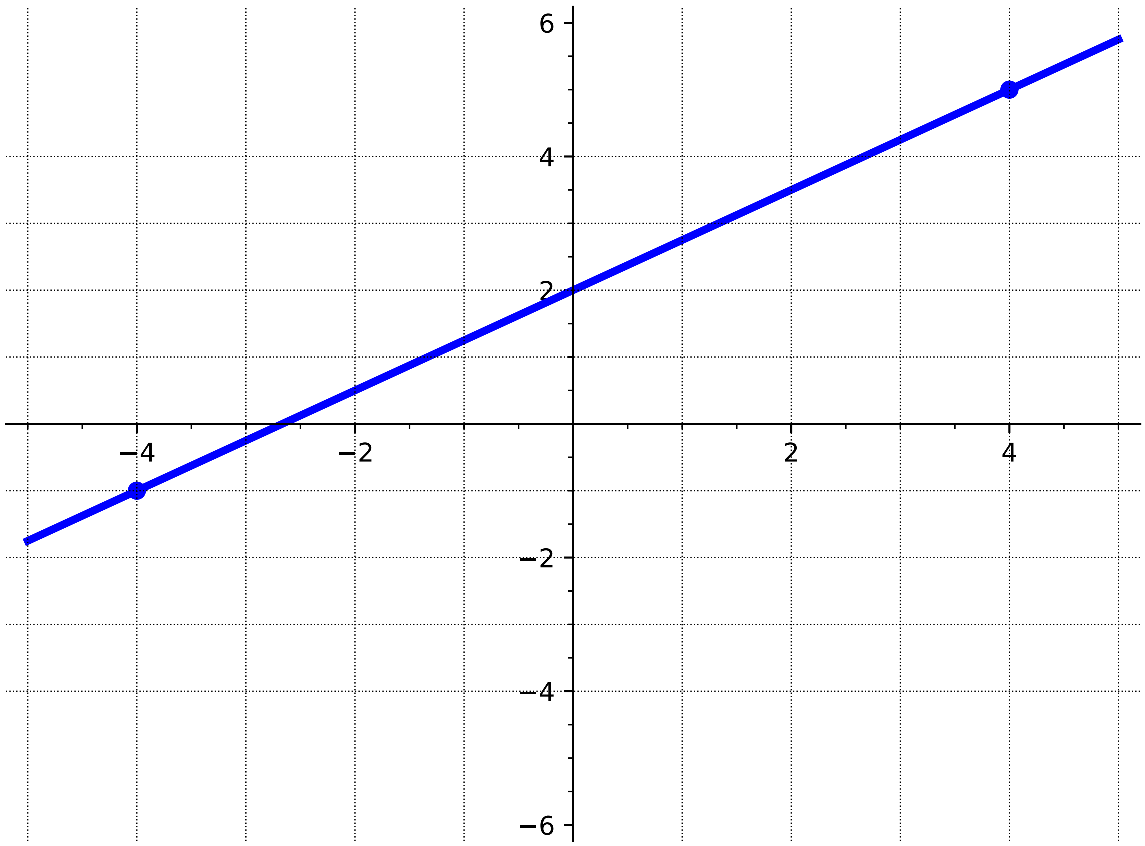
(b) Solve the point-slope form of the equation for y to rewrite the equation in slope-intercept form. Identify the slope and intercept of the line.

Remark 3.2.10 Notice that it was possible to use either point to find an equation of the line in point-slope form. But, when rewritten in slope-intercept form the equation is unique.

Equations of Lines (LF2)

Activity 3.2.11 For each of the following lines, determine which form (point-slope or slope-intercept) would be "easier" and why. Then, write the equation of each line.

(a)



(b) The slope is $-\frac{1}{2}$ and $(1, -3)$ is a point on the line.

(c) Two points on the line are $(0, 3)$ and $(2, 0)$.

Remark 3.2.12 It is always possible to use both forms to write the equation of a line and they are both valid. Although, sometimes the given information lends itself to make one form easier.

Equations of Lines (LF2)

Activity 3.2.13 Write the equation of each line.

(a) The slope is 0 and $(-1, -7)$ is a point on the line.

A. $y = -7$

B. $y = 7x$

C. $y = -x$

D. $x = -1$

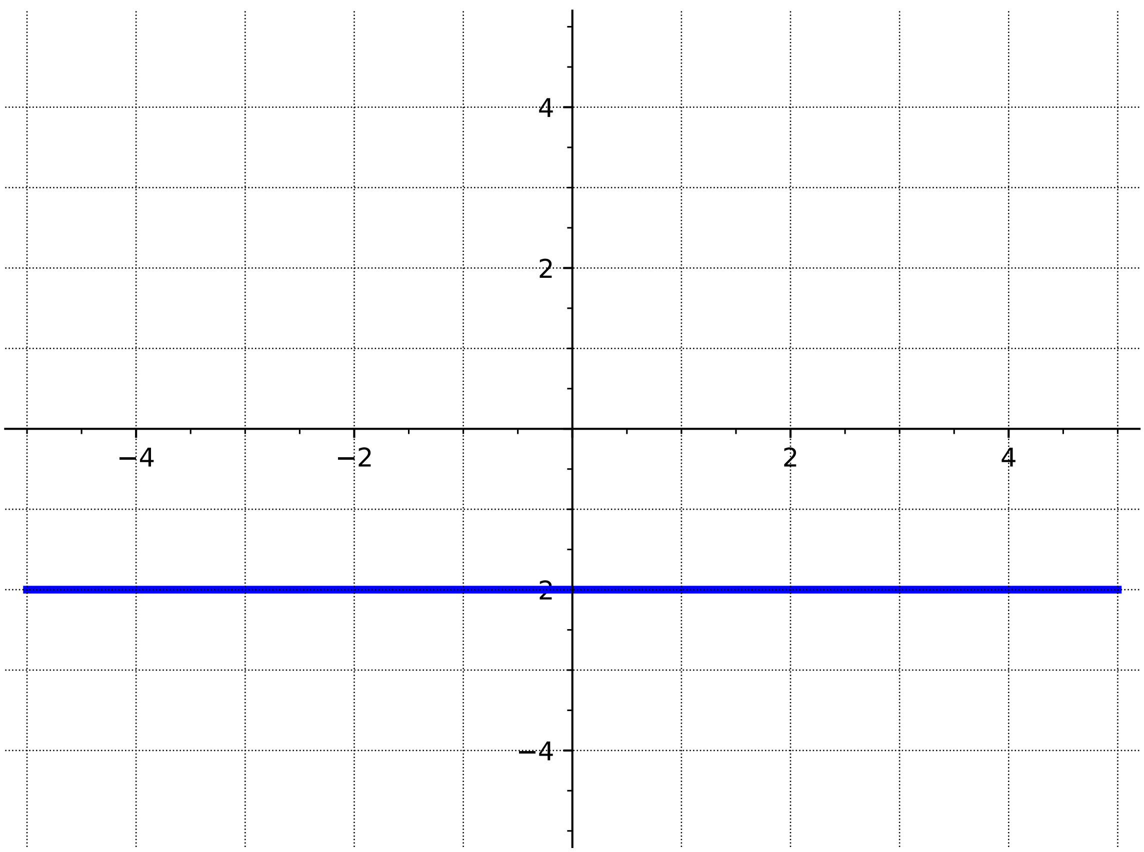
(b) Two points on the line are $(3, 0)$ and $(3, 5)$.

A. $y = 3x + 3$

B. $y = 3x + 5$

C. $x = 3$

D. $y = 3$



(c) A. $x = -2$

B. $y - 2 = x$

C. $y = -2x - 2$

D. $y = -2$

Equations of Lines (LF2)

Definition 3.2.14 A **horizontal line** has a slope of zero and has the form $y = k$ where k is a constant. A **vertical line** has an undefined slope and has the form $x = h$ where h is a constant. $\diamondsuit$

Equations of Lines (LF2)

Definition 3.2.15 The equation of a line can also be written in **standard form**. Standard form looks like $Ax + By = C$. ◊

Remark 3.2.16 It is possible to rearrange a line written in standard form to slope-intercept form by solving for y.

Equations of Lines (LF2)

Activity 3.2.17 Given a line in standard form

$$5x + 4y = 2.$$

Find the slope and y-intercept of the line.

3.3 Graphs of Linear Equations (LF3)

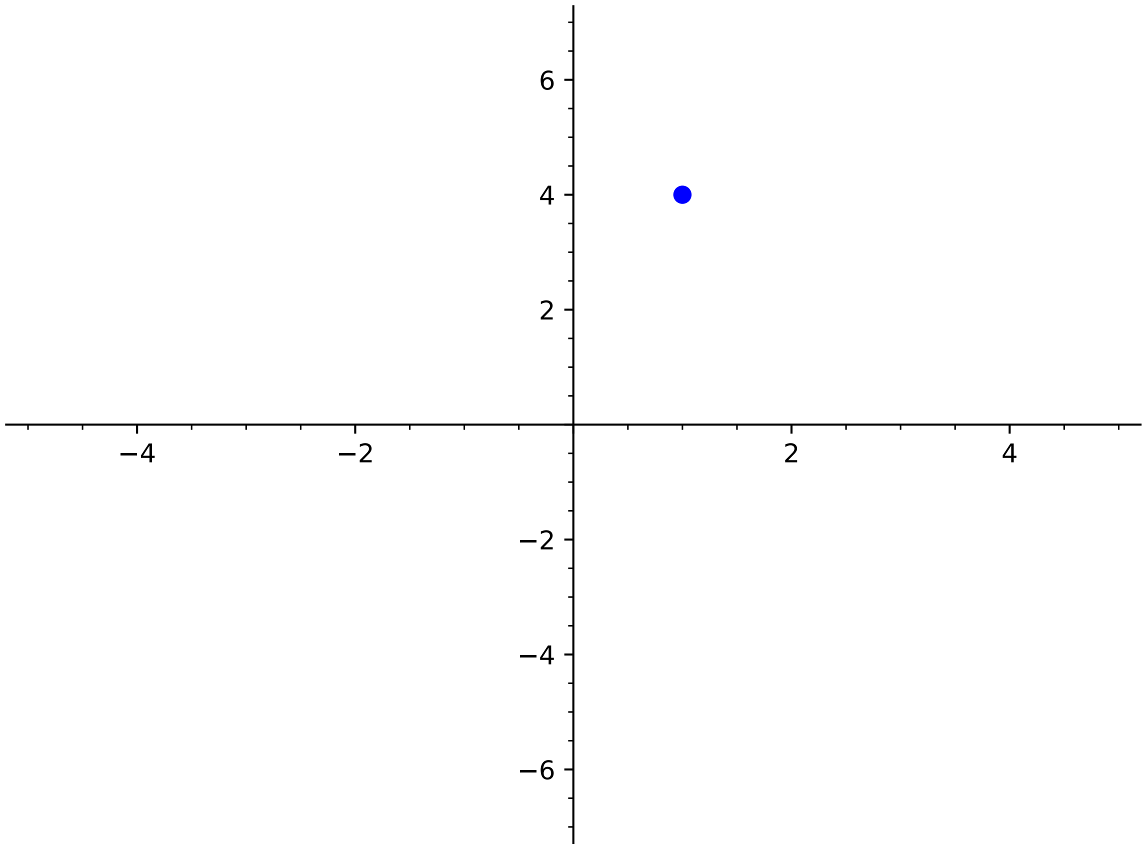
Objectives

- Graph a line given its equation or some combination of characteristics, such as points on the graph, a table of values, the slope, or the intercepts.

Graphs of Linear Equations (LF3)

Activity 3.3.1

(a) Draw a line that goes through the point $(1, 4)$.



(b) Was this the only possible line that goes through the point $(1, 4)$?

A. Yes. The line is unique.

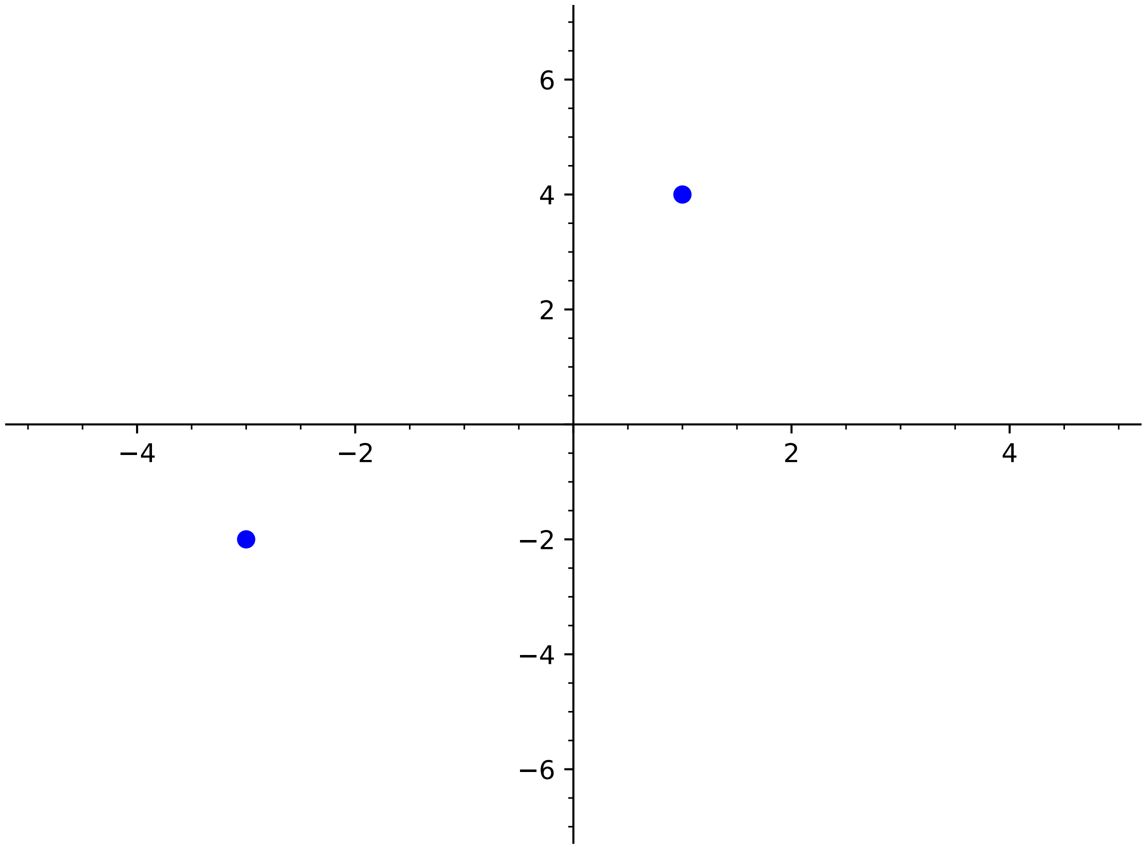
B. No. There is exactly one more line possible.

C. No. There are a lot of lines that go through $(1, 4)$.

D. No. There are an infinite number of lines that go through $(1, 4)$.

(c) Now draw a line that goes through the points $(1, 4)$ and $(-3, -2)$.

Graphs of Linear Equations (LF3)



(d) Was this the only possible line that goes through the points $(1, 4)$ and $(-3, -2)$?

A. Yes. The line is unique.

B. No. There is exactly one more line possible.

C. No. There are a lot of lines that go through $(1, 4)$ and $(-3, -2)$.

D. No. There are an infinite number of lines that go through $(1, 4)$ and $(-3, -2)$.

Observation 3.3.2 If you are given two points, then you can always graph the line containing them by plotting them and connecting them with a line.

Activity 3.3.3

(a) Graph the line containing the points $(-7, 1)$ and $(6, -2)$.

(b) Graph the line containing the points $(-3, 0)$ and $(0, 8)$.

(c) Graph the line given by the table below.

x	y
-3	-12
-2	-9
-1	-6
0	-3
1	0
2	3

(d) Let's say you are given a table that listed six points that are on the same line. How many of those points are necessary to use to graph the line?

A. One point is enough.

B. Two points are enough.

C. Three points are enough.

D. You need to plot all six points.

E. You can use however many you want.

Remark 3.3.4 In Activity 3.3.3, we were given at least two points in each question. However, sometimes we are not directly given two points to graph a line. Instead we are given some combination of characteristics about the line that will help us *find* two points. These characteristics could include a point, the intercepts, the slope, or an equation.

Activity 3.3.5 A line has a slope of $-\frac{1}{3}$ and its y-intercept is 4.

(a) We were given the y-intercept. What point does that correspond to?

A. $(4, 0)$

B. $(0, 4)$

C. $\left(4, -\frac{1}{3}\right)$

D. $\left(-\frac{1}{3}, 4\right)$

(b) After we plot the y-intercept, how can we use the slope to find another point?

A. Start at the y-intercept, then move up one space and to the left three spaces to find another point.

B. Start at the y-intercept, then move up one space and to the right three spaces to find another point.

C. Start at the y-intercept, then move down one space and to the left three spaces to find another point.

D. Start at the y-intercept, then move down one space and to the right three spaces to find another point.

(c) Graph the line that has a slope of $-\frac{1}{3}$ and its y-intercept is 4.

Graphs of Linear Equations (LF3)

Activity 3.3.6 A line is given by the equation $y = -2x + 5$.

(a) What form is the equation given in?

A. Standard form

B. Point-slope form

C. Slope-intercept form

D. The form it is in doesn't have a name.

(b) The form gives us one point right away: the y-intercept. Which of the following is the y-intercept?

A. $(-2, 0)$

B. $(0, -2)$

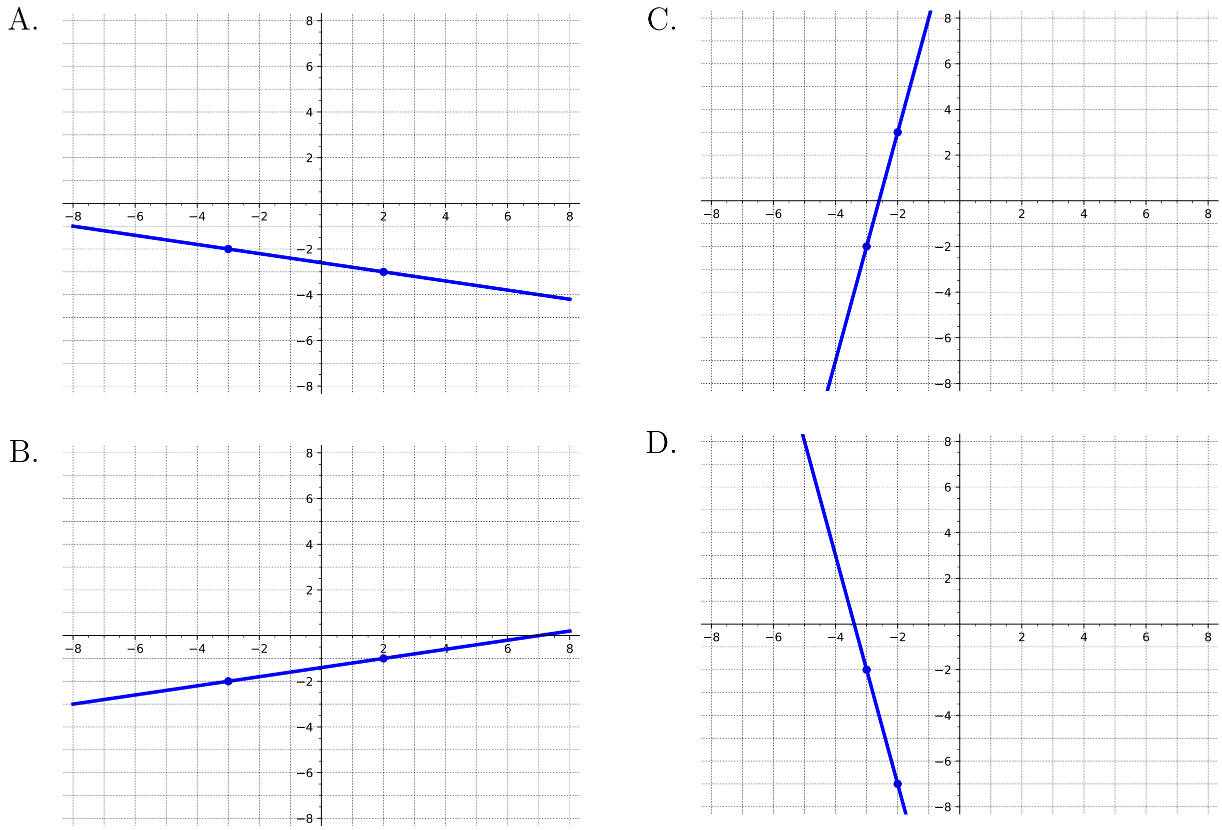
C. $(5, 0)$

D. $(0, 5)$

(c) After we plot the y-intercept, we can use the slope to find another point. Find another point and graph the resulting line.

Graphs of Linear Equations (LF3)

Activity 3.3.7 A line contains the point $(-3, -2)$ and has slope $\frac{1}{5}$. Which of the following is the graph of that line?



Graphs of Linear Equations (LF3)

Activity 3.3.8 A line is given by the equation $y - 6 = -4(x + 2)$.

(a) What form is the equation given in?

A. Standard form

B. Point-slope form

C. Slope-intercept form

D. The form it is in doesn't have a name.

(b) The form gives us one point right away. Which of the following is a point on the line?

A. $(-2, -6)$

B. $(-2, 6)$

C. $(2, -6)$

D. $(2, 6)$

(c) After we plot this point, we can use the slope to find another point. Find another point and graph the resulting line.

Graphs of Linear Equations (LF3)

Activity 3.3.9 Recall from Definition 3.2.14 that the equation of a horizontal line has the form $y = k$ where k is a constant and a vertical line has the form $x = h$ where h is a constant.

(a) Which type of line has a slope of zero?

A. Horizontal

B. Vertical

(b) Which type of line has an undefined slope?

A. Horizontal

B. Vertical

(c) Graph the vertical line that goes through the point $(4, -2)$.

(d) What is the equation of the vertical line through the point $(4, -2)$?

A. $x = 4$

B. $y = 4$

C. $x = -2$

D. $y = -2$

(e) Graph the horizontal line that goes through the point $(4, -2)$.

(f) What is the equation of the horizontal line through the point $(4, -2)$?

A. $x = 4$

B. $y = 4$

C. $x = -2$

D. $y = -2$

Graphs of Linear Equations (LF3)

Activity 3.3.10 Graph each line described below.

(a) The line containing the points $(-3, 4)$ and $(5, -2)$.

(b) The line whose x-intercept is -2 and whose y-intercept is 7.

(c) The line whose slope is $\frac{2}{5}$ that goes through the point $(4, 6)$.

(d) The line whose slope is $-\frac{1}{3}$ and whose y-intercept is -4.

(e) The vertical line through the point $(-2, -7)$.

(f) The horizontal line through the point $(-6, 3)$.

(g) The line with equation $y = -\frac{5}{3}x - 6$.

(h) The line with equation $y - 5 = \frac{7}{2}(x - 2)$.

(i) The line with equation $3x - 6y = 8$.

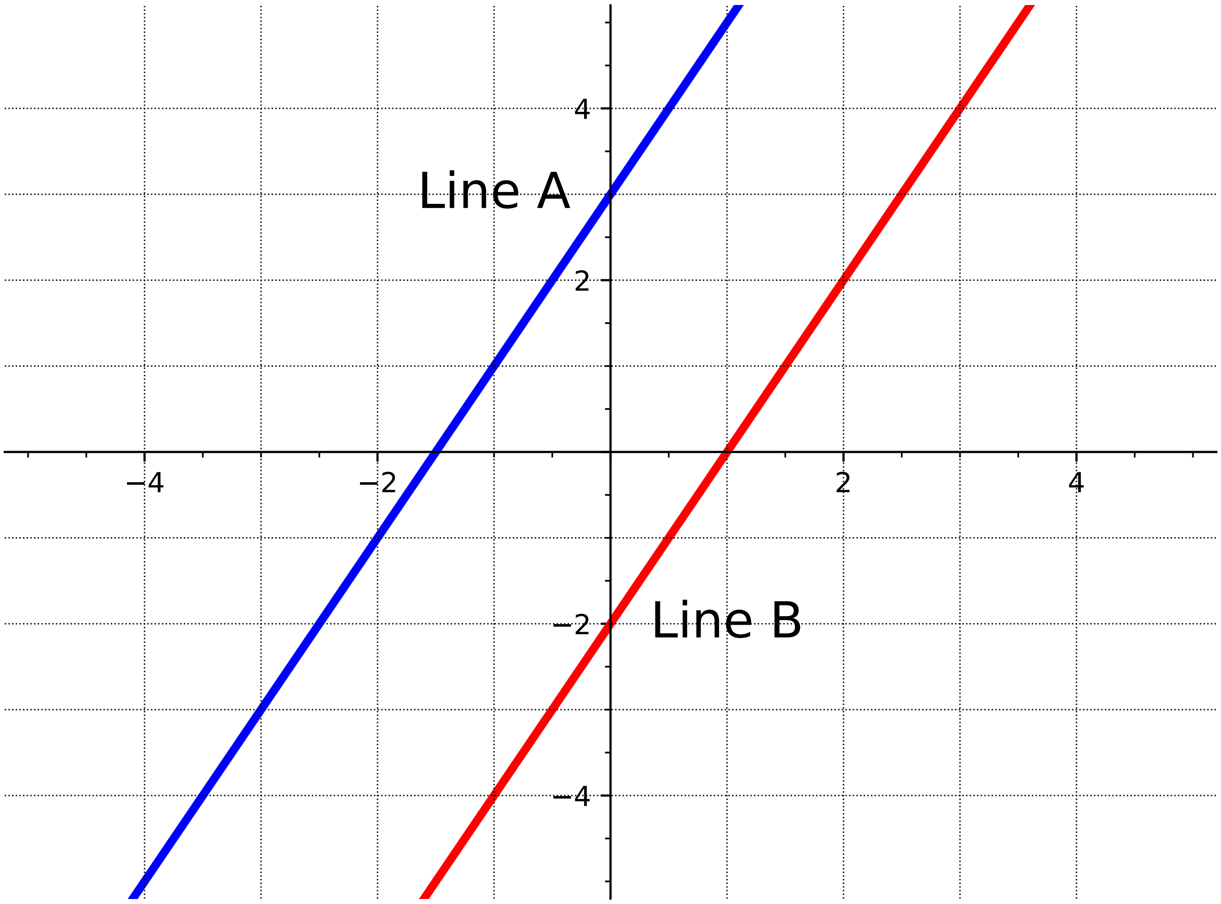
3.4 Parallel and Perpendicular Lines (LF4)

Objectives

- Use slope relationships to determine whether two lines are parallel or perpendicular, and find the equation of lines parallel or perpendicular to a given line through a given point.

Parallel and Perpendicular Lines (LF4)

Activity 3.4.1 Let's revisit Activity 3.2.1 to investigate special types of lines.



(a) What is the slope of line A?

A. 1

B. 2

C. $\frac{1}{2}$

D. -2

(b) What is the slope of line B?

A. 1

B. 2

C. $\frac{1}{2}$

D. -2

(c) What is the y-intercept of line A?

A. -2

B. -1.5

C. 1

D. 3

(d) What is the y-intercept of line B?

A. -2

B. -1.5

C. 1

D. 3

(e) What is the same about the two lines?

(f) What is different about the two lines?

Remark 3.4.2 Notice that in Activity 3.4.1 the two lines never touch.

Parallel and Perpendicular Lines (LF4)

Definition 3.4.3 Parallel lines are lines that always have the same distance apart (equidistant) and will never meet. Parallel lines have the same slope, but different y-intercepts. $\Diamond$

Parallel and Perpendicular Lines (LF4)

Activity 3.4.4 Suppose you have the function,

$$f(x) = -\frac{1}{2}x - 1$$

(a) What is the slope of $f(x)$?

A. -1

B. 2

C. 1

D. $-\frac{1}{2}$

(b) Applying Definition 3.4.3, what would the slope of a line parallel to $f(x)$ be?

A. -1

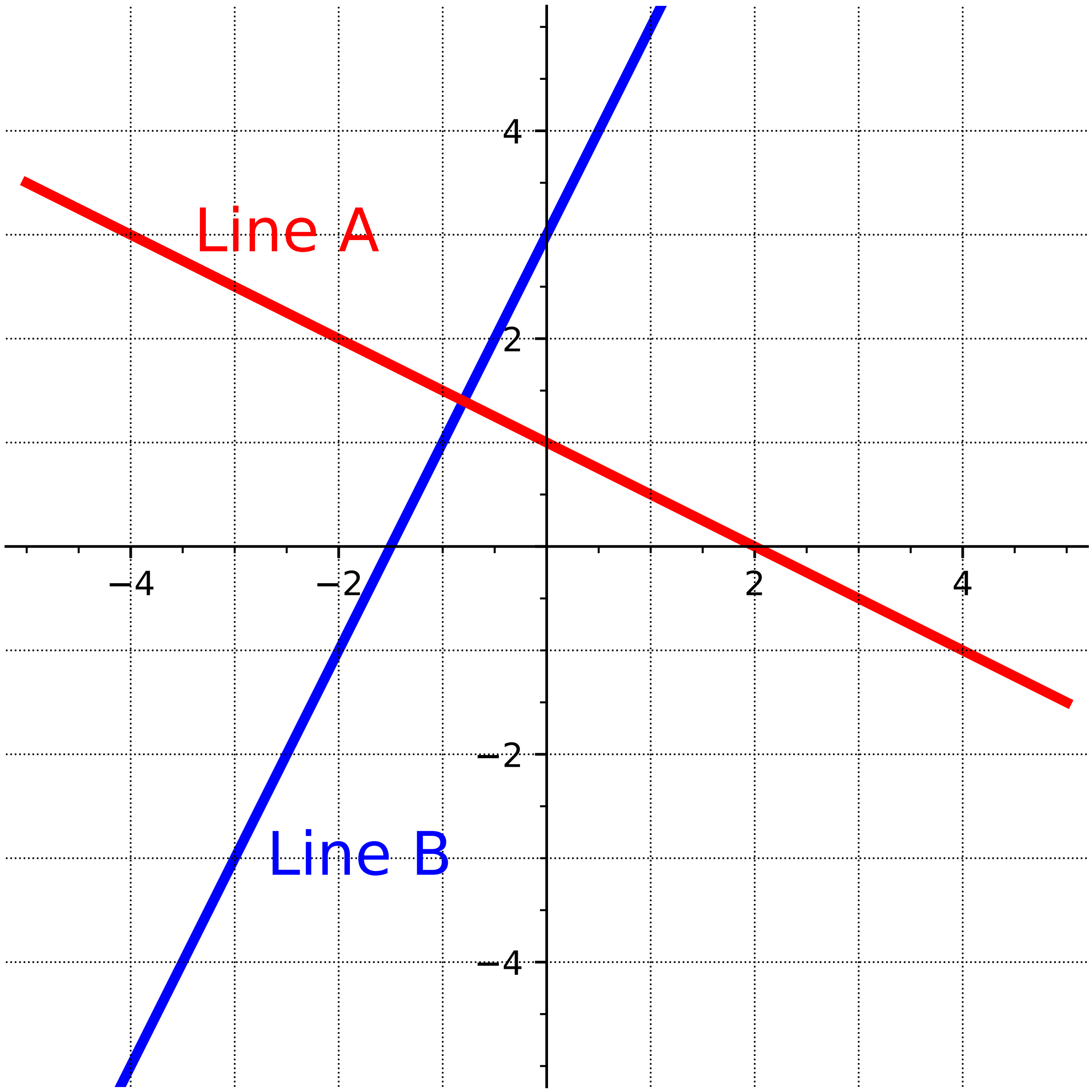
B. 2

C. 1

D. $-\frac{1}{2}$

(c) Find the equation of a line parallel to $f(x)$ that passes through the point $(-4, 2)$.

Activity 3.4.5 Consider the graph of the two lines below.



(a) What is the slope of line A?

A. 3

B. 2

C. $-\frac{1}{2}$

D. -2

(b) What is the slope of line B?

A. 3

B. 2

C. $-\frac{1}{2}$

D. -2

(c) What is the y-intercept of line A?

A. -2

B. $-\frac{1}{2}$

C. 2

D. 3

(d) What is the y-intercept of line B?

A. -2

B. $-\frac{1}{2}$

C. 1

D. 3

(e) If you were to think of slope as ”rise over run,” how would you write the slope of each line?

(f) How would you compare the slopes of the two lines?

Remark 3.4.6 Notice in Activity 3.4.5, that even though the two lines have different slopes, the slopes are somewhat similar. For example, if you take the slope of Line A $\left(-\frac{1}{2}\right)$ and flip and negate it, you will get the slope of Line B $\left(\frac{2}{1}\right)$.

Parallel and Perpendicular Lines (LF4)

Definition 3.4.7 Perpendicular lines are two lines that meet or intersect each other at a right angle. The slopes of two perpendicular lines are *negative reciprocals* of each other (given that the slope exists!). $\Diamond$

Parallel and Perpendicular Lines (LF4)

Activity 3.4.8 Suppose you have the function,

$$f(x) = 3x + 5$$

(a) What is the slope of $f(x)$?

A. $-\frac{1}{3}$

B. 3

C. 5

D. $-\frac{1}{5}$

(b) Applying Definition 3.4.7, what would the slope of a line perpendicular to $f(x)$ be?

A. $-\frac{1}{3}$

B. 3

C. 5

D. $-\frac{1}{5}$

(c) Find an equation of the line perpendicular to $f(x)$ that passes through the point $(3, 6)$.

Parallel and Perpendicular Lines (LF4)

Activity 3.4.9 For each pair of lines, determine if they are parallel, perpendicular, or neither.

(a)

$$f(x) = -3x + 4$$
$$g(x) = 5 - 3x$$

(b)

$$f(x) = 2x - 5$$
$$g(x) = 6x - 5$$

(c)

$$f(x) = 6x - 5$$
$$g(x) = \frac{1}{6}x + 8$$

(d)

$$f(x) = \frac{4}{5}x + 3$$
$$g(x) = -\frac{5}{4}x - 1$$

Parallel and Perpendicular Lines (LF4)

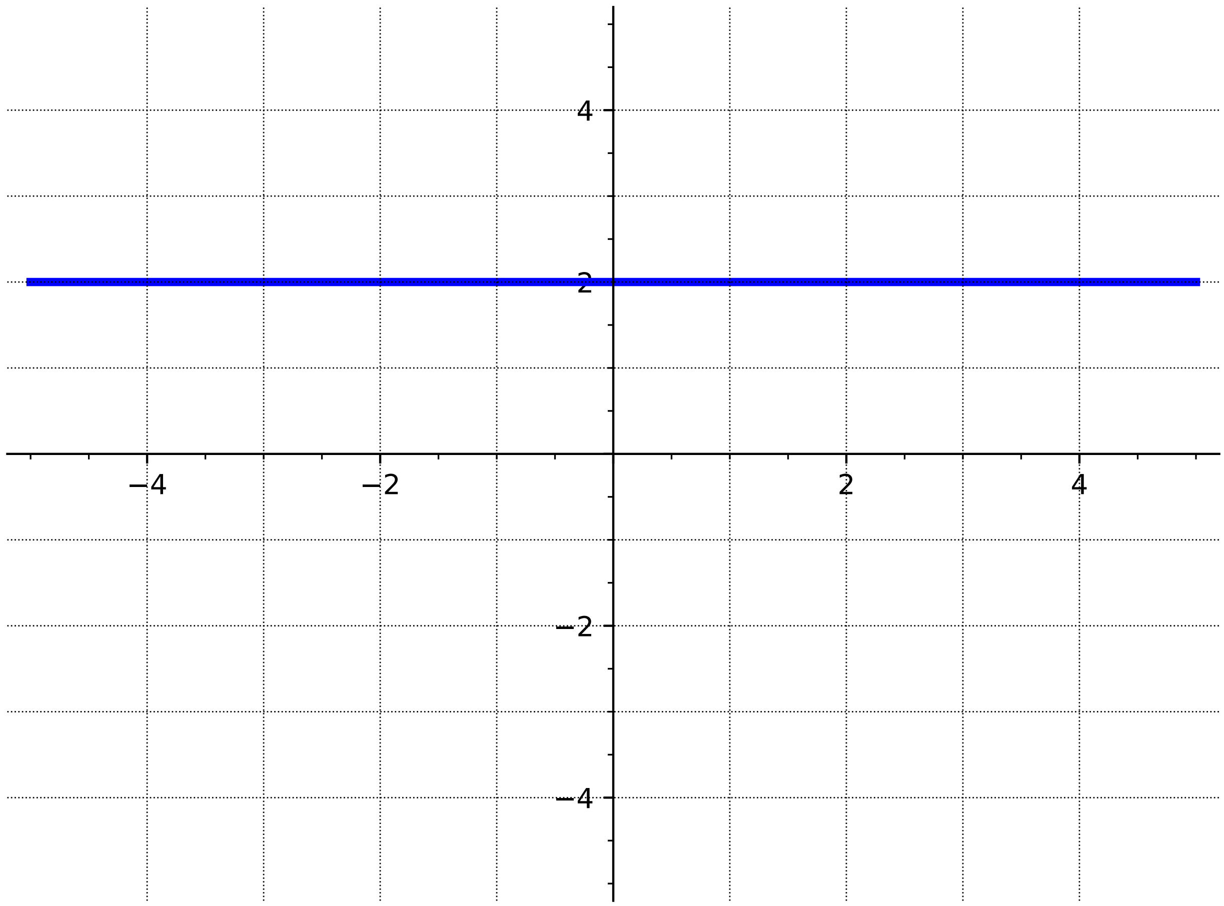
Activity 3.4.10 Consider the linear equation, $f(x) = -\frac{2}{3}x - 4$ and the point A: $(-6, 4)$.

(a) Find an equation of the line that is parallel to $f(x)$ and passes through the point A.

(b) Find an equation of the line that is perpendicular to $f(x)$ and passes through the point A.

Parallel and Perpendicular Lines (LF4)

Activity 3.4.11 Consider the line, $y = 2$, as shown in the graph below.



(a) What is the slope of the line $y = 2$?

A. undefined

B. 0

C. 1

D. $-\frac{1}{2}$

(b) What is the slope of a line that is parallel to $y = 2$?

A. undefined

B. 0

C. 1

D. $-\frac{1}{2}$

(c) Find an equation of the line that is parallel to $y = 2$ and passes through the point $(-1, -4)$.

(d) What is the slope of a line that is perpendicular to $y = 2$?

A. undefined

B. 0

C. 1

D. $-\frac{1}{2}$

Parallel and Perpendicular Lines (LF4)

(e) Find an equation of the line that is perpendicular to $y = 2$ and passes through the point $(-1, 2)$.

3.5 Linear Models and Meanings (LF5)

Objectives

- Build linear models from verbal descriptions, and use the models to establish conclusions, including by contextualizing the meaning of slope and intercept parameters.

Linear Models and Meanings (LF5)

Remark 3.5.1 We begin by revisiting Activity 2.2.14 from Section 2.2.

Linear Models and Meanings (LF5)

Activity 3.5.2 Ellie has \$13 in her piggy bank, and she gets an additional \$1.50 each week for her allowance. Assuming she does not spend any money, the total amount of allowance, $A(w)$, she has after w weeks can be modeled by the function

$$A(w) = 13 + 1.50w.$$

(a) How much money will be in her piggy bank after 5 weeks?

(b) After how many weeks will she have \$40 in her piggy bank?

Linear Models and Meanings (LF5)

Remark 3.5.3 The function in the previous activity is an example of a **linear model**. A linear model is a linear function that describes, or models, a real-life application. In this section we will build and use linear models.

Linear Models and Meanings (LF5)

Activity 3.5.4 Jack bought a package containing 40 cookies. Each day he takes two in his lunch to work.

(a) How many cookies are left in the package after 3 days?

A. 46

B. 42

C. 38

D. 36

E. 34

(b) Fill out the following table to represent the number of cookies left in the package after the given number of days.

number of days	**number of cookies left**
0	
2	
5	
10	
20	

(c) What is the y-intercept? Explain what it represents in the context of the problem.

(d) What is the rate of change? Explain what it represents in the context of the problem.

(e) Write a linear function to model the situation. Let d represent the number of days elapsed and $C(d)$ represent the number of cookies in the package. (Hint: Use the previous two questions to help!)

(f) Find $C(6)$. Explain what that means in the context of the problem.

(g) How many days will it take to empty the package? What does this correspond to on the graph?

Linear Models and Meanings (LF5)

Activity 3.5.5 Daisy's Doughnut Shop sells delicious doughnuts. Each month, they incur a fixed cost of \$2000 for rent, insurance, and other expenses. Then, for each doughnut they produce, it costs them an additional \$0.25.

(a) In January, Daisy's Doughnut Shop produced 1000 doughnuts. What was their total monthy cost to run the shop?

A. \$2000.25

B. \$2002.50

C. \$2025.00

D. \$2250.00

E. \$4500.00

(b) Fill out the following table to represent the cost for producing various amounts of doughnuts.

number of doughnuts	cost
0	
500	
1000	
1500	
2000	

(c) What is the y-intercept? Explain what it represents in the context of the problem.

(d) What is the rate of change? Explain what it represents in the context of the problem.

(e) Write a linear function to model the situation. Let x represent the number of doughnuts produced and $C(x)$ represent the total cost. (Hint: Use the previous two questions to help!)

(f) Find $C(1300)$. Explain what that means in the context of the problem.

(g) Find the x-intercept. Explain what it means in the context of the problem.

Linear Models and Meanings (LF5)

Activity 3.5.6 A taxi costs \$5.00 up front, and then charges \$0.73 per mile traveled.

(a) Write a linear function to model this situation.

(b) How much will it cost for a 13 mile taxi ride?

(c) If the taxi ride cost \$11.06, how many miles did it travel?

Linear Models and Meanings (LF5)

Activity 3.5.7 When reporting the weather, temperature is given in degrees Fahrenheit (F) or degrees Celsius (C). The two scales are related linearly, which means we can find a linear model to describe their relationship. This model lets us convert between the two scales.

(a) Water freezes at $0°C$ and $32°F$. Water boils at $100°C$ and $212°F$. Use this information to write two ordered pairs.

Hint. Choose Celsius to be your input value, and Fahrenheit to be the output value.

(b) Use the two points to write a linear model for this situation. Use C and F as your variables.

(c) If the temperature outside is $25°C$, what is the temperature in Fahrenheit?

(d) If the temperature outside is $50°F$, what is the temperature in Celsius?

(e) What temperature value is the same in Fahrenheit as it is in Celsius?

Activity 3.5.8 Erin needs to print t-shirts for her company retreat. She has found two businesses that produce quality shirts, but they have different pricing structures. Shirts-R-Us requires a \$60 set up fee, then charges \$7 for each shirt produced. Graphix! has no set up fee and charges \$8.50 per shirt produced.

(a) Write a linear function $S(x)$ that models the pricing structure for producing x shirts at Shirts-R-Us.

(b) Write a linear function $G(x)$ that models the pricing structure for producing x shirts at Graphix!.

(c) At which business would it be less expensive to buy 1 shirt? 10 shirts? 100 shirts? Explain your reasoning.

(d) It depended on how many shirts were needed to determine which business was less expensive. For what range of number of shirts should Erin choose Shirts-R-Us and for what range should she choose Graphix!?

Hint. Try finding where the cost is the same at both businesses. And looking at a graph may help as well!

3.6 Systems of Linear Equations (LF6)

Objectives

- Solve a system of two linear equations in two variables.

Systems of Linear Equations (LF6)

Observation 3.6.1 Often times when solving a real-world application, more than one equation is necessary to describe the information. We'll investigate some of those in this section.

Systems of Linear Equations (LF6)

Activity 3.6.2 Admission into a carnival for 4 children and 2 adults is \$128.50. For 6 children and 4 adults, the admission is \$208. Assuming a different price for children and adults, what is the price of the child's admission and the price of the adult admission?

(a) Let c represent the cost of a child's admission and a the cost of an adult admission. Write an equation to represent the total cost for 4 children and 2 adults.

A. $4a + 2c = 128.50$

B. $a + c = 128.50$

C. $4c + 2a = 128.50$

D. $a + c = 336.50$

(b) Now write an equation to represent the total cost for 6 children and 4 adults.

A. $6c + 4a = 208$

B. $a + c = 208$

C. $4a + 6c = 208$

D. $a + c = 336.50$

(c) Using the above equations, check by substitution which admission prices would satisfy both equations?

A. $c = \$20$ and $a = \$24.25$

B. $c = \$24.50$ and $a = \$15.25$

C. $c = \$20$ and $a = \$22$

D. $c = \$14.50$ and $a = \$30.25$

Systems of Linear Equations (LF6)

Definition 3.6.3 A **system of linear equations** consists of two or more linear equations made up of two or more variables. A **solution to a system of equations** is a value for each of the variables that satisfies all the equations at the same time. $\Diamond$

Systems of Linear Equations (LF6)

Activity 3.6.4 Consider the following system of equations.

$$\begin{cases} y = 2x + 4 \\ 3x + 2y = 1 \end{cases}$$

Which of the ordered pairs is a solution to the system?

A. $(3, 10)$

B. $(0, 4)$

C. $(1, -1)$

D. $(-1, 2)$

D

Remark 3.6.5 While we can test points to determine if they are solutions, it is not feasible to test every possible point to find a solution. We need a method to solve a system.

Systems of Linear Equations (LF6)

Activity 3.6.6 Consider the following system of equations.

$$\begin{cases} 3x - y = 2 \\ x + 4y = 5 \end{cases}$$

(a) Rewrite the first equation in terms of y.

(b) Rewrite the second equation in terms of y.

(c) Graph the two equations on the same set of axes. Where do the lines intersect?

(d) Check that the point of intersection of the two lines is a solution to the system of equations.

Remark 3.6.7 Sometimes it is difficult to determine the exact intersection point of two lines using a graph. Let's explore another possible method for solving a system of equations.

Systems of Linear Equations (LF6)

Activity 3.6.8 Consider the following system of equations.

$$\begin{cases} 3x + y = 4 \\ x + 3y = 10 \end{cases}$$

(a) Graph the two equations on the same set of axes. Is it possible to determine exactly where the lines intersect?

(b) Solve the first equation for y and substitute into the second equation. What is the resulting equation?

A. $x + 4 - 3x = 10$

B. $x + 3(4 - 3x) = 10$

C. $4 - 3x + 3y = 10$

D. $3x + (4 - 3x) = 4$

(c) Solve the resulting equation from part (b) for x.

A. $x = -3$

B. $x = \frac{1}{4}$

C. $x = \frac{7}{3}$

D. $x = 0$

(d) Substitute the value of x into the first equation to find the value of y.

A. $y = -5$

B. $y = \frac{13}{4}$

C. $y = -3$

D. $y = \frac{4}{3}$

(e) Write the solution to the system of equations (the found values of x and y) as an ordered pair.

Remark 3.6.9 This method of solving a system of equations is referred to as the **Substitution Method**.

1. Solve one of the equations for one variable.
2. Substitute the expression into the other equation to solve for the remaining variable.
3. Substitute that value into either equation to find the value of the first variable.

Systems of Linear Equations (LF6)

Activity 3.6.10 Solve the following system of equations using the substitution method.

$$\begin{cases} x + 2y = -1 \\ -x + y = 3 \end{cases}$$

Remark 3.6.11 While the substitution method will always work, sometimes the resulting equations will be difficult to solve. Let's explore a third method for solving a system of two linear equations with two variables.

Systems of Linear Equations (LF6)

Activity 3.6.12 Consider the following system of equations.

$$\begin{cases} 5x + 7y = 12 \\ 3x - 7y = 37 \end{cases}$$

(a) Add the two equations together. What is the resulting equation?

A. $2x = -15$

B. $14y = 49$

C. $8x + 14y = 49$

D. $8x = 49$

(b) Use the resulting equation after addition, to solve for the variable.

A. $x = -\dfrac{15}{2}$

B. $y = \dfrac{49}{14}$

C. $x = \dfrac{49}{22}$

D. $x = \dfrac{49}{8}$

(c) Use the value to find the solution to the system of equations.

Remark 3.6.13 This method of solving a system of equations is referred to as the **Elimination Method**.

1. Combine the two equations using addition or subtraction to eliminate one of the variables.

2. Solve the resulting equation.

3. Substitute that value into either equation to find the value of the other variable.

Systems of Linear Equations (LF6)

Activity 3.6.14 Solve the following system of equations using the elimination method.

$$\begin{cases} 7x - 4y = 3 \\ 3y - 7x = 8 \end{cases}$$

Systems of Linear Equations (LF6)

Activity 3.6.15 Consider the following system of equations.

$$\begin{cases} 5x - 9y = 6 \\ -10x + 4y = 2 \end{cases}$$

Notice that if you simply add the two equations together, it will not eliminate a variable. Substitution will also be difficult since it involves fractions.

(a) What value can you multiply the first equation by so that when you add the result to the second equation one variable cancels?

A. -1 and the x will cancel

B. 2 and the x will cancel

C. 3 and the y will cancel

D. -2 and the y will cancel

(b) Perform the multiplication and add the two equations. What is the resulting equation?

A. $-5y = 8$

B. $-14y = 14$

C. $-14y = 8$

D. $-5x = 4$

(c) What is the solution to the system of equations?

A. $(-1, -1.6)$

B. $(-1, -0.6)$

C. $(-0.6, -1)$

D. $(-1.6, -1)$

Systems of Linear Equations (LF6)

Activity 3.6.16 For each system of equations, determine which method (graphical, substitution, or elimination) might be best for solving.

(a)

$$\begin{cases} 5x + 9y = 16 \\ x + 2y = 4 \end{cases}$$

A. Graphical

B. Substitution

C. Elimination

(b)

$$\begin{cases} y = 4x - 6 \\ y = -5x + 21 \end{cases}$$

A. Graphical

B. Substitution

C. Elimination

(c)

$$\begin{cases} x + y = 10 \\ x - y = 12 \end{cases}$$

A. Graphical

B. Substitution

C. Elimination

Activity 3.6.17 Solve each of the systems of equations from Activity 3.6.16 using the method you chose.

3.7 Applications of Systems of Linear Equations (LF7)

Objectives

- Solve questions involving applications of systems of equations.

Applications of Systems of Linear Equations (LF7)

Remark 3.7.1 Now that we have explored multiple methods for solving systems of linear equations, let's put those in to practice using some real-world application problems.

Applications of Systems of Linear Equations (LF7)

Activity 3.7.2 Let's begin by revisiting the carnival admission problem from Section 3.6.

Admission into a carnival for 4 children and 2 adults is \$128.50. For 6 children and 4 adults, the admission is \$208. Assuming a different price for children and adults, what is the price of the child's admission and the price of the adult admission?

(a) First, set up a system of equations representing the given information. Use x to represent the child admission price and y for the adult admission price.

A. $\begin{cases} x + y = 128.50 \\ x + y = 208 \end{cases}$

B. $\begin{cases} 2x + 4y = 128.50 \\ 4x + 6y = 208 \end{cases}$

C. $\begin{cases} 4x + 2y = 128.50 \\ 6x + 4y = 208 \end{cases}$

D. $\begin{cases} 6x + 4y = 128.50 \\ 4x + 2y = 208 \end{cases}$

(b) Solve the system of equations.

A. $(27, 10.25)$

B. $(15.25, 24.5)$

C. $(24.5, 15.25)$

D. $(10, 37)$

(c) Write your solution in terms of the price of admission for children and adults.

Applications of Systems of Linear Equations (LF7)

Activity 3.7.3 Let's revisit another application we've encountered before in Section 1.2, Activity 1.2.7.

Ammie's favorite snack to share with friends is candy salad, which is a mixture of different types of candy. Today she chooses to mix Nerds Gummy Clusters, which cost \$8.38 per pound, and Starburst Jelly Beans, which cost \$7.16 per pound. If she makes seven pounds of candy salad and spends a total of \$55.61, how many pounds of each candy did she buy?

(a) Set up a system of equations to represent the mixture problem. Let N represent the pounds of Nerds Gummy Clusters and S represent the pounds of Starburst Jelly Beans in the mixture.

A. $\begin{cases} N + S = 7 \\ 7.16N + 8.38S = 55.61 \end{cases}$

B. $\begin{cases} N + S = 7 \\ 8.38N + 7.16S = 55.61 \end{cases}$

C. $\begin{cases} N + S = 55.61 \\ 8.38N + 7.16S = 389.27 \end{cases}$

D. $\begin{cases} N + S = 7 \\ 7N + 7S = 55.61 \end{cases}$

(b) Now solve the system of equations and put your answer in the context of the problem.

A. Ammie bought 2.5 lbs of Nerds Gummy Clusters and 4.5 lbs of Starburst Jelly Beans.

B. Ammie bought 3.5 lbs of Nerds Gummy Clusters and 3.5 lbs of Starburst Jelly Beans.

C. Ammie bought 4.5 lbs of Nerds Gummy Clusters and 2.5 lbs of Starburst Jelly Beans.

D. Ammie bought 5.5 lbs of Nerds Gummy Clusters and 1.5 lbs of Starburst Jelly Beans.

Applications of Systems of Linear Equations (LF7)

Activity 3.7.4 A couple has a total household income of \$104,000. The wife earns \$16,000 less than twice what the husband earns. How much does the wife earn?

(a) Set up a system of equations to represent the situation. Let w represent the wife's income and h represent the husband's income.

A. $\begin{cases} w + h = 104000 \\ h = 2w - 16000 \end{cases}$

B. $\begin{cases} w + h = 104000 \\ 2h = w - 16000 \end{cases}$

C. $\begin{cases} w + 2h = 104000 \\ w = h - 16000 \end{cases}$

D. $\begin{cases} w + h = 104000 \\ w = 2h - 16000 \end{cases}$

(b) Solve the system of equations. How much does the wife earn?

A. The wife earns \$29,300.

B. The wife earns \$40,000.

C. The wife earns \$64,000.

D. The wife earns \$74,600.

Applications of Systems of Linear Equations (LF7)

Activity 3.7.5 Kenneth currently sells suits for Company A at a salary of \$22,000 plus a \$10 commission for each suit sold. Company B offers him a position with a salary of \$28,000 plus a \$4 commission for each suit sold. How many suits would Kenneth need to sell for the options to be equal?

Set-up and solve a system of equations to represent the situation.

Chapter 4

Polynomial and Rational Functions (PR)

Objectives

How do we model polynomial or rational change?
By the end of this chapter, you should be able to...

1. Graph quadratic functions and identify their axis of symmetry, and maximum or minimum point.
2. Use quadratic models to solve an application problem and establish conclusions.
3. Determine the zeros and their multiplicities of a polynomial in factored form. Describe and graph the behavior of a polynomial function at the intercepts and the ends.
4. Rewrite a rational function as a polynomial plus a proper rational function.
5. Determine the zeros of a polynomial function with real coefficients.
6. Find the domain and range, vertical and horizontal asymptotes, and intercepts of a rational function and use this information to sketch the graph.

4.1 Graphing Quadratic Functions (PR1)

Objectives

- Graph quadratic functions and identify their axis of symmetry, and maximum or minimum point.

Graphing Quadratic Functions (PR1)

Observation 4.1.1 Quadratic functions have many different applications in the real world. For example, say we want to identify a point at which the maximum profit or minimum cost occurs. Before we can interpret some of these situations, however, we will first need to understand how to read the graphs of quadratic functions to locate these least and greatest values.

Activity 4.1.2 Use the graph of the quadratic function $f(x) = 3(x-2)^2 - 4$ to answer the questions below.

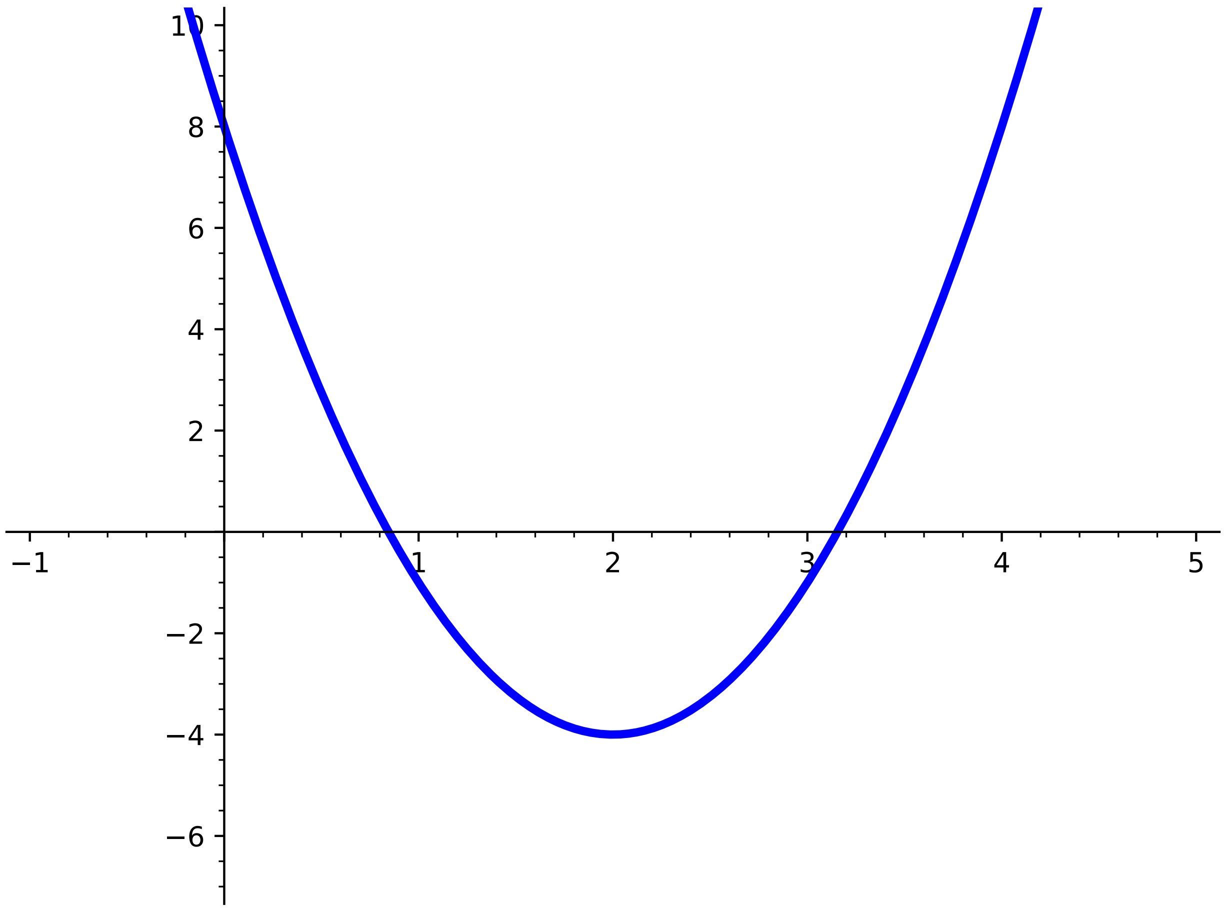


Figure 4.1.3

(a) Make a table for values of $f(x)$ corresponding to the given x-values. What is happening to the y-values as the x-values increase? Do you notice any other patterns of the y-values of the table?

Table 4.1.4

x	$f(x)$
-2	
-1	
0	
1	
2	
3	
4	
5	

(b) At which point (x, y) does $f(x)$ have a minimum value? That is, is there a point on the graph that is lower than all other points?

A. The minimum value appears to occur near $(0, 8)$.

B. The minimum value appears to occur near $(-\frac{1}{5}, 10)$.

C. The minimum value appears to occur near $(2, -4)$.

D. There is no minimum value of this function.

(c) At which point (x, y) does $f(x)$ have a maximum value? That is, is there a point on the graph that is higher than all other points?

A. The maximum value appears to occur near $(-2, 44)$.

B. The maximum value appears to occur near $(-\frac{1}{5}, 10)$.

C. The maximum value appears to occur near $(2, -4)$.

D. There is no maximum value of this function.

Graphing Quadratic Functions (PR1)

Definition 4.1.5 The point at which a quadratic function has a maximum or minimum value is called the **vertex**. The **vertex form** of a quadratic function is given by

$$f(x) = a(x - h)^2 + k,$$

where (h, k) is the **vertex** of the parabola. $\Diamond$

Graphing Quadratic Functions (PR1)

Definition 4.1.6 The **axis of symmetry**, also known as the line of symmetry, is the line that makes the shape of an object symmetrical. For a quadratic function, the axis of symmetry always passes through the vertex (h, k) and so is the vertical line $x = h$. $\diamondsuit$

Activity 4.1.7 Use the given quadratic function, $f(x) = 3(x-2)^2 - 4$, to answer the following:

(a) Applying Definition 4.1.6, what is the vertex and axis of symmetry of $f(x)$?

A. vertex: $(2, -4)$; axis of symmetry: $x = 2$

B. vertex: $(-2, 4)$; axis of symmetry: $x = -2$

C. vertex: $(-2, -4)$; axis of symmetry: $x = -2$

D. vertex: $(2, 4)$; axis of symmetry: $x = 2$

(b) Compare what you got in part a with the values you found in Activity 4.1.2. What do you notice?

Graphing Quadratic Functions (PR1)

Definition 4.1.8 The **standard form** of a quadratic function is given by

$$f(x) = ax^2 + bx + c,$$

where a, b, and c are real coefficients. $\Diamond$

Observation 4.1.9 Just as with the vertex form of a quadratic, we can use the standard form of a quadratic to find the **axis of symmetry** and the **vertex** by using the values of a, b, and c. Given the standard form of a quadratic, the axis of symmetry is the vertical line $x = \frac{-b}{2a}$ and the vertex is at the point $(\frac{-b}{2a}, f(\frac{-b}{2a}))$.

Graphing Quadratic Functions (PR1)

Activity 4.1.10 Use the graph of the quadratic function to answer the questions below.

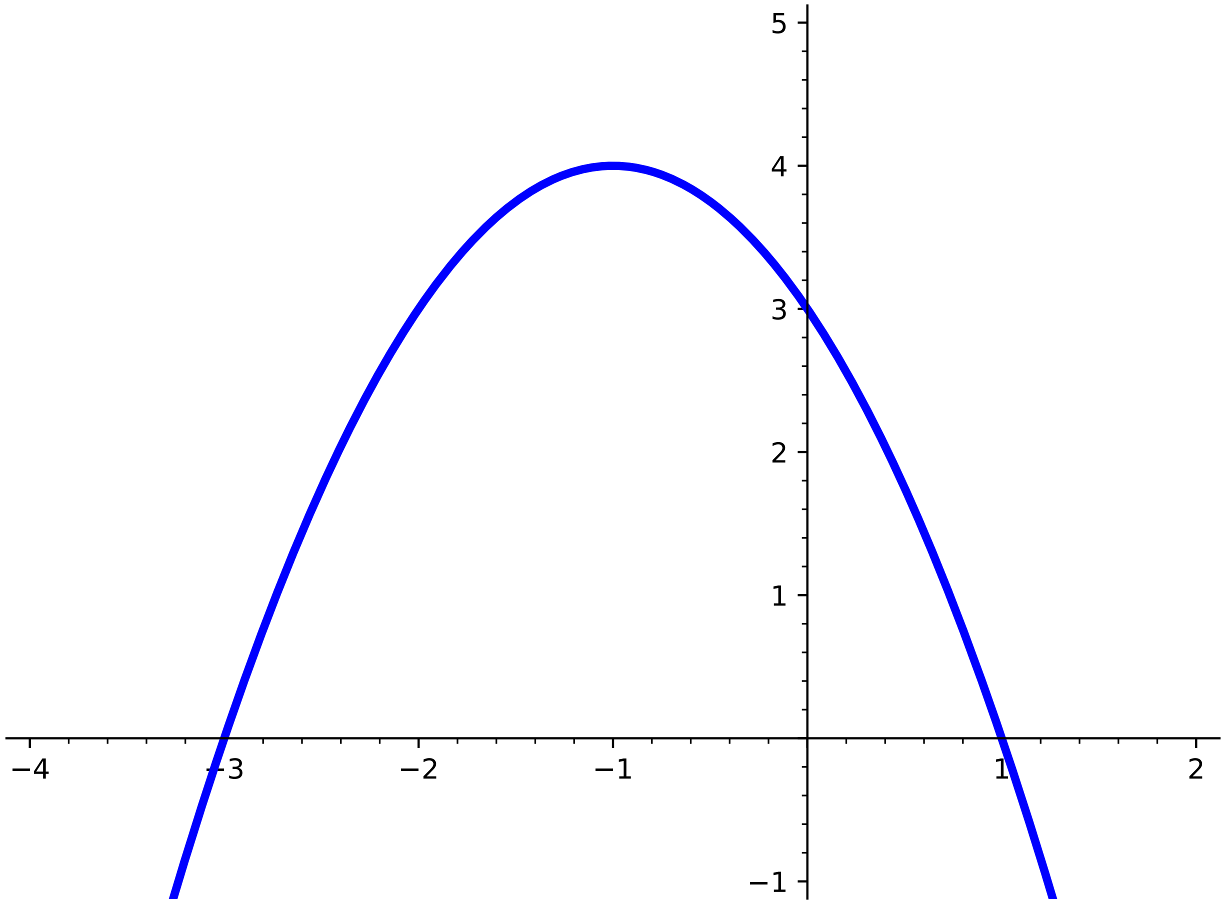


Figure 4.1.11

(a) Which of the following quadratic functions could be the graph shown in the figure?

A. $f(x) = x^2 + 2x + 3$

B. $f(x) = -(x+1)^2 + 4$

C. $f(x) = -x^2 - 2x + 3$

D. $f(x) = (x+1)^2 + 4$

(b) What is the maximum or minimum value?

A. -1

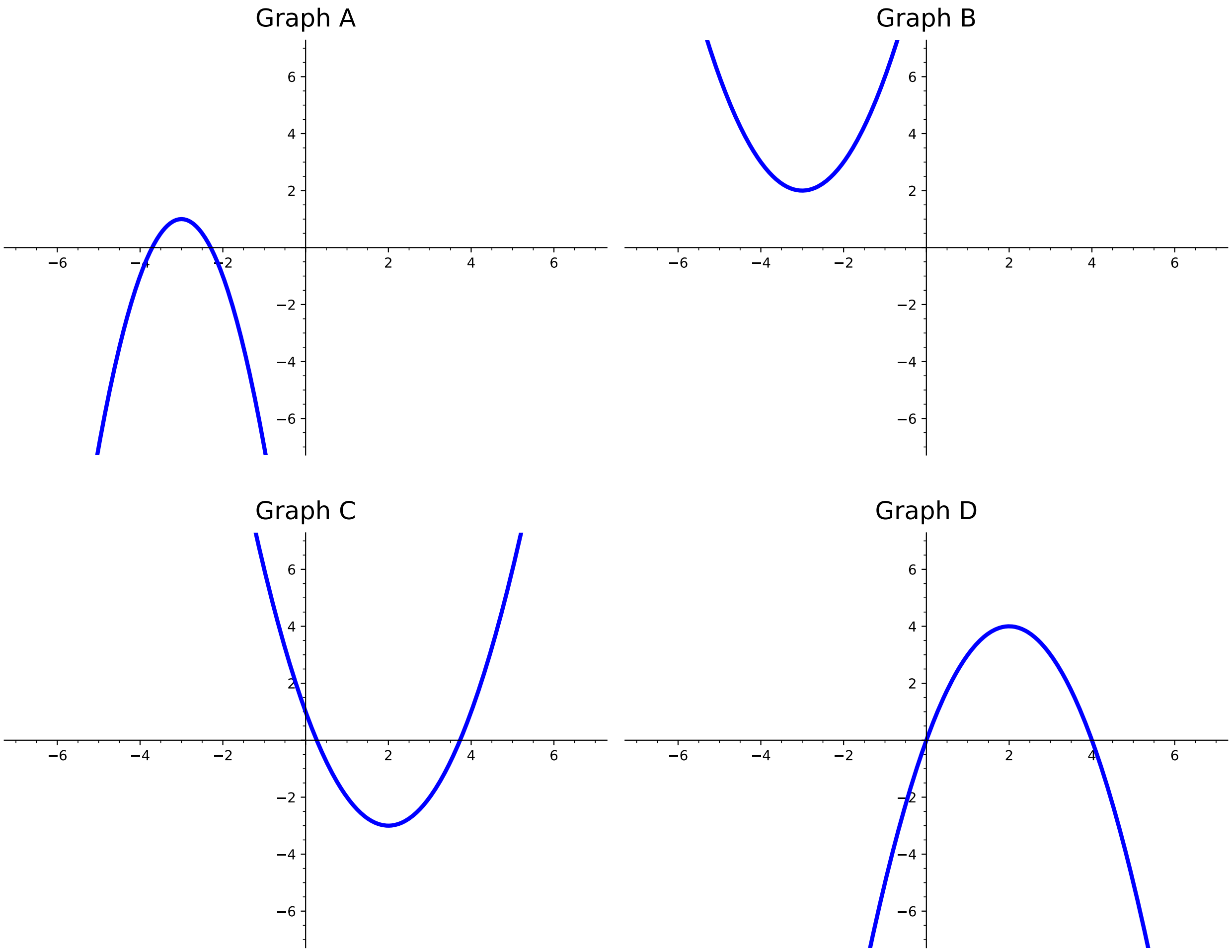
B. 4

C. -3

D. 1

Graphing Quadratic Functions (PR1)

Activity 4.1.12 Consider the following four graphs of quadratic functions:



(a) Which of the graphs above have a maximum?

A. Graph A
B. Graph B
C. Graph C
D. Graph D

(b) Which of the graphs above have a minimum?

A. Graph A
B. Graph B
C. Graph C
D. Graph D

(c) Which of the graphs above have an axis of symmetry at $x = 2$?

A. Graph A
B. Graph B
C. Graph C
D. Graph D

(d) Which of the graphs above represents the function $f(x) = -(x-2)^2 + 4$?

A. Graph A

B. Graph B

C. Graph C

D. Graph D

(e) Which of the graphs above represents the function $f(x) = x^2 - 4x + 1$?

A. Graph A

B. Graph B

C. Graph C

D. Graph D

Remark 4.1.13 Notice that the maximum or minimum value of the quadratic function is the y-value of the vertex.

Graphing Quadratic Functions (PR1)

Activity 4.1.14 A function $f(x)$ has a maximum value at 7 and its axis of symmetry at $x = -2$.

(a) Sketch a graph of a function that meets the criteria for $f(x)$.

(b) Was your graph the only possible answer? Try to sketch another graph that meets this criteria.

Remark 4.1.15 Other points, such as x- and y-intercepts, may be helpful in sketching a more accurate graph of a quadratic function.

Graphing Quadratic Functions (PR1)

Activity 4.1.16 Consider the following two quadratic functions $f(x) = x^2 - 4x + 20$ and $g(x) = 2x^2 - 8x + 24$ and answer the following questions:

(a) Applying Definition 4.1.9, what is the vertex and axis of symmetry of $f(x)$?

A. vertex: $(2, -16)$; axis of symmetry: $x = 2$

B. vertex: $(-2, 16)$; axis of symmetry: $x = -2$

C. vertex: $(-2, -16)$; axis of symmetry: $x = -2$

D. vertex: $(2, 16)$; axis of symmetry: $x = 2$

(b) Applying Definition 4.1.9, what is the vertex and axis of symmetry of $g(x)$?

A. vertex: $(2, -16)$; axis of symmetry: $x = 2$

B. vertex: $(-2, 16)$; axis of symmetry: $x = -2$

C. vertex: $(-2, -16)$; axis of symmetry: $x = -2$

D. vertex: $(2, 16)$; axis of symmetry: $x = 2$

(c) What do you notice about $f(x)$ and $g(x)$?

(d) Now graph both $f(x)$ and $g(x)$ and draw a sketch of each graph on one coordinate plane. How are they similar/different?

4.2 Quadratic Models and Meanings (PR2)

Objectives

- Use quadratic models to solve an application problem and establish conclusions.

Quadratic Models and Meanings (PR2)

Activity 4.2.1 A water balloon is tossed vertically from a fifth story window. It's height $h(t)$, in feet, at a time t, in seconds, is modeled by the function

$$h(t) = -16t^2 + 40t + 50$$

(a) Complete the following table. Do all the values have meaning in terms of the model?

Table 4.2.2

t	$h(t)$
0	
1	
2	
3	
4	
5	

(b) Compute the slope of the line joining $t = 0$ and $t = 1$. Then, compute the slope of the line joining $t = 1$ and $t = 2$. What do you notice about the slopes?

(c) What is the meaning of $h(0) = 50$?

A. The initial height of the water balloon is 50 feet.

B. The water balloon reaches a maximum height of 50 feet.

C. The water balloon hits the ground after 50 seconds.

D. The water balloon travels 50 feet before hitting the ground.

(d) Find the vertex of the quadratic function $h(t)$.

A. $(0, 50)$

B. $(1, 74)$

C. $(1.25, 75)$

D. $(3.4, 0)$

(e) What is the meaning of the vertex?

A. The water balloon reaches a maximum height of 50 feet at the start.

B. After 1 second, the water balloon reaches a maximum height of 74 feet.

C. After 1.25 seconds, the water balloon reaches the maximum height.

D. After 3.4 seconds, the water balloon hits the ground.

Quadratic Models and Meanings (PR2)

Activity 4.2.3 The population of a small city is given by the function $P(t) = -50t^2 + 1200t + 32000$, where t is the number of years after 2015.

(a) When will the population of the city reach a maximum?

A. 2020

B. 2022

C. 2025

D. 2027

(b) Determine when the population of the city is increasing and when it is decreasing.

(c) When will the population of the city reach 36,000 people?

A. 2019

B. 2025

C. 2027

D. 2035

Quadratic Models and Meanings (PR2)

Activity 4.2.4 The unit price of an item affects its supply and demand. That is, if the unit price increases, the demand for the item will usually decrease. For example, an online streaming service currently has 84 million subscribers at a monthly charge of \$6. Market research has suggested that if the owners raise the price to \$8, they would lose 4 million subscribers. Assume that subscriptions are linearly related to the price.

(a) Which of the following represents a linear function which relates the price of the streaming service p to the number of subscribers Q (in millions)?

A. $Q(p) = -2p$

B. $Q(p) = -2p + 84$

C. $Q(p) = -2p - 4$

D. $Q(p) = -2p + 96$

(b) Using the fact that revenue R is price times the number of items sold, $R = pQ$, which of the following represents the revenue in terms of the price?

A. $R(p) = -2p^2$

B. $R(p) = -2p^2 + 84p$

C. $R(p) = -2p^2 - 4p$

D. $R(p) = -2p^2 + 96p$

(c) What price should the streaming service charge for a monthly subscription to maximize their revenue?

A. \$10

B. \$19.50

C. \$24

D. \$28.25

(d) How many subscribers would the company have at this price?

A. 39.5 million

B. 48 million

C. 57 million

D. 76 million

(e) What is the maximum revenue?

A. 760 million

B. 1112 million

C. 1152 million

D. 1116 million

Activity 4.2.5 The owner of a ranch decides to enclose a rectangular region with 240 feet of fencing. To help the fencing cover more land, he plans to use one side of his barn as part of the enclosed region. What is the maximum area the rancher can enclose?

(a) Draw a picture to represent the fenced area against the barn. Use w to represent the length of fence parallel to the barn and l to represent the two sides perpendicular to the barn.

(b) Find an equation for the area of the fence in terms of the length l. It may be useful to find an equation for the total amount of fencing in terms of the length l and width w.

A. $A = lw$

B. $A = l^2$

C. $A = l(240 - 2l)$

D. $A = l\left(120 - \frac{l}{2}\right)$

(c) Use the area equation to find the maximum area the rancher can enclose.

4.3 Graphs of Polynomial Functions (PR3)

Objectives

- Determine the zeros and their multiplicities of a polynomial in factored form. Describe and graph the behavior of a polynomial function at the intercepts and the ends.

Graphs of Polynomial Functions (PR3)

Remark 4.3.1 Just like with quadratic functions, we should be able to determine key characteristics that will help guide us in creating a sketch of any polynomial function. We can start by finding both x and y-intercepts and then explore other characteristics polynomial functions can have. Recall that the **zeros** of a function are the x**-intercepts** - i.e., the values of x that cross or touch the x-axis. Just like with quadratic functions, we can find the zeros of a function by setting the function equal to 0 and solving for x.

Graphs of Polynomial Functions (PR3)

Activity 4.3.2 Given the function, $f(x) = (x-2)(x+1)(x-3)^2$, determine the following characteristics.

(a) How many zeros does $f(x)$ have?

A. 1

B. 2

C. 3

D. 4

(b) What are the zeros of $f(x)$?

A. $1, 2, 3$

B. $-1, 2, 3$

C. $1, -2, -3$

D. $-1, 2, -3$

(c) What is the y-intercept of $f(x)$?

A. -1

B. -6

C. 6

D. -18

(d) Now that we have found both the x and y-intercepts of $f(x)$, do we have enough information to draw a possible sketch of $f(x)$? What other characteristics would be useful to help us draw an accurate sketch of $f(x)$?

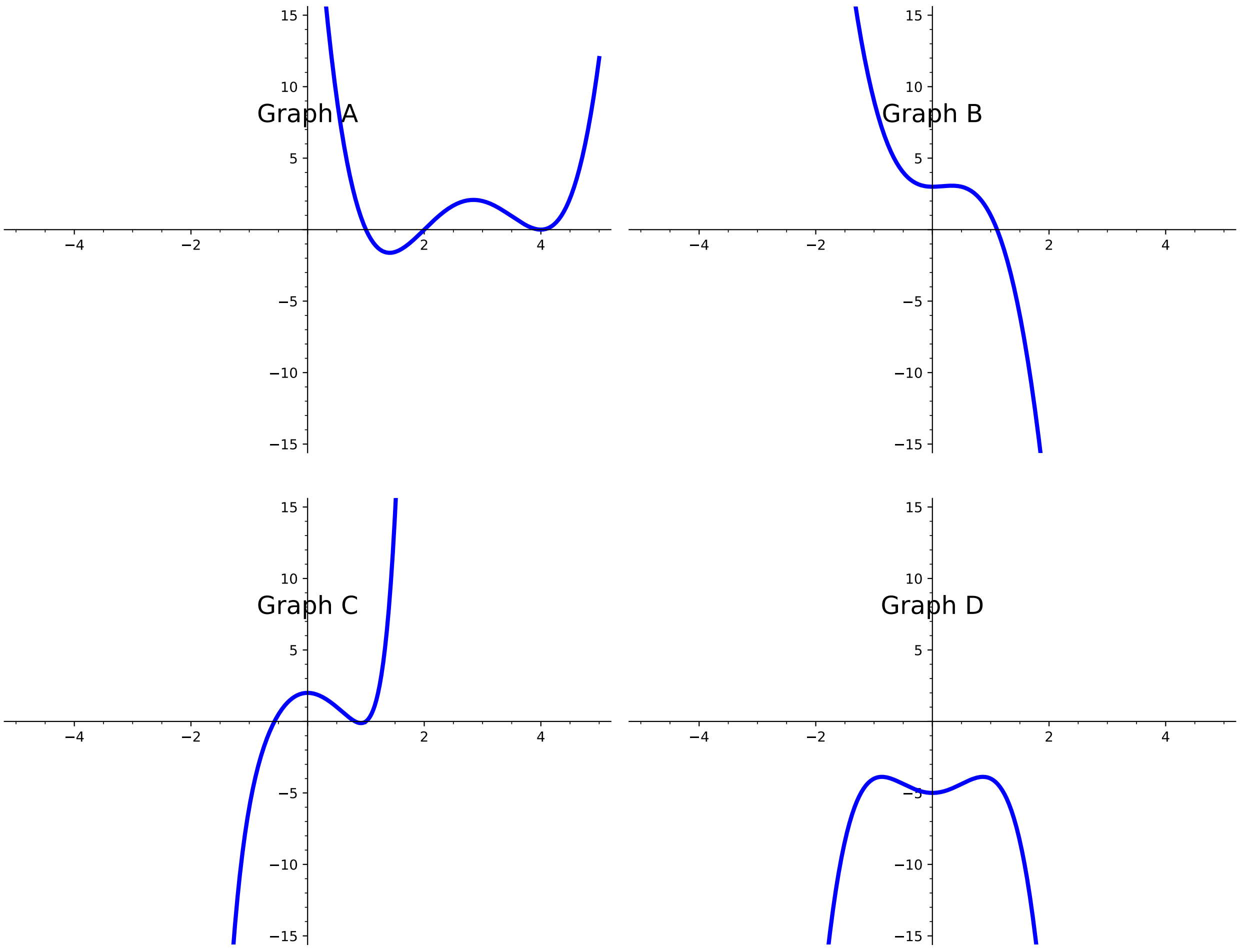
Definition 4.3.3 The **end behavior** of a polynomial function describes the behavior of the graph at the "ends" of the function. In other words, as we move to the right of the graph (as the x values increase), what happens to the y values? Similarly, as we move to the left of the graph (as the x values decrease), what happens to the y values?

Although we are looking at the "ends" to determine the end behavior, note that polynomials do not actually have ends. In other words, polynomial functions have y-values that exist for every x-value.

$\Diamond$

Graphs of Polynomial Functions (PR3)

Activity 4.3.4 Use the graphs of the following polynomial functions to answer the questions below.



(a) How would you describe the behavior of Graph A as you approach the ends?

A. Graph A rises on the left and on the right.

B. Graph A rises on the left, but falls on the right.

C. Graph A rises on the right, but falls on the left.

D. Graph A falls on the left and on the right.

(b) How would you describe the behavior of Graph B as you approach the ends?

A. Graph B rises on the left and on the right.

B. Graph B rises on the left, but falls on the right.

C. Graph B rises on the right, but falls on the left.

D. Graph B falls on the left and on the right.

(c) How would you describe the behavior of Graph C as you approach the ends?

A. Graph C rises on the left and on the right.

B. Graph C rises on the left, but falls on the right.

C. Graph C rises on the right, but falls on the left.

D. Graph C falls on the left and on the right.

(d) How would you describe the behavior of Graph D as you approach the ends?

A. Graph D rises on the left and on the right.

B. Graph D rises on the left, but falls on the right.

C. Graph D rises on the right, but falls on the left.

D. Graph D falls on the left and on the right.

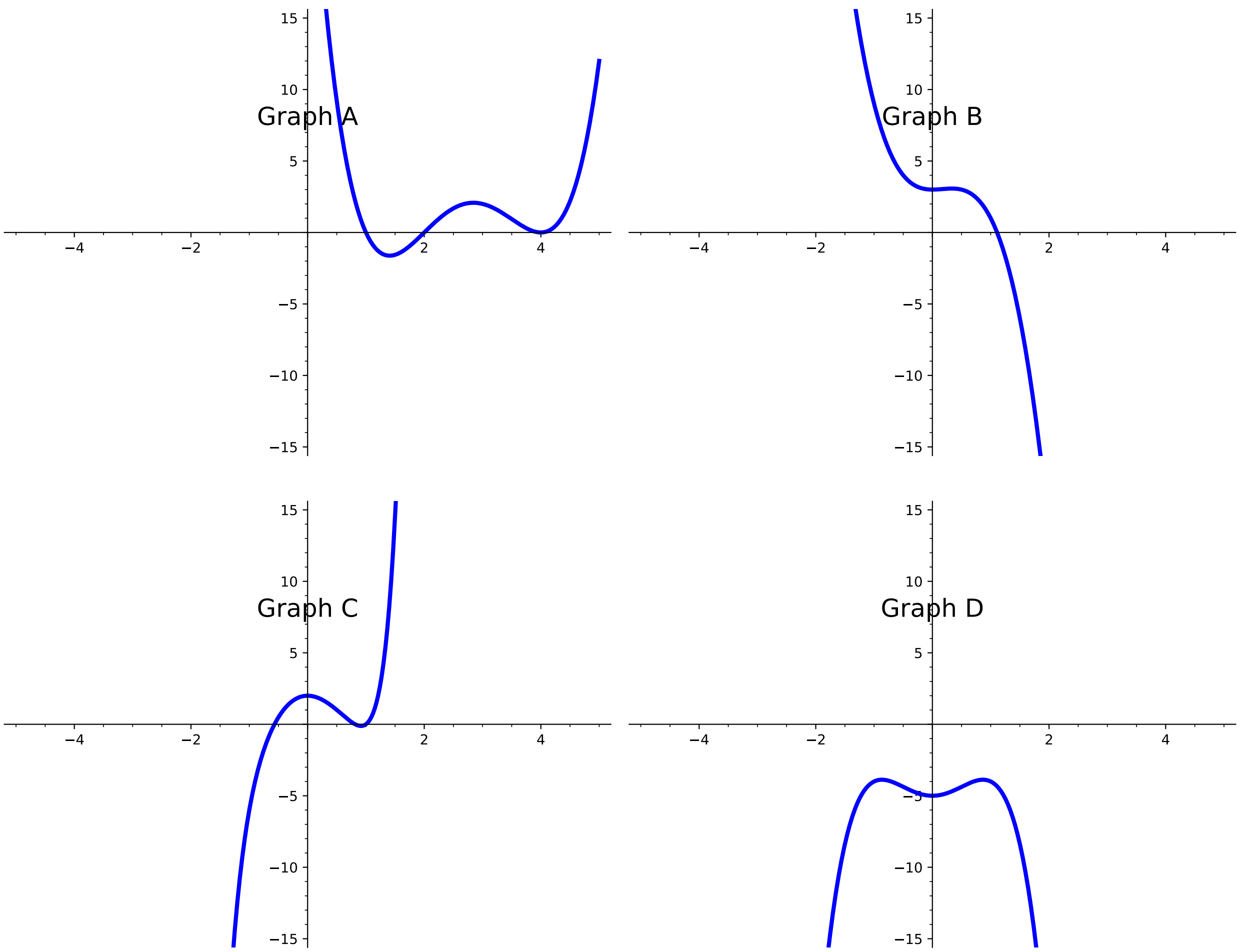
Graphs of Polynomial Functions (PR3)

Definition 4.3.5 Typically, when given an equation of a polynomial function, we look at the **degree** and **leading coefficient** to help us determine the behavior of the ends. The **degree** is the highest exponential power in the polynomial. The **leading coefficient** is the number written in front of the variable with the highest exponential power. ◊

Graphs of Polynomial Functions (PR3)

Activity 4.3.6 Let's refer back to the graphs in Activity 4.3.4 and look at the equations of those polynomial functions. Let's apply Definition 4.3.5 to see if we can determine how the degree and leading coefficients of those graphs affect their end behavior.

- Graph A: $f(x) = -11x^3 + 32 + 42x^2 + x^4 - 64x$
- Graph B: $g(x) = 2x^2 + 3 - 4x^3$
- Graph C: $h(x) = x^7 + 2x^3 - 5x^2 + 2$
- Graph D: $j(x) = 3x^2 - 2x^4 - 5$



(a) What is the degree and leading coefficient of Graph A?

A. Degree: -64; Leading Coefficient: 4

B. Degree: 4; Leading Coefficient: 0

C. Degree: 1; Leading Coefficient: -64

D. Degree: 4; Leading Coefficient: 1

(b) What is the degree and leading coefficient of Graph B?

A. Degree: 3; Leading Coefficient: -4
B. Degree: -4; Leading Coefficient: 3
C. Degree: 2; Leading Coefficient: 3
D. Degree: 3; Leading Coefficient: 4

(c) What is the degree and leading coefficient of Graph C?

A. Degree: -5; Leading Coefficient: 2
B. Degree: 0; Leading Coefficient: 7
C. Degree: -5; Leading Coefficient: 3
D. Degree: 7; Leading Coefficient: 1

(d) What is the degree and leading coefficient of Graph D?

A. Degree: -2; Leading Coefficient: 4
B. Degree: 3; Leading Coefficient: 2
C. Degree: 4; Leading Coefficient: -2
D. Degree: -5; Leading Coefficient: 4

(e) Notice that Graph A and Graph D have their ends going in the same direction. What conjectures can you make about the relationship between their degrees and leading coefficients with the behavior of their graphs?

(f) Notice that Graph B and Graph C have their ends going in opposite directions. What conjectures can you make about the relationship between their degrees and leading coefficients with the behavior of their graphs?

Remark 4.3.7 From Activity 4.3.6, we saw that the degree and leading coefficient of a polynomial function can give us more clues about the behavior of the function. In summary, we know:

- If the degree is even, the ends of the polynomial function will be going in the same direction. If the leading coefficient is positive, both ends will be pointing up. If the leading coefficient is negative, both ends will be pointing down.

- If the degree is odd, the ends of the polynomial function will be going in opposite directions. If the leading coefficient is positive, the left end will fall and the right end will rise. If the leading coefficient is negative, the left end will rise and the right end will fall.

Definition 4.3.8 When describing end behavior, mathematicians typically use **arrow notation**. Just as the name suggests, arrows are used to indicate the behavior of certain values on a graph.

For end behavior, students are often asked to determine the behavior of y-values as x-values either increase or decrease. The statement "As $x \to \infty$, $f(x) \to -\infty$" can be translated to "As x approaches infinity (or as x increases), $f(x)$ (or the y-values) go to negative infinity (i.e., it decreases)." ♢

Activity 4.3.9 Use the graph of $f(x)$ to answer the questions below.

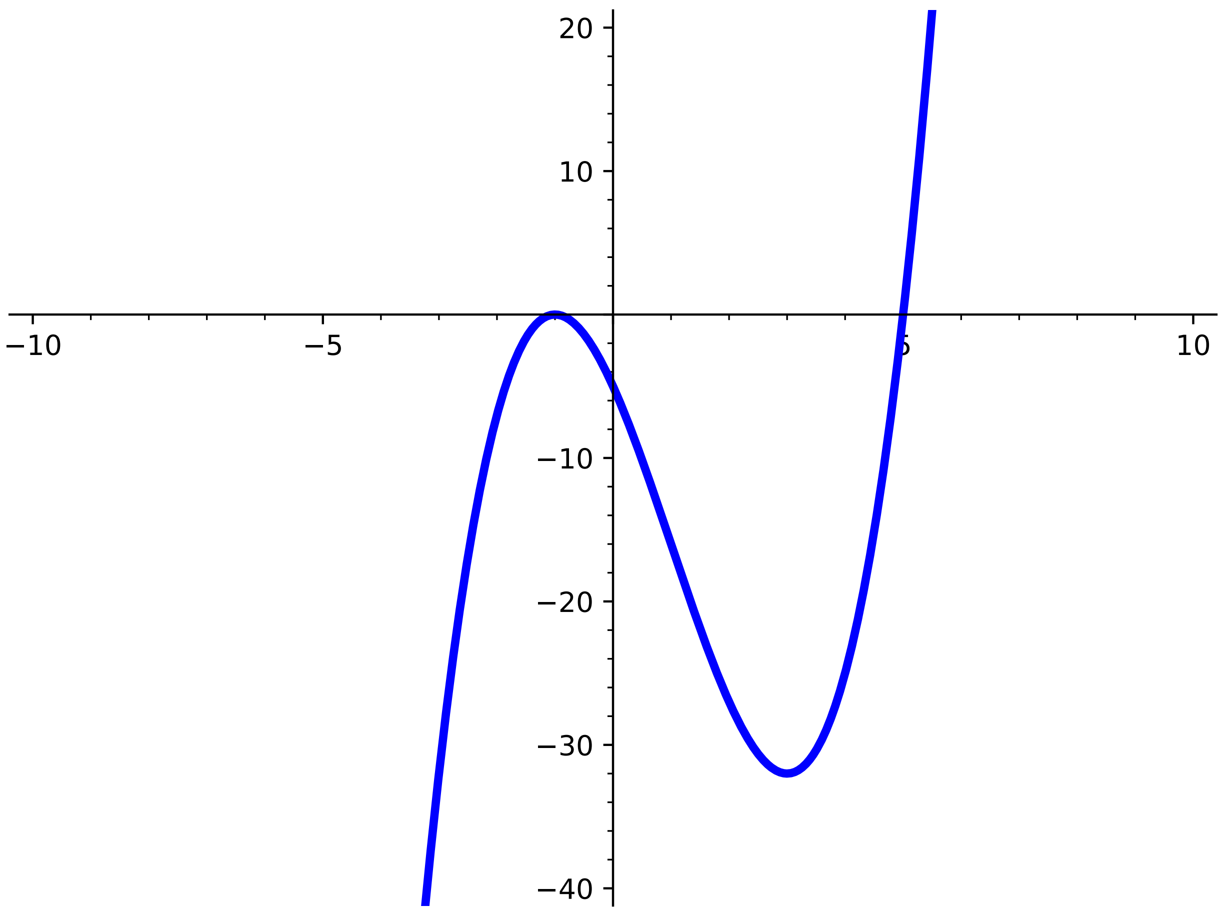


Figure 4.3.10

(a) How would you describe the end behavior of $f(x)$?

A. $f(x)$ rises on the left and on the right.
B. $f(x)$ rises on the left, but falls on the right.
C. $f(x)$ rises on the right, but falls on the left.
D. $f(x)$ falls on the left and on the right.

(b) How would you describe the end behavior of $f(x)$ using arrow notation?

A. As $x \to -\infty$, $f(x) \to -\infty$
As $x \to \infty$, $f(x) \to -\infty$
B. As $x \to -\infty$, $f(x) \to -\infty$
As $x \to \infty$, $f(x) \to \infty$
C. As $x \to -\infty$, $f(x) \to \infty$
As $x \to \infty$, $f(x) \to -\infty$
D. As $x \to -\infty$, $f(x) \to \infty$
As $x \to \infty$, $f(x) \to \infty$

Activity 4.3.11 Let's now look at the graph of $f(x) = (x-2)(x+1)(x-3)^2$ to answer the questions below.

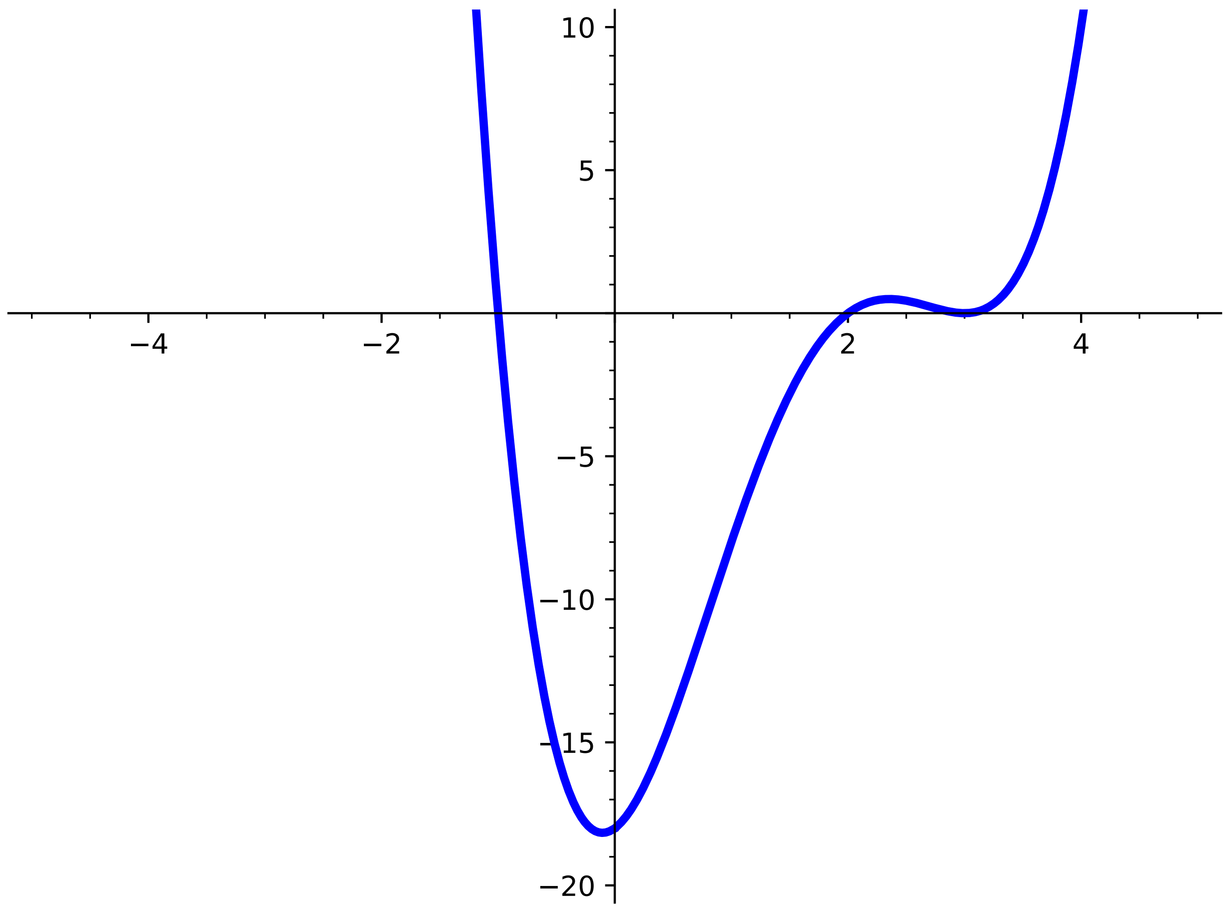


Figure 4.3.12

(a) What are the zeros of $f(x)$?

A. $1, 2, 3$

B. $-1, 2, 3$

C. $1, -2, -3$

D. $-1, 2, -3$

(b) Describe the behavior at each zero. What do you notice?

Definition 4.3.13 The **multiplicity** of a polynomial function is the number of times a given factor appears in the factored form of the equation of a polynomial.

The zero, $x = 3$ in Activity 4.3.11 has multiplicity 2 because the factor $(x-3)$ occurs twice (the exponent is a 2). ♢

Graphs of Polynomial Functions (PR3)

Activity 4.3.14 Use the function, $f(x) = (x-3)^2(x-1)$ to answer the following questions:

(a) What are the zeros of $f(x)$?

A. $-1, -3$

B. $-1, 3$

C. $1, -3$

D. $1, 3$

(b) Using Definition 4.3.13, determine the multiplicities of the zeros you found in part (a).

(c) Now graph $f(x)$ and look at the zeros on the graph. What do you notice about the behavior of the graph at each zero?

Remark 4.3.15 Refer back to Activity 4.3.11 and Activity 4.3.14. Notice that when the graph crosses the x-axis at the zero, the mulitplicity of that zero is odd. When the graph touches the x-axis at the zero, the multiplicity is even. In other words, factors of $f(x)$ with odd exponents will cross the x-axis and factors of $f(x)$ with even exponents will touch the x-axis.

Graphs of Polynomial Functions (PR3)

Definition 4.3.16 When graphing polynomial functions, you may notice that these functions have some ”hills” and ”valleys.” These characteristics of the graph are known as the **local maxima** and **local minima** of the graph - similar to what we've already seen with quadratic functions. Unlike quadratic functions, however, a polynomial graph can have many local maxima/minima (quadratic functions only have one). ♢

Activity 4.3.17 Now that we know all the different characterisitics of polynomials, we should also be able to identify them from a graph. Use the graph below to find the given charactertistics.

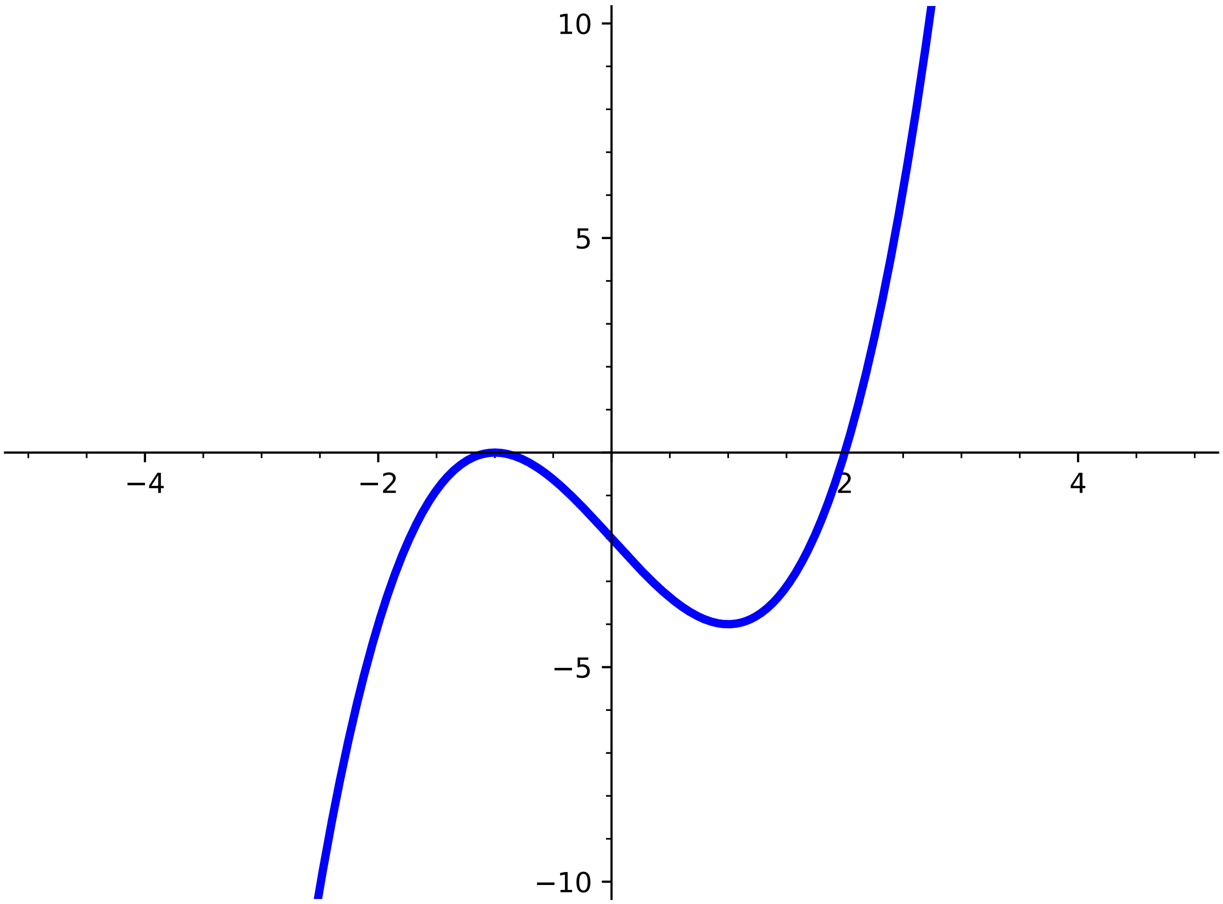


Figure 4.3.18

(a) What are the x-intercept(s) of the polynomial function? Select all that apply.

A. $(1, 0)$
B. $(-1, 0)$
C. $(2, 0)$
D. $(0, -2)$

(b) What are the y-intercept(s) of the polynomial function?

A. $(1, 0)$
B. $(-1, 0)$
C. $(2, 0)$
D. $(0, -2)$

(c) How many zeros does this polynomial function have?

A. 0
B. 1
C. 2
D. 3

(d) At what point is the local minimum located?

A. $(2, -4)$
B. $(-1, 0)$
C. $(-2, 0)$
D. $(1, -4)$
E. $(2, 0)$

(e) At what point is the local maximum located?

A. $(2, -4)$
B. $(-1, 0)$
C. $(-2, 0)$
D. $(1, -4)$
E. $(2, 0)$

(f) How do you describe the behavior of the polynomial function as $x \to \infty$?

A. the y-values go to negative infinity
B. $f(x) \to \infty$
C. the y-values go to positive infinity
D. $f(x) \to -\infty$

(g) How do you describe the behavior of the polynomial function as $x \to -\infty$?

A. the y-values go to negative infinity
B. $f(x) \to \infty$
C. the y-values go to positive infinity
D. $f(x) \to -\infty$

Graphs of Polynomial Functions (PR3)

Activity 4.3.19 Sketch the function, $f(x) = (x-2)(x+1)(x-3)^2$, by first finding the given characteristics.

(a) Find the zeros of $f(x)$.

(b) Find the multiplicities and describe the behavior at each zero.

(c) Find the y-intercept of $f(x)$.

(d) Describe the end behavior of $f(x)$.

(e) Estimate where any local maximums and minimums may occur.

Activity 4.3.20 Sketch the graph of a function $f(x)$ that meets all of the following criteria. Be sure to scale your axes and label any important features of your graph.

- The x-intercepts of $f(x)$ are $0, 2$, and 5.
- 0 and 2 have even multiplicity and 5 has odd multiplicity.
- The end behavior of $f(x)$ is given as: As $x \to \infty$, $f(x) \to \infty$ and As $x \to -\infty$, $f(x) \to -\infty$

One such graph is

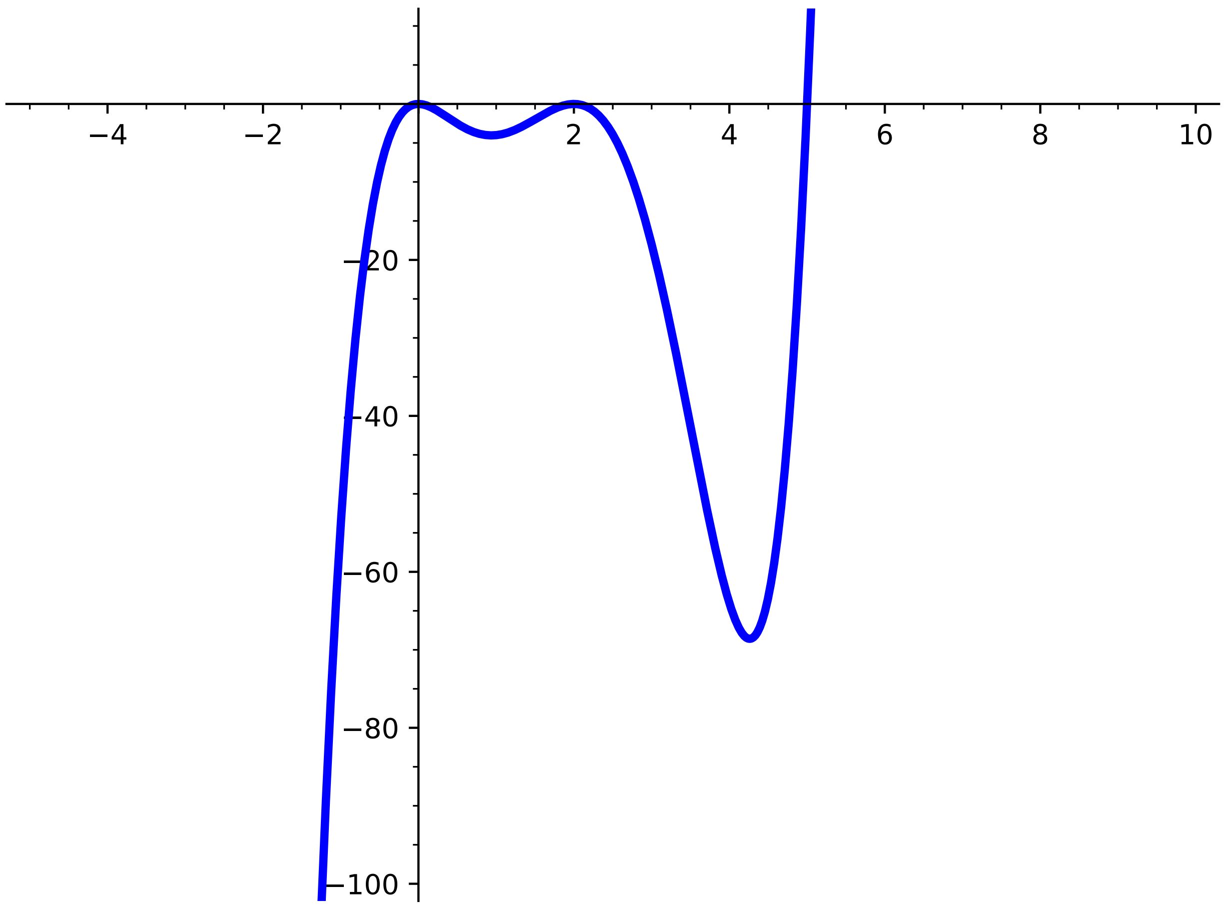


Figure 4.3.21

Graphs of Polynomial Functions (PR3)

Activity 4.3.22 Use the given function, $f(x) = (x+1)^2(x-5)$, to answer the following questions.

(a) What are the zeros of $f(x)$?

A. $-1, -5$

B. $-1, 5$

C. $1, -5$

D. $1, 5$

(b) What are the multiplicities at each zero?

A. At $x = -1$, the mulitplicity is even.
At $x = 5$, the mulitplicity is even.

B. At $x = -1$, the mulitplicity is even.
At $x = 5$, the mulitplicity is odd.

C. At $x = -1$, the mulitplicity is odd.
At $x = 5$, the mulitplicity is even.

D. At $x = -1$, the mulitplicity is odd.
At $x = 5$, the mulitplicity is odd.

(c) What is the end behavior of $f(x)$?

A. $f(x)$ rises on the left and on the right.

B. $f(x)$ rises on the left, but falls on the right.

C. $f(x)$ rises on the right, but falls on the left.

D. $f(x)$ falls on the left and on the right.

(d) Using what you know about the zeros, multiplicities, and end behavior, where on the graph can we estimate the local maxima and minima to be?

(e) Now look at the graph of $f(x)$. At which zero does a local maximum or local minimum occur? Explain how you know.

Remark 4.3.23 We can estimate where these local maxima and minima occur by looking at other characteristics, such as multiplicities and end behavior.

From Activity 4.3.22, we saw that when the function touches the x-axis at a zero, then that zero could be either a local maximum or a local minimum of the graph. When the function crosses the x-axis, however, the local maximum or local minimum occurs between the zeros.

4.4 Polynomial Long Division (PR4)

Objectives

- Rewrite a rational function as a polynomial plus a proper rational function.

Polynomial Long Division (PR4)

Observation 4.4.1 We have seen previously that we can reduce rational functions by factoring, for example

$$\frac{x^2+5x+4}{x^3+3x^2+2x} = \frac{(x+1)(x+4)}{x(x+2)(x+1)} = \frac{x+4}{x(x+2)}.$$

In this section, we will explore the question: what can we do to simplify rational functions if we are not able to reduce by easily factoring?

Polynomial Long Division (PR4)

Definition 4.4.2 Recall that a fraction is called **proper** if its numerator is smaller than its denominator, and **improper** if the numerator is larger than the denominator (so $\frac{3}{5}$ is a proper fraction, but $\frac{32}{7}$ is an improper fraction). Similarly, we define a **proper rational function** to be a rational function where the degree of the numerator is less than the degree of the denominator. $\diamondsuit$

Polynomial Long Division (PR4)

Activity 4.4.3 Label each of the following rational functions as either **proper** or **improper**.

A. $\dfrac{x^3 + x}{x^2 + 4}$

B. $\dfrac{3}{x^2 + 3x + 4}$

C. $\dfrac{7 + x^3}{x^2 + x + 1}$

D. $\dfrac{x^4 + x + 1}{x^4 + 4x^2}$

Polynomial Long Division (PR4)

Observation 4.4.4 When dealing with an improper fraction such as $\dfrac{32}{7}$, it is sometimes useful to rewrite this as an integer plus a proper fraction, e.g. $\dfrac{32}{7} = 4 + \dfrac{4}{7}$. Similarly, it will sometimes be useful to rewrite an improper rational function as the sum of a polynomial and a proper rational function, such as $\dfrac{x^3 + x}{x^2 + 4} = x - \dfrac{3x}{x^2 + 4}$.

Polynomial Long Division (PR4)

Activity 4.4.5 Consider the improper fraction $\dfrac{357}{11}$.

(a) Use long division to write $\dfrac{357}{11}$ as an integer plus a proper fraction.

(b) Now we will carefully redo this process in a way that we can generalize to rational functions. Note that we can rewrite 357 as $357 = 3 \cdot 10^2 + 5 \cdot 10 + 7$, and 11 as $11 = 1 \cdot 10 + 1$. By comparing the leading terms in these expansions, we see that to knock off the leading term of 357, we need to multiply 11 by $3 \cdot 10^1$.

Using the fact that $357 = 11 \cdot 30 + 27$, rewrite $\dfrac{357}{11}$ as $\dfrac{357}{11} = 30 + \dfrac{?}{11}$.

(c) Note now that if we can rewrite $\dfrac{27}{11}$ as an integer plus a proper fraction, we will be done, since $\dfrac{357}{11} = 30 + \dfrac{27}{11}$.

Rewrite $\dfrac{27}{11} = ? + \dfrac{?}{11}$ as an integer plus a proper fraction.

(d) Combine your work in the previous two parts to rewrite $\dfrac{357}{11}$ as an integer plus a proper fraction. How does this compare to what you obtained in part (a)?

Polynomial Long Division (PR4)

Activity 4.4.6 Now let's consider the rational function $\dfrac{3x^2+5x+7}{x+1}$. We want to rewrite this as a polynomial plus a proper rational function.

(a) Looking at the leading terms, what do we need to multiply $x+1$ by so that it would have the same leading term as $3x^2+5x+7$?

A. 3

B. x

C. $3x$

D. $3x+5$

(b) Rewrite $3x^2+5x+7 = 3x(x+1)+?$, and use this to rewrite $\dfrac{3x^2+5x+7}{x+1} = 3x + \dfrac{?}{x+1}$.

(c) Now focusing on $\dfrac{2x+7}{x+1}$, what do we need to multiply $x+1$ by so that it would have the same leading term as $2x+7$?

A. 2

B. x

C. $2x$

D. $2x+7$

(d) Rewrite $\dfrac{2x+7}{x+1} = 2 + \dfrac{?}{x+1}$.

(e) Combine this with the previous parts to rewrite $\dfrac{3x^2+5x+7}{x+1} = 3x + ? + \dfrac{?}{x+1}$.

Polynomial Long Division (PR4)

Activity 4.4.7 Next we will use the notation of long division to rewrite the rational function $\dfrac{3x^2+5x+7}{x+1}$ as a polynomial plus a proper rational function.

(a) First, let's use long division notation to write the quotient.

$$\begin{array}{r} ? \\ (x+1)\;\overline{)\,3x^2+5x+7} \end{array}$$

What do we need to multiply $x+1$ by so that it would have the same leading term as $3x^2+5x+7$?

(b) Now to rewrite $3x^2+5x+7$ as $3x(x+1)+?$, place the product $3x(x+1)$ below and subtract.

$$\begin{array}{r} 3x \\ (x+1)\;\overline{)\,3x^2+5x+7} \\ 3x^2+3x \\ \hline ?\;+? \end{array}$$

(c) Now focusing on $2x+7$, what do we need to multiply $x+1$ by so that it would have the same leading term as $2x+7$?

$$\begin{array}{r} 3x+? \\ (x+1)\;\overline{)\,3x^2+5x+7} \\ 3x^2+3x \\ \hline 2x+7 \end{array}$$

(d) Now, subtract $2(x+1)$ to finish the long division.

Polynomial Long Division (PR4)

$$
\begin{array}{r}
3x+2 \\
(x+1)\ \overline{)\,3x^2+5x+7} \\
\underline{3x^2+3x} \\
2x+7 \\
\underline{?\ \ +?} \\
?
\end{array}
$$

(e) This long division calculation has shown that

$$3x^2 + 5x + 7 = (x + 1)(3x + 2) + 5.$$

Use this to rewrite $\dfrac{3x^2 + 5x + 7}{x + 1}$ as a polynomial plus a proper rational function.

Remark 4.4.8 Note that in Activity 4.4.6 and Activity 4.4.7 we performed the same computations, but just organized our work a little differently.

Polynomial Long Division (PR4)

Activity 4.4.9 Rewrite $\frac{x^2+1}{x-1}$ as a polynomial plus a proper rational function.

Hint. Note that $x^2 + 1 = x^2 + 0x + 1$.

Polynomial Long Division (PR4)

Activity 4.4.10 Rewrite $\dfrac{x^5 + x^3 + 2x^2 - 6x + 7}{x^2 + x - 1}$ as a polynomial plus a proper rational function.

Polynomial Long Division (PR4)

Activity 4.4.11 Rewrite $\dfrac{3x^4 - 5x^2 + 2}{x - 1}$ as a polynomial plus a proper rational function.

4.5 Zeros of Polynomial Functions (PR5)

Objectives

- Determine the zeros of a polynomial function with real coefficients.

Zeros of Polynomial Functions (PR5)

Activity 4.5.1 Consider the function $f(x) = x^3 - 7x^2 + 7x + 15$.

(a) Use polynomial division from Section 4.4 to divide $f(x)$ by $x - 2$. What is the remainder?

(b) Find $f(2)$. What do you notice?

Zeros of Polynomial Functions (PR5)

Activity 4.5.2 Again consider the function $f(x) = x^3 - 7x^2 + 7x + 15$.

(a) Divide $f(x)$ by $x - 3$. What is the remainder?

(b) Find $f(3)$. What do you notice?

Zeros of Polynomial Functions (PR5)

Remark 4.5.3 If we know one zero, then we can divide by $x - a$ where a is a zero. After this, the quotient will have smaller degree and we can work on factoring the rest. We can "chip away" at the polynomial one zero at a time.

Activity 4.5.4 One more time consider the function $f(x) = x^3 - 7x^2 + 7x + 15$.

(a) We already know from Activity 4.5.4 that $x - 3$ is a factor of the polynomial $f(x)$. Use division to express $f(x)$ as $(x - 3) \cdot q(x)$, where $q(x)$ is a quadratic function.

A. $q(x) = x^2 - 2x - 3$

B. $q(x) = x^2 - 10x - 37$

C. $q(x) = x^2 - 4x - 5$

D. $q(x) = x^2 + 4x - 5$

(b) Notice that $q(x)$ is something we can factor. Factor this quadratic and find the remaining zeros.

A. -5

B. 5

C. 4

D. -1

E. 3

F. 1

Remark 4.5.5 We were able to find all the zeros of the polynomial in Activity 4.5.7 because we were given one of the zeros. If we don't have a zero to help us get started (or need more than one zero for a function of higher degree), we have a couple of options.

Zeros of Polynomial Functions (PR5)

Activity 4.5.6 Consider the function $f(x) = 18x^4 + 67x^3 - 81x^2 - 202x + 168$.

(a) Graph the function. According to the graph, what value(s) seem to be zeros?

(b) Use the Remainder Theorem to confirm that your guesses are actually zeros.

(c) Now use these zeros along with polynomial division to rewrite the function as $f(x) = (x-a)(x-b)q(x)$ where a and b are zeros and $q(x)$ is the remaining quadratic function.

(d) Solve the quadratic $q(x)$ to find the remaining zeros.

(e) List all zeros of $f(x)$.

(f) Rewrite $f(x)$ as a product of linear factors.

Zeros of Polynomial Functions (PR5)

Remark 4.5.7 Using the graph to find an initial zero can be helpful, but they may not always be easy to identify.

Zeros of Polynomial Functions (PR5)

Activity 4.5.8 Consider the quadratic function $f(x) = (2x - 5)(3x - 8) = 6x^2 - 31x + 40$.

(a) What are the roots of this quadratic?

(b) What do you notice about these roots in relation to the factors of $a = 6$ and $c = 40$ in $f(x) = 6x^2 - 31x + 40$?

Remark 4.5.9 In Activity 4.5.11 we found that the roots were both factors of the constant term divided by factors of the leading coefficient. This can be extended to polynomials of larger degree.

Zeros of Polynomial Functions (PR5)

Activity 4.5.10 Consider the polynomial $f(x) = 5x^3 - 2x^2 + 20x - 8$.

(a) List the factors of the constant term.

(b) List the factors of the leading coefficient.

(c) Use parts (a) and (b) to list all the possible rational roots.

(d) Use the Remainder Theorem to determine at least one root of $f(x)$.

Zeros of Polynomial Functions (PR5)

Activity 4.5.11 Consider the polynomial $f(x) = 6x^4 + 5x^3 - 6x - 5$

(a) Use the graph and the Rational Root Theorem (Theorem 4.5.13) to find the rational zeros of $f(x)$.

(b) Use the roots, along with the Factor Theorem, to simplify the polynomial into linear and quadratic factors.

(c) Find the zeros of the quadratic factor.

(d) List the roots of the polynomial.

Remark 4.5.12 Notice that the zeros of the quadratic factor were imaginary and are related. This also occured in Activity 1.5.21.

Zeros of Polynomial Functions (PR5)

Activity 4.5.13 Consider the function $f(x) = x^5 + 3x^4 + 4x^3 + 8x^2 - 16$.

(a) Use a graphing utility to graph $f(x)$.

(b) Find *all* the zeros of $f(x)$ and their corresponding multiplicities.

Zeros of Polynomial Functions (PR5)

Activity 4.5.14 Consider the following information about a polynomial $f(x)$:

- $x = 2$ is a zero with multiplicity 1
- $x = -1$ is a zero with multiplicity 2
- $x = i$ is a zero with multiplicity 1

(a) What is the smallest possible degree of such a polynomial $f(x)$ with real coefficients?

A. 2
B. 3
C. 4
D. 5
E. 6

(b) Write an expression for such a polynomial $f(x)$ with real coefficients of smallest possible degree.

4.6 Properties and Graphs of Rational Functions (PR6)

Objectives

- Find the domain and range, vertical and horizontal asymptotes, and intercepts of a rational function and use this information to sketch the graph.

Properties and Graphs of Rational Functions (PR6)

Definition 4.6.1 A function r is **rational** provided that it is possible to write r as the ratio of two polynomials, p and q. That is, r is rational provided that for some polynomial functions p and q, we have

$$r(x) = \frac{p(x)}{q(x)}.$$

$\Diamond$

Properties and Graphs of Rational Functions (PR6)

Observation 4.6.2 Rational functions occur in many applications, so our goal in this lesson is to learn about their properties and be able to graph them. In particular we want to investigate the domain, end behavior, and zeros of rational functions.

Activity 4.6.3 Consider the rational function

$$r(x) = \frac{x^2 - 3x + 2}{x^2 - 4x + 3}.$$

(a) Find $r(1)$, $r(2)$, $r(3)$, and $r(4)$.

(b) Label each of these four values as giving us information about the DOMAIN of $r(x)$, information about the ZEROES of $r(x)$, or NEITHER.

Properties and Graphs of Rational Functions (PR6)

Observation 4.6.4 Let p and q be polynomial functions so that $r(x) = \dfrac{p(x)}{q(x)}$ is a rational function. The domain of r is the set of all real numbers except those for which $q(x) = 0$.

Recall that the **domain** of a function was defined in Definition 2.1.1.

Activity 4.6.5 Let's investigate the domain of $r(x)$ more closely. We will be using the same function from the previous activity:

$$r(x) = \frac{x^2 - 3x + 2}{x^2 - 4x + 3}.$$

(a) Rewrite $r(x)$ by factoring the numerator and denominator, but do not try to simplify any further. What do you notice about the relationship between the values that are not in the domain and how the function is now written?

(b) The function was not defined for $x = 3$. Make a table for values of $r(x)$ near $x = 3$.

Table 4.6.6

x	$r(x)$
2	
2.9	
2.99	
2.999	
3	undefined
3.001	
3.01	
3.1	

(c) Which of the following describe the behavior of the graph near $x = 3$?

A. As $x \to 3$, $r(x)$ approaches a finite number

B. As $x \to 3$ from the left, $r(x) \to \infty$

C. As $x \to 3$ from the left, $r(x) \to -\infty$

D. As $x \to 3$ from the right, $r(x) \to \infty$

E. As $x \to 3$ from the right, $r(x) \to -\infty$

(d) The function was also not defined for $x = 1$. Make a table for values of $r(x)$ near $x = 1$.

Table 4.6.7

x	$r(x)$
0	
0.9	
0.99	
0.999	
1	undefined
1.001	
1.01	
1.1	

(e) Which of the following describe the behavior of the graph near $x = 1$?

A. As $x \to 1$, $r(x)$ approaches a finite number

B. As $x \to 1$ from the left, $r(x) \to \infty$

C. As $x \to 1$ from the left, $r(x) \to -\infty$

D. As $x \to 1$ from the right, $r(x) \to \infty$

E. As $x \to 1$ from the right, $r(x) \to -\infty$

(f) The function is behaving differently near $x = 1$ than it is near $x = 3$. Can you see anything in the factored form of $r(x)$ that may help you account for the difference?

Remark 4.6.8 Features of a rational function. Let $r(x) = \dfrac{p(x)}{q(x)}$ be a rational function.

- If $p(a) = 0$ and $q(a) \neq 0$, then $r(a) = 0$, so r has a **zero** at $x = a$.
- If $q(a) = 0$ and $p(a) \neq 0$, then $r(a)$ is undefined and r has a **vertical asymptote** at $x = a$.
- If $p(a) = 0$ and $q(a) = 0$ and we can show that there is a finite number L such that $r(x) \to L$, then $r(a)$ is not defined and r has a **hole** at the point (a, L).

Properties and Graphs of Rational Functions (PR6)

Observation 4.6.9 Another property of rational functions we want to explore is the end behavior. This means we want to explore what happens to a given rational function $r(x)$ when x goes toward positive infinity or negative infinity.

Activity 4.6.10 Consider the rational function $r(x) = \dfrac{1}{x^3}$.

(a) Plug in some very large positive numbers for x to see what $r(x)$ is tending toward. Which of the following best describes the behavior of the graph as x approaches positive infinity?

A. As $x \to \infty$, $r(x) \to \infty$.

B. As $x \to \infty$, $r(x) \to -\infty$.

C. As $x \to \infty$, $r(x) \to 0$.

D. As $x \to \infty$, $r(x) \to 1$.

(b) Now let's look at $r(x)$ as x tends toward negative infinity. Plug in some very large negative numbers for x to see what $r(x)$ is tending toward. Which of the following best describes the behavior of the graph as x approaches negative infinity?

A. As $x \to -\infty$, $r(x) \to \infty$.

B. As $x \to -\infty$, $r(x) \to -\infty$.

C. As $x \to -\infty$, $r(x) \to 0$.

D. As $x \to -\infty$, $r(x) \to 1$.

Observation 4.6.11 We can generalize what we have just found to any function of the form $\frac{1}{x^n}$, where $n > 0$. Since x^n increases without bound as $x \to \infty$, we find that $\frac{1}{x^n}$ will tend to 0. In fact, the numerator can be any constant and the function will still tend to 0!

Similarly, as $x \to -\infty$, we find that $\frac{1}{x^n}$ will tend to 0 too.

Activity 4.6.12 Consider the rational function $r(x) = \dfrac{3x^2 - 5x + 1}{7x^2 + 2x - 11}$.

Observe that the largest power of x that's present in $r(x)$ is x^2. In addition, because of the dominant terms of $3x^2$ in the numerator and $7x^2$ in the denominator, both the numerator and denominator of r increase without bound as x increases without bound.

(a) In order to understand the end behavior of r, we will start by writing the function in a different algebraic form.

Multiply the numerator and denominator of r by $\dfrac{1}{x^2}$. Then distribute and simplify as much as possible in both the numerator and denominator to write r in a different algebraic form. Which of the following is that new form?

A. $\dfrac{3x^4 - 5x^3 + x^2}{7x^4 + 2x^3 - 11x^2}$

B. $\dfrac{3 - \dfrac{5}{x} + \dfrac{1}{x^2}}{7 + \dfrac{2}{x} - \dfrac{11}{x^2}}$

C. $\dfrac{\dfrac{3x^2}{x^2} - \dfrac{5x}{x^2} + \dfrac{1}{x^2}}{\dfrac{7x^2}{x^2} + \dfrac{2x}{x^2} - \dfrac{11}{x^2}}$

D. $\dfrac{3x^2 - 5x + 1}{7x^4 + 2x^3 - 11x^2}$

(b) Now determine the end behavior of each piece of the numerator and each piece of the denominator.

Hint. Use Observation 4.6.11 to help!

(c) Simplify your work from the previous step. Which of the following best describes the end behavior of $r(x)$?

A. As $x \to \pm\infty$, $r(x)$ goes to 0.

B. As $x \to \pm\infty$, $r(x)$ goes to $\dfrac{3}{7}$.

C. As $x \to \pm\infty$, $r(x)$ goes to ∞.

D. As $x \to \pm\infty$, $r(x)$ goes to $-\infty$.

Properties and Graphs of Rational Functions (PR6)

Observation 4.6.13 If the end behavior of a function tends toward a specific value a, then we say that the function has a **horizontal asymptote** at $y = a$.

Activity 4.6.14 Find the horizontal asymptote (if one exists) of the following rational functions. Follow the same method we used in Activity 4.6.12.

(a) $f(x) = \dfrac{4x^3 - 3x^2 + 6}{9x^3 + 7x - 5}$

A. $y = 0$

B. $y = \dfrac{4}{9}$

C. $y = -\dfrac{3}{7}$

D. $y = -\dfrac{6}{5}$

E. There is no horizontal asymptote.

(b) $g(x) = \dfrac{4x^3 - 3x^2 + 6}{9x^5 + 7x - 5}$

A. $y = 0$

B. $y = \dfrac{4}{9}$

C. $y = -\dfrac{3}{7}$

D. $y = -\dfrac{6}{5}$

E. There is no horizontal asymptote.

(c) $h(x) = \dfrac{4x^5 - 3x^2 + 6}{9x^3 + 7x - 5}$

A. $y = 0$

B. $y = \dfrac{4}{9}$

C. $y = -\dfrac{3}{7}$

D. $y = -\dfrac{6}{5}$

E. There is no horizontal asymptote.

Activity 4.6.15 Some patterns have emerged from the previous problem. Fill in the rest of the sentences below to describe how to find horizontal asymptotes of rational functions.

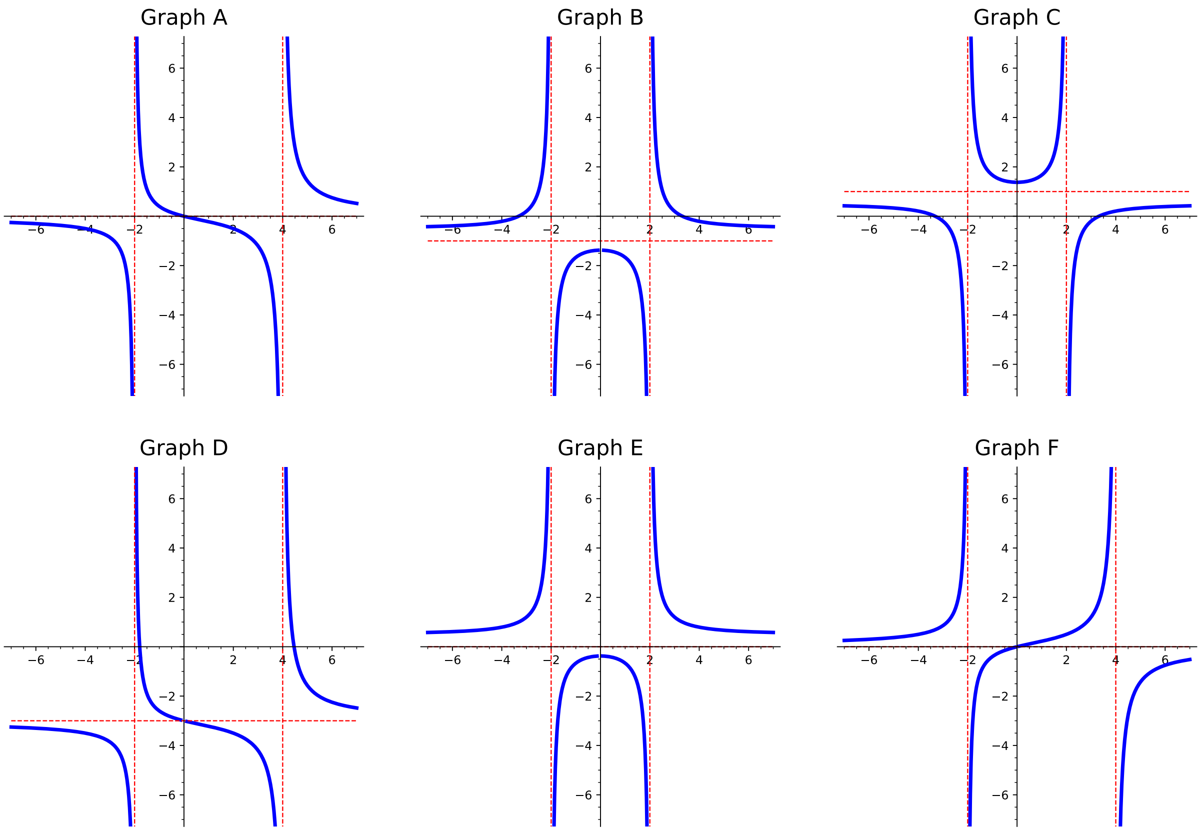
(a) If the degree of the numerator is the same as the degree of the denominator, then...

(b) If the degree of the numerator is less than the degree of the denominator, then...

(c) If the degree of the numerator is greater than the degree of the denominator, then...

Properties and Graphs of Rational Functions (PR6)

Activity 4.6.16 Consider the following six graphs of rational functions:

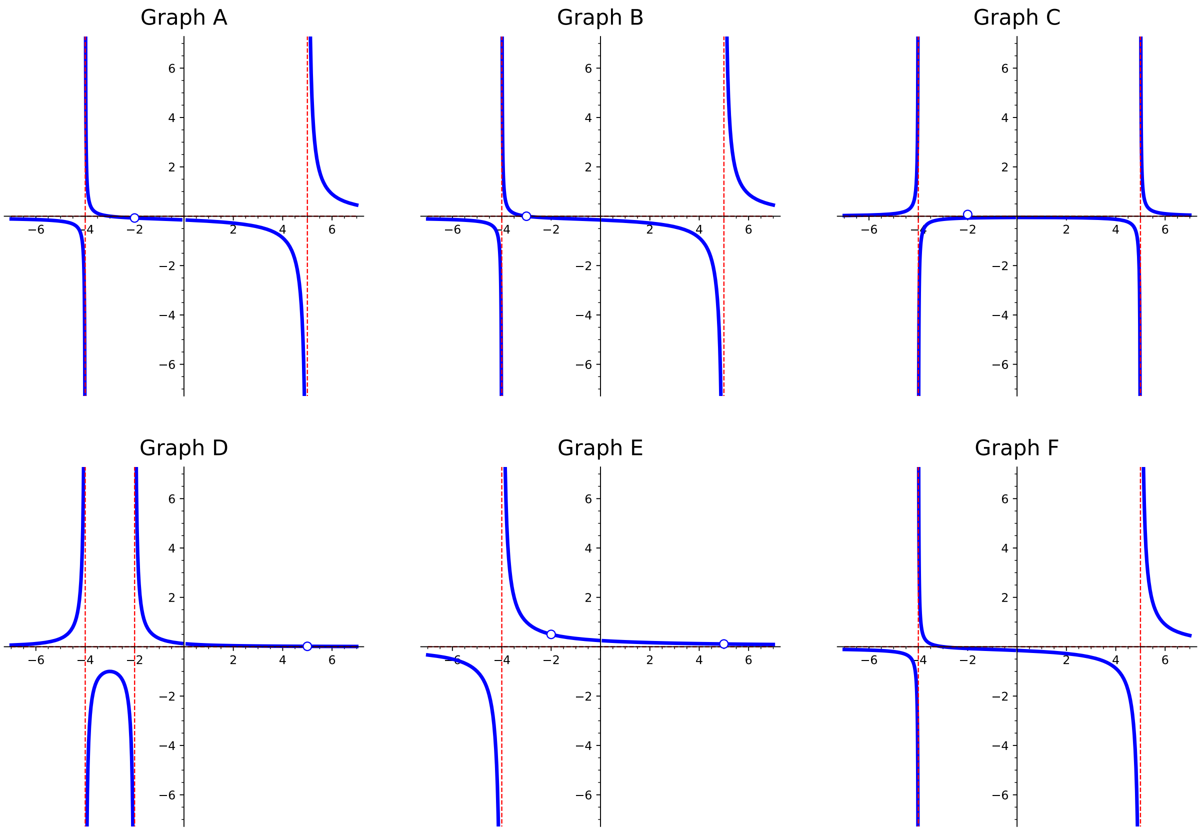


(a) Which of the graphs above represents the function $f(x) = \dfrac{2x}{x^2 - 2x - 8}$?

(b) Which of the graphs above represents the function $g(x) = \dfrac{x^2 + 3}{2x^2 - 8}$?

Properties and Graphs of Rational Functions (PR6)

Activity 4.6.17 Consider the following six graphs of rational functions:



(a) Which of the graphs above represents the function $f(x) = \dfrac{x^2+5x+6}{(x^2-3x-10)(x+4)}$?

(b) Which of the graphs above represents the function $g(x) = \dfrac{x^2-3x-10}{(x+2)(x^2-x-20)}$?

Properties and Graphs of Rational Functions (PR6)

Activity 4.6.18 Let $f(x) = \dfrac{-(x-1)(x-4)}{2\,(x+3)^2(x-1)}$.

(a) Find the roots of $f(x)$.

(b) Find the y-intercept of the graph of $f(x)$.

(c) Find any horizontal asymptotes on the graph of $f(x)$.

(d) Find any vertical asymptotes on the graph of $f(x)$.

(e) Find any holes on the graph of $f(x)$.

(f) Sketch the graph of $f(x)$.

Properties and Graphs of Rational Functions (PR6)

Activity 4.6.19 For each of the following rational functions, identify the location of any potential hole in the graph. Then, create a table of function values for input values near where the hole should be located. Use your work to decide whether or not the graph indeed has a hole, with written justification.

(a) $r(x) = \dfrac{x^2 - 16}{x + 4}$

(b) $s(x) = \dfrac{(x-2)^2(x+3)}{x^2 - 5x + 6}$

(c) $u(x) = \dfrac{(x-2)^3(x+3)}{x^2 - 5x + 6}$

(d) $w(x) = \dfrac{(x-2)(x+3)}{(x^2 - 5x + 6)^2}$

Activity 4.6.20 Suppose you are given a function $r(x) = \frac{p(x)}{q(x)}$, and you know that $p(3) = 0$ and $q(3) = 0$. What can you conlude about the function $r(x)$ at $x = 3$?

A. $r(x)$ has a hole at $x = 3$.

B. $r(x)$ has an asymptote at $x = 3$.

C. $r(x)$ has either a hole or an asymptote at $x = 3$.

D. $r(x)$ has neither a hole nor an asymptote at $x = 3$.

Chapter 5

Exponential and Logarithmic Functions (EL)

Objectives

How do we model exponential growth?
By the end of this chapter, you should be able to...

1. Determine if a given function is exponential. Find an equation of an exponential function. Evaluate exponential functions (including base e).
2. Graph exponential functions and determine the domain, range, and asymptotes.
3. Convert between exponential and logarithmic form. Evaluate a logarithmic function, including common and natural logarithms.
4. Graph logarithmic functions and determine the domain, range, and asymptotes.
5. Use properties of logarithms to condense or expand logarithmic expressions.
6. Solve exponential and logarithmic equations.
7. Solve application problems using exponential and logarithmic equations.

5.1 Introduction to Exponentials (EL1)

Objectives

- Determine if a given function is exponential. Find an equation of an exponential function. Evaluate exponential functions (including base e).

Introduction to Exponentials (EL1)

Remark 5.1.1 Linear functions have a constant rate of change - that is a constant change in output for every change in input. Let's consider functions which do not fit this model - those which grow more rapidly and change by a varying amount for every change in input.

Introduction to Exponentials (EL1)

Activity 5.1.2 You have two job offers on the horizon. One has offered to pay you \$10,000 per month while the other is offering \$0.01 the first month, \$0.02 the second month, \$0.04 the third month and doubles every month. Which job would you rather take?

(a) Make a table representing how much money you will be paid each month for the first two years from the first job - paying \$10,000 per month.

(b) Make a table representing how much money you will be paid each month for the first two years from the second job - paying \$0.01 the first month and doubling every month after.

(c) Which job is earning more money per month after one year?

(d) Which job is earning more money per month after 18 months?

(e) According to your tables, does the second job ever earn more money per month than the first job?

Remark 5.1.3 This idea of a function that grows very rapidly by a factor, ratio, or percent each time, like the second job in Activity 5.1.2, is considered exponential growth.

Introduction to Exponentials (EL1)

Definition 5.1.4 Let a be a non-zero real number and $b \neq 1$ a positive real number. An **exponential function** takes the form

$$f(x) = ab^x$$

a is the initial value and b is the base. $\diamond$

Introduction to Exponentials (EL1)

Activity 5.1.5 Evaluate the following exponential functions.

(a) $f(x) = 4^x$ for $f(3)$

(b) $f(x) = \left(\frac{1}{3}\right)^x$ for $f(3)$

(c) $f(x) = 3\,(5)^x$ for $f(-2)$

(d) $f(x) = -2^{3x-4}$ for $f(4)$

Remark 5.1.6 Notice that in Activity 5.1.5 part (a) the ouput value is larger than the base, while in part (b) the output value is smaller than the base. This is similar to the difference between a positive and negative slope for linear functions.

Introduction to Exponentials (EL1)

Activity 5.1.7 Consider two exponential functions $f(x) = 100(2)^x$ and $g(x) = 100\left(\frac{1}{2}\right)^x$.

(a) Fill in the table of values for $f(x)$.

x	$f(x)$
0	
1	
2	
3	
4	

(b) Fill in the table of values for $g(x)$.

x	$g(x)$
0	
1	
2	
3	
4	

(c) How do the values in the tables compare?

Remark 5.1.8 In Activity 5.1.7, the only difference between the two exponential functions was the base. $f(x)$ has a base of 2, while $g(x)$ has a base of $\frac{1}{2}$. Let's use this fact to update Definition 5.1.4.

Remark 5.1.9 An exponential function of the form $f(x) = ab^x$ will grow (or increase) if $b > 1$ and decay (or decrease) if $0 < b < 1$.

Introduction to Exponentials (EL1)

Activity 5.1.10 For each year t, the population of a certain type of tree in a forest is represented by the function $F(t) = 856(0.93)^t$.

(a) How many of that certain type of tree are in the forest initially?

(b) Is the number of trees of that type growing or decaying?

Activity 5.1.11 To begin creating equations for exponential functions using a and b, let's compare a linear function and an exponential function. The tables show outputs for two different functions r and s that correspond to equally spaced input.

x	$r(x)$
0	12
3	10
6	8
9	6

x	$s(x)$
0	12
3	9
6	6.75
9	5.0625

(a) Which function is linear?

(b) What is the initial value of the linear function?

(c) What is the slope of the linear function?

(d) What is the initial value of the exponential function?

(e) What is the ratio of consecutive outputs in the exponential function?

A. $\frac{4}{3}$

B. $\frac{3}{4}$

C. $-\frac{4}{3}$

D. $-\frac{3}{4}$

Remark 5.1.12 In a linear function the differences are constant, while in an exponential function the ratios are constant.

Activity 5.1.13 Find an equation for an exponential function passing through the points $(0, 4)$ and $(1, 6)$.

(a) Find the initial value.

A. 0

B. 4

C. 1

D. 6

(b) Find the common ratio.

A. $\frac{3}{2}$

B. 6

C. $\frac{2}{3}$

D. $\frac{1}{6}$

(c) Find the equation.

A. $f(x) = 6\left(\frac{3}{2}\right)^x$

B. $f(x) = 4\,(6)^x$

C. $f(x) = 4\left(\frac{2}{3}\right)^x$

D. $f(x) = 4\left(\frac{3}{2}\right)^x$

Remark 5.1.14 Recall the negative rule of exponents which states that for any nonzero real number a and natural number n

$$a^{-n} = \frac{1}{a^n}$$

Activity 5.1.15 Let's consider the two exponential functions $f(x) = 2^{-x}$ and $g(x) = \left(\frac{1}{2}\right)^x$.

(a) Fill in the table of values for $f(x)$.

x	$f(x)$
-2	
-1	
0	
1	
2	

(b) Fill in the table of values for $g(x)$.

x	$g(x)$
-2	
-1	
0	
1	
2	

(c) What do you notice about the two functions?

(d) Use Remark 5.1.14 and other properties of exponents to try and rewrite $f(x)$ as $g(x)$.

Remark 5.1.16 Similar to how π arises naturally in geometry, there is an irrational number called e that arises naturally when working with exponentials. We usually use the approximation $e \approx 2.718282$. e is also found on most scientific and graphing calculators.

See Activity 5.7.11 for an exploration of how e arises naturally.

Introduction to Exponentials (EL1)

Activity 5.1.17 Use a calculator to evaluate the following exponentials involving the base e.

(a) $f(x) = -2e^x - 2$ for $f(-2)$

A. -0.0366

B. -2.2707

C. -1.7293

D. -16.778

(b) $f(x) = \frac{1}{3}e^{x+1}$ for $f(-1)$

A. 1

B. 0

C. 1.122

D. $\frac{1}{3}$

5.2 Graphs of Exponential Functions (EL2)

Objectives

- Graph exponential functions and determine the domain, range, and asymptotes.

Graphs of Exponential Functions (EL2)

Activity 5.2.1 Consider the function $f(x) = 2^x$.

(a) Fill in the table of values for $f(x)$. Then plot the points on a graph.

x	$f(x)$
-2	
-1	
0	
1	
2	

(b) What seems to be happening with the graph as x goes toward infinity? Plug in large positive values of x to test your guess, then describe the end behavior.

A. As $x \to \infty$, $f(x) \to -\infty$.

B. As $x \to \infty$, $f(x) \to -2$.

C. As $x \to \infty$, $f(x) \to 0$.

D. As $x \to \infty$, $f(x) \to 2$.

E. As $x \to \infty$, $f(x) \to \infty$.

(c) What seems to be happening with the graph as x goes toward negative infinity? Plug in large negative values of x to test your guess, then describe the end behavior.

A. As $x \to -\infty$, $f(x) \to -\infty$.

B. As $x \to -\infty$, $f(x) \to -2$.

C. As $x \to -\infty$, $f(x) \to 0$.

D. As $x \to -\infty$, $f(x) \to 2$.

E. As $x \to -\infty$, $f(x) \to \infty$.

(d) Complete the graph you started in Task 5.2.1.a, connecting the points and including the end behavior you've just determined.

(e) Does your graph seem to have any asymptotes?

A. No. There are no asymptotes.

B. There is a vertical asymptote but no horizontal one.

C. There is a horizontal asymptote but no vertical one.

D. The graph has both a horizontal and vertical asymptote.

(f) What the equation for each asymptote of $f(x)$? Select all that apply.

A. There are no asymptotes.

B. $x = 0$

C. $x = 3$

D. $y = 0$

E. $y = 3$

(g) Find the domain and range of $f(x)$. Write your answers using interval notation.

(h) Find the interval(s) where $f(x)$ is increasing and the interval(s) where $f(x)$ is decreasing. Write your answers using interval notation.

Graphs of Exponential Functions (EL2)

Remark 5.2.2 The graph of an exponential function $f(x) = b^x$ where $b > 1$ has the following characteristics:

- Its domain is $(-\infty, \infty)$ and its range is $(0, \infty)$.
- It is an exponential growth function; that is it is increasing on $(-\infty, \infty)$.
- There is a horizontal asymptote at $y = 0$. There is no vertical asymptote.
- There is a y-intercept at $(0, 1)$. There is no x-intercept.

Graphs of Exponential Functions (EL2)

Activity 5.2.3 Consider the function $g(x) = \left(\frac{1}{2}\right)^x$.

(a) Fill in the table of values for $g(x)$. Then plot the points on a graph.

x	$g(x)$
-2	
-1	
0	
1	
2	

(b) What seems to be happening with the graph as x goes toward infinity? Plug in large positive values of x to test your guess, then describe the end behavior.

A. As $x \to \infty$, $g(x) \to -\infty$.

B. As $x \to \infty$, $g(x) \to -2$.

C. As $x \to \infty$, $g(x) \to 0$.

D. As $x \to \infty$, $g(x) \to 2$.

E. As $x \to \infty$, $g(x) \to \infty$.

(c) What seems to be happening with the graph as x goes toward negative infinity? Plug in large negative values of x to test your guess, then describe the end behavior.

A. As $x \to -\infty$, $g(x) \to -\infty$.

B. As $x \to -\infty$, $g(x) \to -2$.

C. As $x \to -\infty$, $g(x) \to 0$.

D. As $x \to -\infty$, $g(x) \to 2$.

E. As $x \to -\infty$, $g(x) \to \infty$.

(d) Complete the graph you started in Task 5.2.3.a, connecting the points and including the end behavior you've just determined.

(e) What are the equations of the asymptote(s) of the graph?

(f) Find the domain and range of $f(x)$. Write your answers using interval notation.

(g) Find the interval(s) where $f(x)$ is increasing and the interval(s) where $f(x)$ is decreasing. Write your answers using interval notation.

Graphs of Exponential Functions (EL2)

Activity 5.2.4 Consider the two exponential functions we've just graphed: $f(x) = 2^x$ and $g(x) = \left(\frac{1}{2}\right)^x$.

(a) How are the graphs of $f(x)$ and $g(x)$ similar?

(b) How are the graphs of $f(x)$ and $g(x)$ different?

Graphs of Exponential Functions (EL2)

Remark 5.2.5 We can now update Remark 5.2.2 so that it includes all values of the base of an exponential function.

The graph of an exponential function $f(x) = b^x$ has the following characteristics:

- Its domain is $(-\infty, \infty)$ and its range is $(0, \infty)$.

- Remember, exponential functions are only defined when $b > 0$ and $b \neq 1$ as we saw in Definition 5.1.4.

  If $b > 1$, $f(x)$ is increasing on $(-\infty, \infty)$ and is an exponential growth function. If $0 < b < 1$, $f(x)$ is decreasing on $(-\infty, \infty)$ and is an exponential decay function.

- There is a horizontal asymptote at $y = 0$. There is no vertical asymptote.

- There is a y-intercept at $(0, 1)$. There is no x-intercept.

Graphs of Exponential Functions (EL2)

Activity 5.2.6 Let's look at a third exponential function, $h(x) = 2^{-x}$.

(a) Before plotting any points or graphing, what do you think the graph might look like? What sort of characteristics might it have?

(b) Fill in the table of values for $h(x)$. Then plot the points on a graph.

x	$h(x)$
-2	
-1	
0	
1	
2	

(c) This function $h(x)$ looks to be the same as a function we looked at previously. Use properties of exponents to rewrite $h(x)$ in a different way.

(d) In addition to plotting points, we can use transformations to graph. If we consider $f(x) = 2^x$ to be the parent function, what transformation is needed to graph $h(x) = 2^{-x}$?

A. A vertical stretch.

B. A horizontal stretch.

C. A reflection over the x-axis.

D. A reflection over the y-axis.

Graphs of Exponential Functions (EL2)

Remark 5.2.7 For a reminder of transformations, see Section 2.4 and the following definitions:

- Definition 2.4.5
- Definition 2.4.9
- Definition 2.4.14
- Definition 2.4.15
- Definition 2.4.22
- Definition 2.4.23

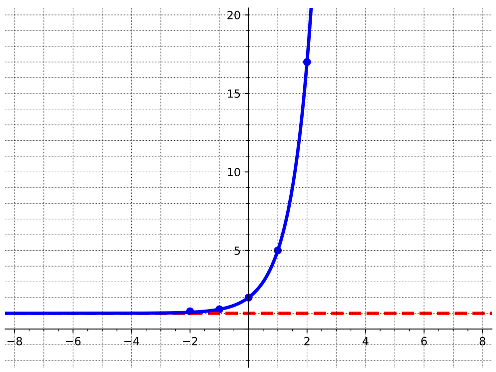
Graphs of Exponential Functions (EL2)

Activity 5.2.8 Let $f(x) = 4^x$.

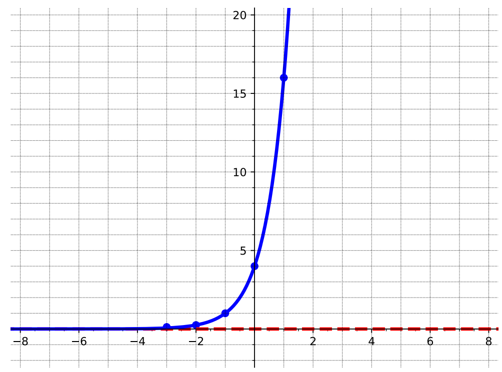
(a) Graph $f(x)$.

(b) Match the following functions to their graphs.

- $g(x) = -4^x$
- $h(x) = 4^{-x}$
- $j(x) = 4^{x+1}$
- $k(x) = 4^x + 1$



A.



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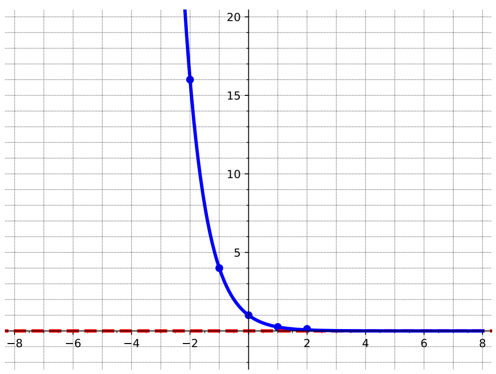

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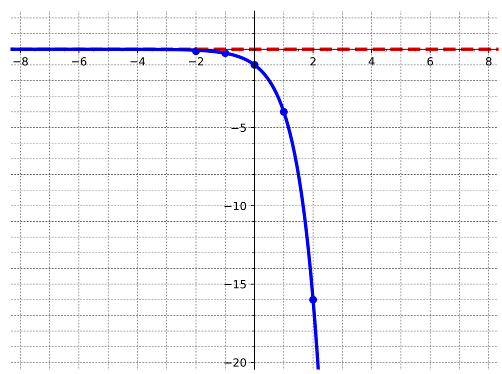


D.

(c) Find the domain, range, and equation of the asymptote for the parent function $(f(x))$ and each of the four transformations $(g(x), h(x), j(x),$ and $k(x))$.

(d) Which of the transformations affected the domain of the exponential function? Select all that apply.

A. A vertical shift.

B. A horizontal shift.

C. A reflection over the x-axis.

D. A reflection over the y-axis.

E. None of these.

Graphs of Exponential Functions (EL2)

(e) Which of the transformations affected the range of the exponential function? Select all that apply.

A. A vertical shift.

B. A horizontal shift.

C. A reflection over the x-axis.

D. A reflection over the y-axis.

E. None of these.

(f) Which of the transformations affected the asymptote of the exponential function? Select all that apply.

A. A vertical shift.

B. A horizontal shift.

C. A reflection over the x-axis.

D. A reflection over the y-axis.

E. None of these.

Graphs of Exponential Functions (EL2)

Activity 5.2.9 Consider the function $f(x) = e^x$.

(a) Graph $f(x) = e^x$. First find $f(0)$ and $f(1)$. Then use what you know about the characteristics of exponential graphs to sketch the rest. Then state the domain, range, and equation of the asymptote. (Recall that $e \approx 2.72$ to help estimate where to put your points.)

(b) Sketch the graph of $g(x) = e^{x-2}$ using transformations. State the transformation(s) used, the domain, the range, and the equation of the asymptote.

(c) Sketch the graph of $h(x) = -3e^x$ using transformations. State the transformation(s) used, the domain, the range, and the equation of the asymptote.

(d) Sketch the graph of $g(x) = e^{-x} - 4$ using transformations. State the transformation(s) used, the domain, the range, and the equation of the asymptote.

Graphs of Exponential Functions (EL2)

Activity 5.2.10 Graph each of the following exponential functions. Include any asymptotes with a dotted line. State the domain, the range, the equation of the asymptote, and whether it is growth or decay.

(a) $f(x) = 3^x$

(b) $f(x) = 6^{-x}$

(c) $f(x) = \frac{1}{5}^{x-2}$

(d) $f(x) = \frac{1}{3}^{x} + 4$

5.3 Introduction to Logarithms (EL3)

Objectives

- Convert between exponential and logarithmic form. Evaluate a logarithmic function, including common and natural logarithms.

Introduction to Logarithms (EL3)

Activity 5.3.1 Let $P(t)$ be the function given by $P(t) = 10^t$.

(a) Fill in the table of values for $P(t)$.

t	$y = P(t)$
-3	
-2	
-1	
0	
1	
2	
3	

(b) Do you think P will have an inverse function? Why or why not?

(c) Since P has an inverse function, we know there exists some other function, say L, such that $y = P(t)$ represent the same relationship between t and y as $t = L(y)$. In words, this means that L reverses the process of raising to the power of 10, telling us the *power* to which we need to raise 10 to produce a desired result. Fill in the table of values for $L(y)$.

y	$L(y)$
10^{-3}	
10^{-2}	
10^{-1}	
10^{0}	
10^{1}	
10^{2}	
10^{3}	

(d) What are the domain and range of P?

(e) What are the domain and range of L?

Remark 5.3.2 The powers of 10 function $P(t)$ has an inverse L. This new function L is called the base 10 logarithm. But, we could have done a similar procedure with any base, which leads to the following definition.

Introduction to Logarithms (EL3)

Definition 5.3.3 The **base b logarithm** of a number is the exponent we must raise b to get that number. We represent this function as $y = \log_b(x)$.

We read the logarithmic expression as "The logarithm with base b of x is equal to y," or "log base b of x is y." ◊

Remark 5.3.4 We can use Definition 5.3.3 to express the relationship between logarithmic form and exponential form as follows:

$$\log_b(x) = y \iff b^y = x$$

whenever $b > 0, b \neq 1$

Introduction to Logarithms (EL3)

Activity 5.3.5 Write the following logarithmic equations in exponential form.

(a) $\log_7(\sqrt{7}) = \dfrac{1}{2}$

A. $7^{\frac{1}{2}} = \sqrt{7}$

B. $7^{\sqrt{7}} = \dfrac{1}{2}$

C. $\sqrt{7}^{\frac{1}{2}} = 7$

D. $\left(\dfrac{1}{2}\right)^7 = \sqrt{7}$

(b) $\log_3(m) = r$

A. $3^m = r$

B. $r^3 = m$

C. $3^r = m$

D. $m^r = 3$

(c) $\log_2(x) = 6$

A. $2^x = 6$

B. $6^2 = x$

C. $x^x = 6$

D. $2^6 = x$

Introduction to Logarithms (EL3)

Activity 5.3.6 Write the following exponential equations in logarithmic form.

(a) $5^2 = 25$

A. $\log_5(2) = 25$

B. $\log_5(25) = 2$

C. $\log_{25}(5) = 2$

D. $\log_2(25) = 5$

(b) $3^{-1} = \frac{1}{3}$

A. $\log_3(-1) = \frac{1}{3}$

B. $\log_{\frac{1}{3}}(-1) = 3$

C. $\log_3\left(\frac{1}{3}\right) = -1$

D. $\log_{-1}\left(\frac{1}{3}\right) = 3$

(c) $10^a = n$

A. $\log_{10}(n) = a$

B. $\log_{10}(a) = n$

C. $\log_n(10) = a$

D. $\log_a(n) = 10$

Introduction to Logarithms (EL3)

Activity 5.3.7 We can use the idea of converting a logarithm to an exponential to evaluate logarithms.

(a) Consider the logarithm $\log_3(9)$. If we want to evaluate this, which question should you try and solve?

A. To what exponent must 9 be raised in order to get 3?

B. What exponent must be raised to the third in order to get 9?

C. To what exponent must 3 be raised in order to get 9?

D. What exponent must be raised to the ninth in order to get 3?

(b) Evaluate the logarithm, $\log_3(9)$, by answering the question from part (a).

Introduction to Logarithms (EL3)

Activity 5.3.8 Evaluate the following logarithms.

(a) $\log_2(8)$

A. 4

B. $\frac{1}{4}$

C. -3

D. 3

(b) $\log_{144}(12)$

A. $\frac{1}{2}$

B. -2

C. 2

D. $-\frac{1}{2}$

(c) $\log_{10}\left(\frac{1}{1000}\right)$

A. $\frac{1}{3}$

B. -3

C. 3

D. $-\frac{1}{3}$

(d) $\log_e\left(e^3\right)$

A. 3

B. e^3

C. -3

D. $\frac{1}{3}$

(e) $\log_7(1)$

A. 7

B. $\frac{1}{7}$

C. 0

D. 1

Remark 5.3.9 Consider the results of Activity 5.3.8 part (d) and (e). Using the rules of exponents and the fact that exponents and logarithms are inverses, these properties hold for any base:

$$\log_b(b^x) = x$$

$$\log_b(1) = 0$$

Remark 5.3.10 There are some logarithms that occur so often, we sometimes write them without noting the base. They are the common logarithm and the natural logarithm.

The **common logarithm** is a logarithm with base 10 and is written without a base.

$$\log_{10}(x) = \log(x)$$

The **natural logarithm** is a logarithm with base e and has its own notation.

Recall the number e was introduced in Remark 5.1.16

$$\log_e(x) = \ln(x)$$

Introduction to Logarithms (EL3)

Activity 5.3.11 Evaluate the following logarithms. Some may be done by inspection and others may require a calculator.

(a) $\log_4 \left(\frac{1}{64}\right)$

(b) $\ln(1)$

(c) $\ln(12)$

(d) $\log(100)$

(e) $\log_5(32)$

(f) $\log_5\left(\sqrt{5}\right)$

(g) $\log(-10)$

Remark 5.3.12 Notice that in Activity 5.3.11 part (g) you were unable to evaluate the logarithm. Given that exponentials and logarithms are inverses, their domain and range are related. The range of an expnential function is $(0, \infty)$ which becomes the domain of a logarithmic function. This means that the argument of any logarithmic function must be greater than zero.

Introduction to Logarithms (EL3)

Activity 5.3.13 Find the domain of the function $\log_3(2x - 4)$.

(a) Set up an inquality that you must solve to find the domain.

(b) Solve the inequality to find the domain. Write your answer in interval notation.

5.4 Graphs of Logarithmic Functions (EL4)

Objectives

- Graph logarithmic functions and determine the domain, range, and asymptotes.

Graphs of Logarithmic Functions (EL4)

Activity 5.4.1 Consider the function $g(x) = \log_2 x$.

(a) Since we are familiar with graphing exponential functions, we'll use that to help us graph logarithmic ones. Rewrite $g(x)$ in exponential form, replacing $g(x)$ with y.

A. $x^y = 2$

B. $2^y = x$

C. $y^2 = x$

D. $2^x = y$

E. $x^2 = y$

(b) Fill in the table of values. Notice you are given y-values, not x-values to plug in since those are easier in the equivalent exponential form. Then plot the points on a graph.

x	y
	-2
	-1
	0
	1
	2

(c) What seems to be happening with the graph as x goes toward infinity? Plug in large positive values of x to test your guess, then describe the end behavior.

A. As $x \to \infty$, $y \to -\infty$.

B. As $x \to \infty$, $y \to 0$.

C. As $x \to \infty$, $y \to 6$.

D. As $x \to \infty$, $y \to \infty$.

E. The graph isn't defined as $x \to \infty$.

(d) What seems to be happening with the graph as x goes toward negative infinity? Plug in large negative values of x to test your guess, then describe the end behavior.

A. As $x \to -\infty$, $y \to -\infty$.

B. As $x \to -\infty$, $y \to 0$.

C. As $x \to -\infty$, $y \to 6$.

D. As $x \to -\infty$, $y \to \infty$.

E. The graph isn't defined as $x \to -\infty$.

(e) What seems to be happening with the graph as we approach x-values closer and closer to zero from the positive direction?

A. As $x \to 0$ from the positive direction, $y \to -\infty$.

B. As $x \to 0$ from the positive direction, $y \to 0$.

C. As $x \to 0$ from the positive direction, $y \to \infty$.

D. As $x \to 0$ from the positive direction, the graph isn't defined.

(f) What seems to be happening with the graph as we approach x-values closer and closer to zero from the negative direction?

A. As $x \to 0$ from the negative direction, $y \to -\infty$.

B. As $x \to 0$ from the negative direction, $y \to 0$.

C. As $x \to 0$ from the negative direction, $y \to \infty$.

D. As $x \to 0$ from the negative direction, the graph isn't defined.

(g) Complete the graph you started in Task 5.4.1.a, connecting the points and including the end behavior and behavior near zero that you've just determined.

(h) Does your graph seem to have any asymptotes?

A. No. There are no asymptotes.

B. There is a vertical asymptote but no horizontal one.

C. There is a horizontal asymptote but no vertical one.

D. The graph has both a horizontal and vertical asymptote.

(i) What the equation for each asymptote of $f(x)$? Select all that apply.

A. There are no asymptotes.

B. $x = 0$

C. $x = 6$

D. $y = 0$

E. $y = 6$

(j) Find the domain and range of $g(x)$. Write your answers using interval notation.

(k) Find the interval(s) where $g(x)$ is increasing and the interval(s) where $g(x)$ is decreasing. Write your answers using interval notation.

Activity 5.4.2 The function we've just graphed, $g(x) = \log_2 x$, and the function $f(x) = 2^x$ (which we graphed in Activity 5.2.1) are inverse functions.

(a) How are the graphs of $f(x)$ and $g(x)$ similar?

(b) How are the graphs of $f(x)$ and $g(x)$ different?

Graphs of Logarithmic Functions (EL4)

Remark 5.4.3 The graph of a logarithmic function $g(x) = \log_b x$ where $b > 0$ and $b \neq 1$ has the following characteristics:

- Its domain is $(0, \infty)$ and its range is $(-\infty, \infty)$.
- There is a vertical asymptote at $x = 0$. There is no horizontal asymptote.
- There is an x-intercept at $(1, 0)$. There is no y-intercept.

Graphs of Logarithmic Functions (EL4)

Remark 5.4.4 Just as with other types of functions, we can use transformations to graph logarithmic functions. For a reminder of these transformations, see Section 2.4 and the following definitions:

- Definition 2.4.5
- Definition 2.4.9
- Definition 2.4.14
- Definition 2.4.15
- Definition 2.4.22
- Definition 2.4.23

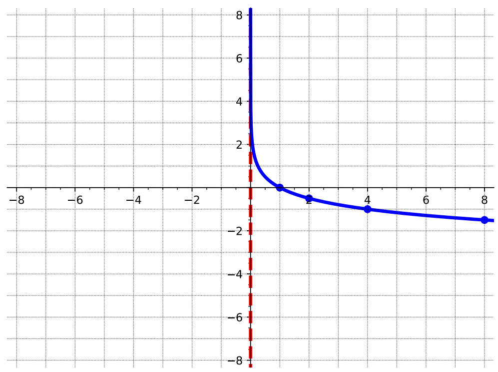
Graphs of Logarithmic Functions (EL4)

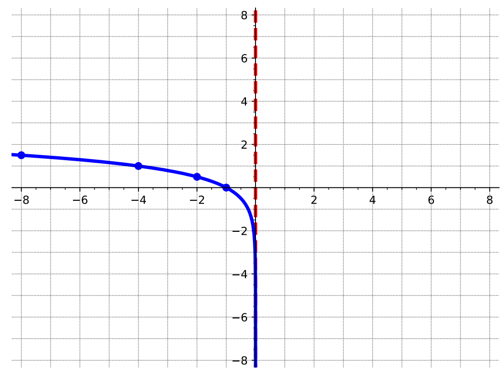
Activity 5.4.5 Let $f(x) = \log_4 x$.

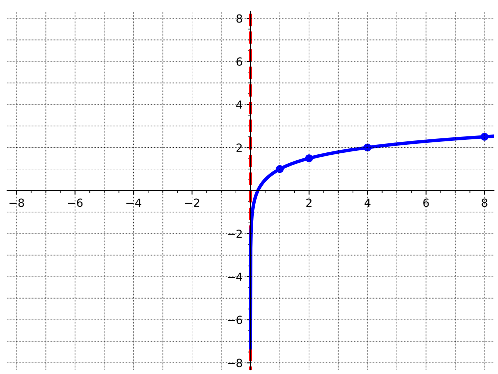
(a) Graph $f(x)$.

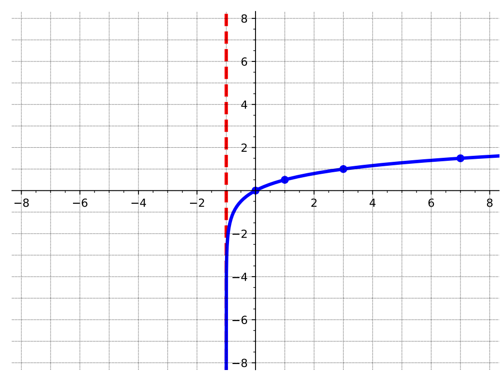
(b) Match the following functions to their graphs.

- $g(x) = -\log_4 x$
- $h(x) = \log_4(-x)$
- $j(x) = \log_4(x + 1)$
- $k(x) = \log_4(x) + 1$

A. 

C. 

B. 

D. 

(c) Find the domain, range, and equation of the asymptote for the parent function ($f(x)$) and each of the four transformations ($g(x), h(x), j(x),$ and $k(x)$).

(d) Which of the transformations affected the domain of the logarithmic function? Select all that apply.

A. A vertical shift.

B. A horizontal shift.

C. A reflection over the x-axis.

D. A reflection over the y-axis.

E. None of these.

(e) Which of the transformations affected the range of the logarithmic function? Select all that apply.

A. A vertical shift.

B. A horizontal shift.

C. A reflection over the x-axis.

D. A reflection over the y-axis.

E. None of these.

(f) Which of the transformations affected the asymptote of the logarithmic function? Select all that apply.

A. A vertical shift.

B. A horizontal shift.

C. A reflection over the x-axis.

D. A reflection over the y-axis.

E. None of these.

Graphs of Logarithmic Functions (EL4)

Activity 5.4.6 Consider the function $f(x) = \ln(x)$.

(a) Graph $f(x) = \ln(x)$. First find $f(1)$ and $f(e)$. Then use what you know about the characteristics of logarithmic graphs to sketch the rest. Then state the domain, range, and equation of the asymptote. (Recall that $e \approx 2.72$ to help estimate where to put your points.)

(b) Sketch the graph of $g(x) = \ln(x-3)$ using transformations. State the transformation(s) used, the domain, the range, and the equation of the asymptote.

(c) Sketch the graph of $h(x) = 3\ln(x)$ using transformations. State the transformation(s) used, the domain, the range, and the equation of the asymptote.

Graphs of Logarithmic Functions (EL4)

Activity 5.4.7 Graph each of the following logarithmic functions. Include any asymptotes with a dotted line. State the domain, the range, and the equation of the asymptote.

(a) $f(x) = \log_3 x$

(b) $f(x) = \log_6(-x)$

(c) $f(x) = \log_{\frac{1}{5}} x$

(d) $f(x) = \log_{\frac{1}{3}} x + 2$

5.5 Properties of Logarithms (EL5)

Objectives

- Use properties of logarithms to condense or expand logarithmic expressions.

Properties of Logarithms (EL5)

Remark 5.5.1 Recall that $\log_b M = \log_b N$ if and only if $M = N$. In addition, because exponentials and logarithms are inverses, we also know that $\log_b(b^k) = k$. In addition, according to the law of exponents, we know that:

$$x^a \cdot x^b = x^{a+b}$$

$$\frac{x^a}{x^b} = x^{a-b}$$

$$(x^a)^b = x^{a \cdot b}$$

Consider all these as you move through the activities in this section.

Properties of Logarithms (EL5)

Activity 5.5.2 Let's begin with the law of exponents to see if we can understand the product property of logs. According to the law of exponents, we know that $10^x \cdot 10^y = 10^{x+y}$. Start with this equation as you move through this activity.

(a) Let $a = 10^x$ and $b = 10^y$. How could you rewrite the left side of the equation $10^x \cdot 10^y$?

A. $a + b$

B. $a - b$

C. 10^{x+y}

D. $a \cdot b$

(b) Recall from Section 5.3 that $\log_b M = \log_b N$ if and only if $M = N$. Use this property to apply the logarithm to both sides of the rewritten equation from part a. What is that equation?

A. $\log_{10}(a + b) = \log_{10}\left(10^{x+y}\right)$

B. $\log_{10}(a \cdot b) = \log_{10}\left(10^{x+y}\right)$

C. $\log_{10}(a \cdot b) = \log_{10}\left(10^{a+b}\right)$

D. $\log_{10}(a + b) = \log_{10}\left(10^{a+b}\right)$

(c) Knowing that $\log_b(b^k) = k$, how could you simplify the right side of the equation?

A. $\log_{10}(a + b)$

B. $\log_{10}(x + y)$

C. $a + b$

D. $x + y$

(d) Recall in part a, we defined $10^x = a$ and $10^y = b$. What would these look like in logarithmic form?

A. $\log_{10} a = x$

B. $\log_x a = 10$

C. $\log_{10} b = y$

D. $\log_y b = 10$

(e) Using your solutions in part d, how can we rewrite the right side of the equation?

A. 10^{a+b}

B. $\log_{10} a - \log_{10} b$

C. $\log_{10} a + \log_{10} b$

D. $10^x + 10^y$

(f) Combining parts a and d, which equation represents $10^x \cdot 10^y = 10^{x+y}$ in terms of logarithms?

A. $\log_{10}(a + b) = 10^{a+b}$

B. $\log_{10}(a \cdot b) = \log_{10} a - \log_{10} b$

C. $\log_{10}(a \cdot b) = \log_{10} a + \log_{10} b$

D. $\log_{10}(a \cdot b) = 10^x + 10^y$

Properties of Logarithms (EL5)

Activity 5.5.3 According to the law of exponents, we know that $\frac{10^x}{10^y} = 10^{x-y}$. Start with this equation as you move through this activity.

(a) Let $a = 10^x$ and $b = 10^y$. How could you rewrite the left side of the equation $\frac{10^x}{10^y}$?

A. $a + b$

B. $a - b$

C. 10^{x+y}

D. $a \cdot b$

(b) Recall from Section 5.3 that $\log_b M = \log_b N$ if and only if $M = N$. Use this property to apply the logarithm to both sides of the rewritten equation from part a. What is that equation?

A. $\log_{10}(a - b) = \log_{10}\left(10^{x-y}\right)$

B. $\log_{10}\left(\frac{a}{b}\right) = \log_{10}\left(10^{x-y}\right)$

C. $\log_{10}\left(\frac{a}{b}\right) = \log_{10}\left(10^{a+b}\right)$

D. $\log_{10}(a - b) = \log_{10}\left(10^{a-b}\right)$

(c) Knowing that $\log_b(b^k) = k$, how could you simplify the right side of the equation?

A. $\log_{10}(a - b)$

B. $\log_{10}(x - y)$

C. $x - y$

D. $a - b$

(d) Recall in part a, we defined $10^x = a$ and $10^y = b$. What would these look like in logarithmic form?

A. $\log_{10} a = x$

B. $\log_x a = 10$

C. $\log_{10} b = y$

D. $\log_y b = 10$

(e) Using your solutions in part d, how can we rewrite the right side of the equation?

A. 10^{a+b}

B. $\log_{10} a - \log_{10} b$

C. $\log_{10} a - \log_{10} b$

D. 10^{x-y}

(f) Combining parts a and d, which equation represents $\frac{10^x}{10^y} = 10^{x-y}$ in terms of logarithms?

A. $\log_{10}(a - b) = 10^{a+b}$

B. $\log_{10}(a - b) = \log_{10} a - \log_{10} b$

C. $\log_{10}\left(\frac{a}{b}\right) = \log_{10} a - \log_{10} b$

D. $\log_{10}\left(\frac{a}{b}\right) = 10^{x-y}$

Properties of Logarithms (EL5)

Remark 5.5.4 In Activity 5.5.2 and Activity 5.5.3, you explored an example of two common properties of logarithms!

Properties of Logarithms (EL5)

Definition 5.5.5 The **product property of logarithms** states

$$\log_a(m \cdot n) = \log_a m + \log_a n.$$

The **quotient property of logarithms** states

$$\log_a\left(\frac{m}{n}\right) = \log_a m - \log_a n.$$

Notice that for each of these properties, the base has to be the same. ◊

Properties of Logarithms (EL5)

Activity 5.5.6 There is still one more property to consider. This activity will investigate the power property. Suppose you are given

$$\log(x^3) = \log(x \cdot x \cdot x).$$

(a) By applying Definition 5.5.5, how could you rewrite the right-hand side of the equation?

A. $\log(3x)$

B. $\log(x) - \log(x) - \log(x)$

C. $\log(x^3)$

D. $\log(x) + \log(x) + \log(x)$

(b) By combining ”like” terms, you can simplify the right-hand side of the equation further. What equation do you have after simplifying the right-hand side?

A. $\log(x^3) = \log(3x)$

B. $\log(x^3) = -3\log(x)$

C. $\log(x^3) = \log(x^3)$

D. $\log(x^3) = 3\log(x)$

Properties of Logarithms (EL5)

Definition 5.5.7 The **power property of logarithms** states

$$\log_a(m^n) = n \cdot \log_a(m).$$

$\Diamond$

Properties of Logarithms (EL5)

Activity 5.5.8 Apply Definition 5.5.5 and Definition 5.5.7 to expand the following. (Note: When you are asked to expand logarithmic expressions, your goal is to express a single logarithmic expression into many individual parts or components.)

(a)

$$\log_3\left(\frac{6}{19}\right)$$

A. $\log_3 6 + \log_3 19$

B. $\log_3(6 - 19)$

C. $\log_3 6 - \log_3 19$

D. $\log_3(6 + 19)$

(b)

$$\log(a \cdot b)^2$$

A. $2(\log a + \log b)$

B. $2\log a - \log b$

C. $\log a + 2\log b$

D. $2\log a + 2\log b$

(c)

$$\ln\left(\frac{x^3}{y}\right)$$

A. $3\ln x + \ln y$

B. $3\ln x - \ln y$

C. $3(\ln x - \ln y)$

D. $\ln x^3 - \ln y$

(d)

$$\log(x \cdot y \cdot z^3)$$

A. $\log x + \log y + 3\log z$

B. $\log x + \log y + \log z^3$

C. $\log x - \log y - 3\log z$

D. $3(\log x + y + z)$

Properties of Logarithms (EL5)

Activity 5.5.9 Apply Definition 5.5.5 and Definition 5.5.7 to condense into a single logarithm.

(a)

$$6\log_6 a + 3\log_6 b$$

A. $6(\log_6 a) + 3(\log_6 b)$

B. $\log_6 a^6 + \log_6 b^3$

C. $(\log_6 a)^6 + (\log_6 b)^3$

D. $\log_6 \left(a^6 \cdot b^3\right)$

(b)

$$\ln x - 4\ln y$$

A. $\ln x - \ln y^4$

B. $\ln\left(\dfrac{x}{y^4}\right)$

C. $\ln\left(\dfrac{x}{y}\right)^4$

D. $\ln(x \cdot y^4)$

(c)

$$2(\log(2x) - \log y)$$

A. $\log\left(\dfrac{4x^2}{y^2}\right)$

B. $\log\left(\dfrac{2x^2}{y^2}\right)$

C. $2 \cdot \log\left(\dfrac{2x}{y}\right)$

D. $\log\left(\dfrac{4x^2}{y}\right)$

(d)

$$\log 3 + 2\log 5$$

A. $\log(3 \cdot 5^2)$

B. $2 \cdot \log 75$

C. $\log 75$

D. $\log 225$

Properties of Logarithms (EL5)

Remark 5.5.10 You might have noticed that a scientific calculator has only ”log” and ”ln” buttons (because those are the most common bases we use), but not all logs have base 10 or e as their bases.

Properties of Logarithms (EL5)

Activity 5.5.11 Suppose you wanted to find the value of $\log_5 3$ in your calculator but you do not know how to input a base other than 10 or e (i.e., you only have the "log" and "ln" buttons on your calculator). Let's explore another helpful tool that can help us find the value of $\log_5 3$.

(a) Let's start with the general statement, $\log_b a = x$. How can we rewrite this logarithmic equation into an exponential equation?

(b) Now take the log of both sides of your equation and apply the power property of logarithms to bring the exponent down. What equation do you have now?

(c) Solve for x. What does x equal?

(d) Recall that when we started, we defined $x = \log_b a$. Substitute $\log_b a$ into your equation you got in part c for x. What is the resulting equation?

(e) Apply what you got in part d to find the value of $\log_5 3$. What is the approximate value of $\log_5 3$?

Remark 5.5.12 Notice in Activity 5.5.11, we were able to calculate $\log_5 3$ using logs of base 10. You should now be able to find the value of a logarithm of any base!

Properties of Logarithms (EL5)

Definition 5.5.13 The **change of base formula** is used to write a logarithm of a number with a given base as the ratio of two logarithms each with the same base that is different from the base of the original logarithm.

$$\log_b a = \frac{\log a}{\log b}$$

◊

Properties of Logarithms (EL5)

Activity 5.5.14 Apply Definition 5.5.13 and a calculator to approximate the value of each logarithm.

(a) $\log_2 30$

A. 0.204

B. 4.907

C. $\dfrac{\log 30}{\log 2}$

D. $\dfrac{\log 2}{\log 30}$

(b) $\ln 183$

A. $\dfrac{\ln 183}{\ln e}$

B. 67.32

C. $\dfrac{\ln e}{\ln 183}$

D. 5.209

5.6 Solving Exponential and Logarithmic Equations (EL6)

Objectives

- Solve exponential and logarithmic equations.

Solving Exponential and Logarithmic Equations (EL6)

Remark 5.6.1 Recall that we can convert between exponential and logarithmic forms.

$$\log_b x = y$$

is equivalent to

$$b^y = x$$

Solving Exponential and Logarithmic Equations (EL6)

Activity 5.6.2 This activity will investigate ways we can solve logarithmic equations using properties and definitions from previous sections.

(a) Given that $\log_3 9 = x$, how can we rewrite this into an exponential equation?

A. $9^x = 3$

B. $\log\left(\frac{9}{3}\right)$

C. $3^x = 9$

D. $x^3 = 9$

(b) Now that $\log_3 9 = x$ is rewritten as an exponential equation, what is the value of x?

A. -2

B. 2

C. $-\frac{1}{2}$

D. $\frac{1}{2}$

Remark 5.6.3 Notice in Activity 5.6.2, you were able to solve a logarithmic equation by converting it into an exponential equation. This is one method in solving logarithmic equations.

Solving Exponential and Logarithmic Equations (EL6)

Activity 5.6.4 For each of the following, solve the logarithmic equations by first converting them to exponential equations.

(a) $\log_{10}(1{,}000{,}000) = x$

A. -6

B. 100,000

C. 6

D. 0.00001

(b) $\log_3(x+3) = 0$

A. 0

B. -2

C. 2

D. -1

(c) $\log_5(2x+4) = 2$

A. 14

B. $\dfrac{29}{2}$

C. 16

D. $\dfrac{21}{2}$

Solving Exponential and Logarithmic Equations (EL6)

Activity 5.6.5 Not all logarithmic equations can be solved by converting to exponential equations. In this activity, we will explore another way to solve logarithmic equations.

(a) Suppose you are given the equation,

$$\log(4x - 5) = \log(2x - 1),$$

and you brought all the logs to one side to get

$$\log(4x - 5) - \log(2x - 1) = 0.$$

Using the quotient property of logs, how could you condense the left side of the equation?

A. $\log\left(\frac{4x-5}{2x-1}\right)$

B. $\log\left(\frac{2x-1}{4x-5}\right)$

C. $\frac{\log(4x-5)}{\log(2x-1)}$

D. $\frac{\log(2x-1)}{\log(4x-5)}$

(b) Now that you have a single logarithm, convert this logarithmic form into exponential form. What does this new equation look like?

A. $\frac{4x-5}{2x-1} = 10^0$

B. $\frac{2x-1}{4x-5} = 1$

C. $\frac{2x-1}{4x-5} = 10^0$

D. $\frac{4x-5}{2x-1} = 1$

(c) Notice that the "log" has disappeared and you now have an equation with just the variable x. Which of the following is equivalent to the equation you got in part b?

A. $4x - 5 = 2x - 1$

B. $4x - 5 = 0$

C. $2x - 1 = 0$

(d) Compare the answer you got in part c to the original equation given $\log(4x - 5) = \log(2x - 1)$. What do you notice?

(e) Solve the equation you got in part d to find the value of x.

A. -3

B. $\frac{5}{4}$

C. $\frac{1}{2}$

D. 2

Remark 5.6.6 Notice in Activity 5.6.5, that you did not have to convert the logarithmic equation into an exponential equation. A faster method, when you have a log on both sides of the equals sign, is to ”drop” the logs and set the arguments equal to one another. Be careful though - you can only have one log on each side before you can ”drop” them!

Solving Exponential and Logarithmic Equations (EL6)

Definition 5.6.7 The **one-to-one property of logarithms** states that if both sides of an equation can be rewritten as a single logarithm with the same base, then the arguments can be set equal to each other (and then solved algebraically). $\Diamond$

Solving Exponential and Logarithmic Equations (EL6)

Activity 5.6.8 Apply Definition 5.6.7 and other properties of logarithms (i.e., product, quotient, and power) to solve the following logarithmic equations.

(a) $\log(-2a+9) = \log(7-4a)$

(b) $\log_9(x+6) - \log_9 x = \log_9 2$

(c) $\log_8 2 + \log_8(4x^2) = 1$

(d) $\ln(4x+1) - \ln 3 = 5$

Activity 5.6.9 In some cases, you will get equations with logs of different bases. Apply properties of logarithms to solve the following logarithmic equations.

(a) $\log_3(x-6)=\log_9 x$

Hint. Use the change of base formula to rewrite $\log_9 x$ so that it has a base of 3.

(b) $\log_2 x=\log_8(4x)$

Solving Exponential and Logarithmic Equations (EL6)

Activity 5.6.10 Now that we've looked at how to solve logarithmic equations, let's see how we can apply similar methods to solving exponential equations.

(a) Suppose you are given the equation $2^x = 48$. There is no whole number value we can raise 2 to to get 48. What two whole numbers must x be between?

(b) We'll use logarithms to isolate the variable in the exponent. How can we convert $2^x = 48$ into a logarithmic equation?

A. $x = \log_{48} 2$

B. $x = \log_2 48$

C. $48 = \log_2 x$

D. $2 = \log_{48} x$

(c) Notice that the answer you got in part b is an exact answer for x. There will be times, though, that it will be helpful to also have an approximation for x. Which of the following is a good approximation for x?

A. $x \approx 5.585$

B. $x \approx 0.179$

C. $x \approx 24$

D. $x \approx \dfrac{1}{24}$

Solving Exponential and Logarithmic Equations (EL6)

Remark 5.6.11 Notice in Activity 5.6.10 we started with an exponential equation and then solved by converting the equation into a logarithmic equation. Logarithms can help us get the variable out of the exponent.

Solving Exponential and Logarithmic Equations (EL6)

Activity 5.6.12 Although rewriting an exponential equation into a logarithmic equation is helpful at times, it is not the only method in solving exponential equations. In this activity, we will explore what happens when we take the log of both sides of an exponential equation and use the properties of logarithms to solve in another way.

(a) Suppose you are given the equation:

$$3^x = 7.$$

Take the log of both sides. What equation do you now have?

(b) Apply the power property of logarithms (Definition 5.5.7) to bring down the exponent. What equation do you now have?

(c) Solve for x. What does x equal?

(d) Using the change-of-base formula (Definition 5.5.13), rewrite your answer from part c so that x is written as a single logarithm. What is the exact value of x?

(e) If you were to solve $3^x = 7$ by converting it into a logarithmic equation, what would it look like?

(f) What do you notice about your answer from parts d and e?

Activity 5.6.13 Use the method of taking the log of both sides (as you saw in Activity 5.6.12) to solve $5^{2x+3} = 8$.

(a) Take the log of both sides and use the power property of logarithms to bring down the exponent. What equation do you have now?

A. $2x + 3\log 5 = \log 8$

B. $2x + (3\log 5) = \log 8$

C. $(2x + 3) \cdot \log 5 = \log 8$

D. $2x\log 5 + 3 = \log 8$

(b) Solve for x.

A. $x = \dfrac{\log_5 8 - 3}{2}$

B. $x = \dfrac{\log 8 - 3\log 5}{2}$

C. $x = \dfrac{\log 8}{3\log 5} - 2$

D. $x = \dfrac{\log 8 - 3}{2\log 5}$

(c) What is the approximate value of x?

A. $x \approx -0.85$

B. $x \approx -1.5$

C. $x \approx -0.60$

D. $x \approx -1.57$

Solving Exponential and Logarithmic Equations (EL6)

Activity 5.6.14 In this activity, we will explore other types of exponential equations, which will require other methods of solving.

(a) Suppose you are given the equation

$$5^{3x} = 5^{7x-2}$$

and you decide to take the log of both sides as your first step to get

$$\log 5^{3x} = \log 5^{7x-2}$$

What would you use next to solve this equation?

A. Quotient property of logarithms

B. Power property of logarithms

C. Product property of logarithms

D. Change of base formula

E. One-to-one property of logarithms

(b) Applying the property you chose in part a, what would the resulting equation be?

A. $3x \log 5 = 7x - 2 \log 5$

B. $3x \log 5 = 7x \log 5 - 2 \log 5$

C. $\log 5^{3x} = \log 5^{7x-2}$

D. $(3x) \log 5 = (7x - 2) \log 5$

(c) Now that you have a logarithmic equation, divide both sides by $\log 5$ to begin to isolate the variable x. After dividing by $\log 5$, what equation do you now have?

A. $3x = \dfrac{7x}{\log 5} - 2$

B. $3x = 7x - 2$

C. $5^{3x} = 5^{7x-2}$

D. $\dfrac{3x \log 5}{\log 5} = \dfrac{(7x - 2) \log 5}{\log 5}$

(d) Compare the equation you got in part c to the original equation given. What do you notice?

(e) Solve for x.

A. 2

B. $\dfrac{1}{2}$

C. -2

D. $\dfrac{1}{2}$

Remark 5.6.15 Notice in Activity 5.6.14, it is much faster to set the exponents equal to one another. Make sure to check that the bases are equal *before* you set the exponents equal! And if the bases are not equal, you might have to use properties of exponents to help you get the bases to be the same.

Solving Exponential and Logarithmic Equations (EL6)

Definition 5.6.16 When you are given an exponential equation with the same bases on both sides, you can simply set the exponents equal to one another and solve. This is known as the **one-to-one property of exponentials**. $\lozenge$

Solving Exponential and Logarithmic Equations (EL6)

Activity 5.6.17 When an exponential equation has the same base on each side, the exponents can be set equal to one another. If the bases aren't the same, we can rewrite them using properties of exponents and use the one-to-one property of exponentials.

(a) Suppose you are given

$$5^x = 625,$$

how could you rewrite this equation so that both sides have a base of 5?

(b) Suppose you are given

$$4^x = 32,$$

how could you rewrite this equation so that both sides have a base of 2?

(c) Suppose you are given

$$3^{1-x} = \frac{1}{27},$$

how could you rewrite this equation so that both sides have a base of 3? (Hint: you many need to revisit properties of exponents)

(d) Suppose you are given

$$6^{\frac{x-3}{4}} = \sqrt{6},$$

how could you rewrite this equation so that both sides have a base of 6? (Hint: you many need to revisit properties of exponents)

Solving Exponential and Logarithmic Equations (EL6)

Activity 5.6.18 For each of the following, use properties of exponentials and logarithms to solve.

(a) $6^{-2x} = 6^{2-3x}$

A. 2

B. $-\frac{2}{5}$

C. -2

D. $-\frac{5}{2}$

(b) $\log_2 256 = x$

A. -8

B. 254

C. 8

D. 128

(c) $5\ln(9x) = 20$

A. e^4

B. $\frac{e^4}{9}$

C. $e^{\left(\frac{4}{9}\right)}$

D. $\frac{4}{\ln 9}$

(d) $10^x = 4.23$

A. $\log 4.23$

B. 42.3

C. 1.44

D. 0.63

(e) $\log_6(5x - 5) = \log_6(3x + 7)$

A. 2

B. 0

C. 6

D. 1

Solving Exponential and Logarithmic Equations (EL6)

Activity 5.6.19 For each of the following, use properties of exponentials and logarithms to solve.

(a) $\log_8 2 + \log_8 4x^2 = 1$

(b) $5^{x+7} = 3$

(c) $8^{\frac{x-6}{6}} = \sqrt{8}$

(d) $\log_6(x+1) - \log_6 x = \log_6 29$

5.7 Applications of Exponential and Logarithmic Functions (EL7)

Objectives

- Solve application problems using exponential and logarithmic equations.

Applications of Exponential and Logarithmic Functions (EL7)

Remark 5.7.1 Now that we have explored multiple methods for solving exponential and logarithmic equations, let's put those in to practice using some real-world application problems.

Activity 5.7.2 A coffee is sitting on Mr. Abacus's desk cooling. It cools according to the function $T = 70(0.80)^x + 20$, where x is the time elapsed in minutes and T is the temperature in degrees Celsius.

(a) What is the initial temperature of Mr. Abacus's coffee?

A. $20°C$
B. $70°C$
C. $0.80°C$
D. $90°C$

(b) What is the temperature of Mr. Abacus's coffee after 10 minutes?

A. $7.5°C$
B. $20°C$
C. $27.5°C$
D. $76°C$

(c) According to the function given, if Mr. Abacus leaves his coffee on his desk all day, what will the coffee eventually cool to?

A. $90°C$
B. $70°C$
C. $20°C$
D. $0°C$

Hint. Think about what happens to the function as $x \to \infty$.

Applications of Exponential and Logarithmic Functions (EL7)

Activity 5.7.3 A video posted on YouTube initially had 80 views as soon as it was posted. The total number of views to date has been increasing exponentially according to the exponential growth function $y = 80e^{0.2t}$, where t represents time measured in days since the video was posted.

(a) How many views will the video have after 3 days? Round to the nearest whole number.

A. 293 views

B. 146 views

C. 98 views

D. 82 views

(b) How many days does it take until 2,500 people have viewed this video?

A. 18 days

B. 156 days

C. 39 days

D. 17 days

Activity 5.7.4 In 2006, 80 deer were introduced into a wildlife refuge. By 2012, the population had grown to 180 deer. The population was growing exponentially. Recall that the general form of an exponential equation is $f(x) = a \cdot b^x$, where a is the initial value, b is the growth/decay factor, and t is time. We want to write a function $N(t)$ to represent the deer population after t years.

(a) What is the initial value for the deer population?

(b) We are not given the growth factor, so we must solve for it. Write an exponential equation using the initial population, the 2012 population, and the time elapsed.

(c) Take your equation in part b and solve for b using logs.

(d) Now that you have found the growth factor (from part b), what is the equation, in terms of $N(t)$ and t that represents the deer population?

(e) If the growth continues according to this exponential function, when will the population reach 250?

Applications of Exponential and Logarithmic Functions (EL7)

Activity 5.7.5 The concentration of salt in ocean water, called salinity, varies as you go deeper in the ocean. Suppose $f(x) = 28.9 + 1.3\log(x+1)$ models salinity of ocean water to depths of 1000 meters at a certain latitude, where x is the depth in meters and $f(x)$ is in grams of salt per kilogram of seawater. (Note that salinity is expressed in the unit g/kg, which is often written as ppt (part per thousand) or ‰ (permil).)

(a) At 50 meters, what is the salinity of the seawater?

A. 31.1 ppt

B. 32.1 ppt

C. 51.6 ppt

D. 30.6 ppt

(b) Approximate the depth (to the nearest tenth of a meter) where the salinity equals 33 ppt.

A. -1.0 meters

B. 1426.1 meters

C. -0.9 meters

D. 1424.1 meters

Activity 5.7.6 The first key on a piano keyboard (called A_0) corresponds to a pitch with a frequency of 27.5 cycles per second. With every successive key, going up the black and white keys, the pitch multiplies by a constant. The formula for the frequency, f of the pitch sounded when the nth note up the keyboard is played is given by

$$n = 1 + 12\log_2 \frac{f}{27.5}$$

(a) A note has a frequency of 220 cycles per second. How many notes up the piano keyboard is this?

A. 37 notes

B. 39 notes

C. 12 notes

D. 96 notes

(b) What frequency does the 12th note have? Round to the nearest tenth.

A. 0.1 cycles per second

B. 227.0 cycles per second

C. 51.9 cycles per second

D. 49.4 cycles per second

Remark 5.7.7 Another application of exponential equations is compound interest. Savings instruments in which earnings are continually reinvested, such as mutual funds and retirement accounts, use compound interest. The term compounding refers to interest earned not only on the original value, but on the accumulated value of the account. Compound interest can be calcuated by using the formula

$$A(t) = P\left(1 + \frac{r}{n}\right)^{nt},$$

where $A(t)$ is the account value, t is measured in years, P is the starting amount of the account (also known as the principal), r is the annual percentage rate (APR) written as a decimal, and n is the number of compounding periods in one year.

Applications of Exponential and Logarithmic Functions (EL7)

Activity 5.7.8 Before we can apply the compound interest formula, we need to understand what "compounding" means. Recall that compounding refers to interest earned not only on the original value, but on the accumulated value of the account. This amount is calculated a certain number of times in a given year.

(a) Suppose an account is compounded quarterly, what value of n would we use in the formula?

A. 1
B. 2
C. 4
D. 12
E. 365

(b) Suppose an account is compounded daily, what value of n would we use in the formula?

A. 1
B. 2
C. 4
D. 12
E. 365

(c) Suppose an account is compounded semi-annually, what value of n would we use in the formula?

A. 1
B. 2
C. 4
D. 12
E. 365

(d) Suppose an account is compounded monthly, what value of n would we use in the formula?

A. 1
B. 2
C. 4
D. 12
E. 365

(e) Suppose an account is compounded annually, what value of n would we use in the formula?

A. 1
B. 2
C. 4
D. 12
E. 365

Activity 5.7.9 A 529 Plan is a college-savings plan that allows relatives to invest money to pay for a child's future college tuition; the account grows tax-free. Lily currently has $10,000 and opens a 529 account that will earn 6% compounded semi-annually.

(a) Which equation could we use to determine how much money Lily will have for her granddaughter after t years?

A. $A(t) = 10{,}000\left(1+\frac{6}{2}\right)^{2t}$

B. $A(t) = 10{,}000\left(1+\frac{0.06}{2}\right)^{2t}$

C. $A(t) = 10{,}000\left(1+\frac{6}{\frac{1}{2}}\right)^{\frac{1}{2}t}$

D. $A(t) = 10{,}000\left(1+\frac{0.06}{2}\right)^{18}$

(b) To the nearest dollar, how much will Lily have in the account in 10 years?

A. $106,090

B. $103,000

C. $13,439

D. $18,061

(c) How many years will it take Lily to have $40,000 in the account for her granddaughter? Round to the nearest tenth.

A. 23.4 years

B. 46.9 years

C. 52.1 years

D. 25.6 years

Applications of Exponential and Logarithmic Functions (EL7)

Activity 5.7.10 For each of the following, determine the appropriate equation to use to solve the problem.

(a) Kathy plans to purchase a car that depreciates (loses value) at a rate of 14% per year. The initial cost of the car is \$21,000. Which equation represents the value, v, of the car after 3 years?

A. $v = 21{,}000(0.14)^3$

B. $v = 21{,}000(0.86)^3$

C. $v = 21{,}000(1.14)^3$

D. $v = 21{,}000(0.86)(3)$

(b) Mr. Smith invested \$2,500 in a savings account that earns 3% interest compounded annually. He made no additional deposits or withdrawals. Which expression can be used to determine the number of dollars in this account at the end of 4 years?

A. $2{,}500(1 + 0.03)^4$

B. $2{,}500(1 + 0.3)^4$

C. $2{,}500(1 + 0.04)^3$

D. $2{,}500(1 + 0.4)^3$

Activity 5.7.11 Suppose you want to invest \$100 in a banking account that has a 100% interest rate. Let's investigate what would happen to the amount of money you have at the end of one year in the account with varying compounding periods. Use the formula, $A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$, to help you solve the following problems.

(a) Suppose the account is compounded annually (i.e., $n = 1$). How much money would you have in the account at the end of the year?

(b) By what factor did the money in your account grow in part a?

(c) Suppose the account is compounded semi-annually. How much money would you have in the account at the end of the year?

(d) By what factor did the money in your account grow in part b?

(e) Let's investigate as the compounding periods increase. Fill in the following table.

n	$A(t)$	**Growth Factor**
1		
2		
5		
10		
100		
1,000		
10,000		
100,000		

(f) What do you notice as the value of n increases?

Observation 5.7.12 In Activity 5.7.11, we can see that as the value of n increases, the factor by which the money increased by tended toward the value of 2.71. This value is a significant mathematical constant and is denoted by the symbol e. It is approximately equal to 2.718.

Applications of Exponential and Logarithmic Functions (EL7)

Remark 5.7.13 For many real-world phenomena, e is used as the base for exponential functions. Exponential models that use e as the base are called continuous growth or decay models. We see these models in finance, computer science, and most of the sciences, such as physics, toxicology, and fluid dynamics. For all real numbers t, and all positive numbers a and r, continuous growth or decay is represented by the formula

$$A(t) = ae^{rt},$$

where a is the initial value, r is the continuous growth rate per of unit time, and t is the elapsed time. If $r > 0$, then the formula represents continuous growth. If $r < 0$, then the formula represents continuous decay.

For business applications, the continuous growth formula is called the continuous compounding formula and takes the form

$$A(t) = Pe^{rt},$$

where P is the principal or the initial invested, r is the growth or interest rate per of unit time, and t is the period or term of the investment.

Applications of Exponential and Logarithmic Functions (EL7)

Activity 5.7.14 Use the continuous formulas to answer the following questions.

(a) A person invested \$1,000 in an account earning 10% per year compounded continuously. How much was in the account at the end of one year?

(b) Radon-222 decays at a continuous rate of 17.3% per day. How much will 100 mg of Radon-222 decay to in 3 days?

Colophon

This book was authored in PreTeXt.