Linear Algebra for Team-Based Inquiry Learning

2025 Edition

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Steven Clontz
University of South Alabama

Drew Lewis

Contributing Authors

Jessalyn Bolkema
California State University, Dominguez Hills

Jeff Ford
Gustavus Adolphus College

Jordan Kostiuk
Brown University

Sharona Krinsky
California State University, Los Angeles

Jennifer Nordstrom
Linfield University

Kate Owens
College of Charleston

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Website: Team-Based Inquiry Learning[1]

[1] tbil.org
[2] github.com/TeamBasedInquiryLearning/library/blob/main/LICENSE.md

Contents

Chapter 1

Systems of Linear Equations (LE)

Learning Outcomes

How can we solve systems of linear equations?
By the end of this chapter, you should be able to...

1. Translate back and forth between a system of linear equations, a vector equation, and the corresponding augmented matrix.

2. Explain why a matrix isn't in reduced row echelon form, and put a matrix in reduced row echelon form.

3. Determine the number of solutions for a system of linear equations or a vector equation.

4. Compute the solution set for a system of linear equations or a vector equation with infinitely many solutions.

Readiness Assurance.

Before beginning this chapter, you should be able to...

1. Determine if a system to a two-variable system of linear equations will have zero, one, or infinitely-many solutions by graphing.
   - Review: Khan Academy[1]
2. Find the unique solution to a two-variable system of linear equations by back-substitution.
   - Review: Khan Academy[2]
3. Describe sets using set-builder notation, and check if an element is a member of a set described by set-builder notation.
   - Review: YouTube[3]

1.1 Linear Systems, Vector Equations, and Augmented Matrices (LE1)

Learning Outcomes

- Translate back and forth between a system of linear equations, a vector equation, and the corresponding augmented matrix.

[1] bit.ly/2l21etm
[2] www.khanacademy.org/math/algebra-basics/alg-basics-systems-of-equations/alg-basics-solving-systems-with-substitution/v/practice-using-substitution-for-systems
[3] youtu.be/xnfUZ-NTsCE

Linear Systems, Vector Equations, and Augmented Matrices (LE1)

Activity 1.1.1 Consider the pairs of lines described by the equations below. Decide which of these are parallel, identical, or transverse (i.e., intersect in a single point).

(a)

$$\begin{aligned} -x_1 + 3x_2 &= 1 \\ 2x_1 - 5x_2 &= 2 \end{aligned}$$

(b)

$$\begin{aligned} -x_1 + 3x_2 &= 1 \\ 2x_1 - 6x_2 &= -2 \end{aligned}$$

(c)

$$\begin{aligned} -x_1 + 3x_2 &= 1 \\ 2x_1 - 6x_2 &= 3 \end{aligned}$$

Definition 1.1.2 A **matrix** is an $m \times n$ array of real numbers with m rows and n columns:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \end{bmatrix}.$$

Frequently we will use matrices to describe an ordered list of its **column vectors**:

$$\begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}, \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}, \cdots, \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} = \vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n.$$

When order is irrelevant, we will use set notation:

$$\left\{ \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}, \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}, \cdots, \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} \right\} = \{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n\}.$$

$\Diamond$

Definition 1.1.3 A **Euclidean vector** is an ordered list of real numbers

$$\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}.$$

We will find it useful to almost always typeset Euclidean vectors vertically, but the notation $\begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix}^T$ is also valid when vertical typesetting is inconvenient. The set of all Euclidean vectors with n components is denoted as $\mathbb{R}^n$, and vectors are often described using the notation $\vec{v}$.

Each number in the list is called a **component**, and we use the following definitions for the sum of two vectors, and the product of a real number and a vector:

$$\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = \begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \\ \vdots \\ a_n + b_n \end{bmatrix} \qquad c\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} ca_1 \\ ca_2 \\ \vdots \\ ca_n \end{bmatrix}$$

◊

Example 1.1.4 Following are some examples of addition and scalar multiplication in $\mathbb{R}^4$.

$$\begin{bmatrix} 3 \\ -3 \\ 0 \\ 4 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \\ 7 \\ 1 \end{bmatrix} = \begin{bmatrix} 3+0 \\ -3+2 \\ 0+7 \\ 4+1 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 7 \\ 5 \end{bmatrix}$$

$$-4 \begin{bmatrix} 0 \\ 2 \\ -2 \\ 3 \end{bmatrix} = \begin{bmatrix} -4(0) \\ -4(2) \\ -4(-2) \\ -4(3) \end{bmatrix} = \begin{bmatrix} 0 \\ -8 \\ 8 \\ -12 \end{bmatrix}$$

□

Definition 1.1.5 A **linear equation** is an equation of the variables x_i of the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b.$$

A **solution** for a linear equation is a Euclidean vector

$$\begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix}$$

that satisfies

$$a_1s_1 + a_2s_2 + \cdots + a_ns_n = b$$

(that is, a Euclidean vector whose components can be plugged into the equation). $\Diamond$

Remark 1.1.6 In previous classes you likely used the variables x, y, z in equations. However, since this course often deals with equations of four or more variables, we will often write our variables as x_i, and assume $x = x_1, y = x_2, z = x_3, w = x_4$ when convenient.

Linear Systems, Vector Equations, and Augmented Matrices (LE1)

Definition 1.1.7 A **system of linear equations** (or a **linear system** for short) is a collection of one or more linear equations.

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\
\vdots \qquad \vdots \qquad\qquad \vdots \quad &\quad \vdots \\
a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m
\end{aligned}$$

Its **solution set** is given by

$$\left\{ \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix} \middle| \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix} \text{ is a solution to all equations in the system} \right\}.$$

$\Diamond$

Remark 1.1.8 When variables in a large linear system are missing, we prefer to write the system in one of the following standard forms:

Original linear system:	Verbose standard form:	Concise standard form:
$x_1 + 3x_3 = 3$	$1x_1 + 0x_2 + 3x_3 = 3$	$x_1 + 3x_3 = 3$
$3x_1 - 2x_2 + 4x_3 = 0$	$3x_1 - 2x_2 + 4x_3 = 0$	$3x_1 - 2x_2 + 4x_3 = 0$
$-x_2 + x_3 = -2$	$0x_1 - 1x_2 + 1x_3 = -2$	$ - x_2 + x_3 = -2$

Remark 1.1.9 It will often be convenient to think of a system of equations as a vector equation.

By applying vector operations and equating components, it is straightforward to see that the vector equation

$$x_1 \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ -2 \\ -1 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ -2 \end{bmatrix}$$

is equivalent to the system of equations

$$\begin{alignedat}{4} x_1 & & & + 3x_3 & &= & &3 \\ 3x_1 & - 2x_2 & & + 4x_3 & &= & &0 \\ & - x_2 & & + x_3 & &= & -&2 \end{alignedat}$$

Definition 1.1.10 A linear system is **consistent** if its solution set is non-empty (that is, there exists a solution for the system). Otherwise it is **inconsistent**. ◊

Linear Systems, Vector Equations, and Augmented Matrices (LE1)

Fact 1.1.11 *All linear systems are one of the following:*

1. Consistent with one solution: *its solution set contains a single vector, e.g.* $\left\{\begin{bmatrix}1\\2\\3\end{bmatrix}\right\}$

2. Consistent with infinitely-many solutions*: its solution set contains infinitely many vectors, e.g.* $\left\{\begin{bmatrix}1\\2-3a\\a\end{bmatrix}\middle|\, a\in\mathbb{R}\right\}$

3. Inconsistent*: its solution set is the empty set, denoted by either* $\{\}$ *or* $\emptyset$.

Activity 1.1.12 All inconsistent linear systems contain a logical **contradiction**. Find a contradiction in this system to show that its solution set is the empty set.

$$\begin{aligned} -x_1 + 2x_2 &= 5 \\ 2x_1 - 4x_2 &= 6 \end{aligned}$$

Activity 1.1.13 Consider the following consistent linear system.

$$\begin{aligned} -x_1 + 2x_2 &= -3 \\ 2x_1 - 4x_2 &= 6 \end{aligned}$$

(a) Find several different solutions for this system:

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix} \qquad \begin{bmatrix} ? \\ 2 \end{bmatrix} \qquad \begin{bmatrix} 0 \\ ? \end{bmatrix} \qquad \begin{bmatrix} ? \\ ? \end{bmatrix} \qquad \begin{bmatrix} ? \\ ? \end{bmatrix}$$

(b) Suppose we let $x_2 = a$ where a is an arbitrary real number. Which of these expressions for x_1 in terms of a satisfies both equations of the linear system?

A. $x_1 = -3a$

B. $x_1 = 3$

C. $x_1 = 2a + 3$

D. $x_1 = -4a + 6$

(c) Given $x_2 = a$ and the expression you found in the previous task, which of these describes the solution set for this system?

A. $\left\{ \begin{bmatrix} 2a+3 \\ a \end{bmatrix} \middle| a \in \mathbb{R} \right\}$

B. $\left\{ \begin{bmatrix} a \\ 2a+3 \end{bmatrix} \middle| a \in \mathbb{R} \right\}$

C. $\left\{ \begin{bmatrix} a \\ b \end{bmatrix} \middle| a \in \mathbb{R} \right\}$

D. $\left\{ \begin{bmatrix} 2a+3 \\ 2b-3 \end{bmatrix} \middle| a \in \mathbb{R} \right\}$

Activity 1.1.14 Consider the following linear system.

$$\begin{aligned} x_1 + 2x_2 \quad - \ x_4 &= 3 \\ x_3 + 4x_4 &= -2 \end{aligned}$$

Substitute $x_2 = a$ and $x_4 = b$, and then solve for x_1 and x_3:

$$x_1 = \ ? \qquad\qquad x_3 = \ ?$$

Then use these to describe the solution set

$$\left\{ \left[\begin{array}{c} ? \\ a \\ ? \\ b \end{array}\right] \middle| \, a, b \in \mathbb{R} \right\}$$

to the linear system.

Observation 1.1.15 Solving linear systems of two variables by graphing or substitution is reasonable for two-variable systems, but these simple techniques won't usually cut it for equations with more than two variables or more than two equations. For example,

$$\begin{aligned} -2x_1 - 4x_2 + x_3 - 4x_4 &= -8 \\ x_1 + 2x_2 + 2x_3 + 12x_4 &= -1 \\ x_1 + 2x_2 + x_3 + 8x_4 &= 1 \end{aligned}$$

has the exact same solution set as the system in the previous activity, but we'll want to learn new techniques to compute these solutions efficiently.

Remark 1.1.16 The only important information in a linear system are its coefficients and constants.

Original linear system:	Verbose standard form:	Coefficients/constants:
$\begin{aligned} x_1 + 3x_3 &= 3 \\ 3x_1 - 2x_2 + 4x_3 &= 0 \\ -x_2 + x_3 &= -2 \end{aligned}$	$\begin{aligned} 1x_1 + 0x_2 + 3x_3 &= 3 \\ 3x_1 - 2x_2 + 4x_3 &= 0 \\ 0x_1 - 1x_2 + 1x_3 &= -2 \end{aligned}$	$\begin{array}{ccc\|c} 1 & 0 & 3 & 3 \\ 3 & -2 & 4 & 0 \\ 0 & -1 & 1 & -2 \end{array}$

Linear Systems, Vector Equations, and Augmented Matrices (LE1)

Definition 1.1.17 A system of m linear equations with n variables is often represented by writing its coefficients and constants in an **augmented matrix**: the $m \times n$ matrix of its coefficients augmented with the m constant values as a final column.

$$\begin{array}{cccc} a_{11}x_1 + & a_{12}x_2 + \ldots + & a_{1n}x_n & = b_1 \\ a_{21}x_1 + & a_{22}x_2 + \ldots + & a_{2n}x_n & = b_2 \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1}x_1 + & a_{m2}x_2 + \ldots + & a_{mn}x_n & = b_m \end{array}$$

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$

Sometimes, we will find it useful to refer only to the coefficients of the linear system (and ignore its constant terms). We call the $m \times n$ array consisting of these coefficients a **coefficient matrix**.

$$\left[\begin{array}{cccc} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{array}\right]$$

$\Diamond$

Example 1.1.18 The corresponding augmented matrix for this system is obtained by simply writing the coefficients and constants in matrix form.

Linear system:

$$\begin{alignedat}{4} x_1 & & & {}+{} & 3x_3 & {}={} & 3 \\ 3x_1 & {}-{} & 2x_2 & {}+{} & 4x_3 & {}={} & 0 \\ & {}-{} & x_2 & {}+{} & x_3 & {}={} & -2 \end{alignedat}$$

Augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 3 \\ 3 & -2 & 4 & 0 \\ 0 & -1 & 1 & -2 \end{array}\right]$$

Vector equation:

$$x_1 \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ -2 \\ -1 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ -2 \end{bmatrix}$$

□

Activity 1.1.19 Consider the following augmented matrices. For each of them, decide how many variables and how many equations the corresponding linear system has.

(a)

$$\left[\begin{array}{ccc|c} 2 & 1 & 3 & 3 \\ 1 & -2 & 4 & 3 \\ 3 & -1 & 7 & -1 \end{array}\right]$$

(b)

$$\left[\begin{array}{ccc|c} 2 & 1 & 3 & 3 \\ 1 & -2 & 4 & 3 \\ 3 & -1 & 7 & -1 \\ 3 & -1 & 7 & -1 \end{array}\right]$$

(c)

$$\left[\begin{array}{ccc|c} 2 & 0 & 3 & 3 \\ 1 & 0 & 4 & 3 \\ 3 & 0 & 7 & -1 \\ 3 & 0 & 7 & -1 \end{array}\right]$$

(d)

$$\left[\begin{array}{ccc|c} 2 & 1 & 3 & 3 \\ 1 & -2 & 4 & 3 \\ 0 & 0 & 0 & 0 \\ 3 & -1 & 7 & -1 \end{array}\right]$$

1.2 Row Reduction of Matrices (LE2)

Learning Outcomes

- Explain why a matrix isn't in reduced row echelon form, and put a matrix in reduced row echelon form.

Row Reduction of Matrices (LE2)

Activity 1.2.1 Consider the following matrices:

$$A = \left[\begin{array}{ccc|c} 1 & 0 & 3 & 3 \\ 3 & -2 & 4 & 0 \\ 0 & -1 & 1 & -2 \end{array}\right], \quad B = \left[\begin{array}{ccc} 2 & 5 & 3 \\ 1 & -2 & 4 \\ 3 & -1 & 7 \end{array}\right]$$

(a) Write down a linear system whose augmented matrix is A. Can you write down another?

(b) Write down a linear system whose coefficient matrix is B. Can you write down another?

Row Reduction of Matrices (LE2)

Definition 1.2.2 Two systems of linear equations (and their corresponding augmented matrices) are said to be **equivalent** if they have the same solution set.

For example, both of these systems share the same solution set $\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$.

$$\begin{aligned} 3x_1 - 2x_2 &= 1 \\ x_1 + 4x_2 &= 5 \end{aligned} \qquad\qquad \begin{aligned} 3x_1 - 2x_2 &= 1 \\ 4x_1 + 2x_2 &= 6 \end{aligned}$$

Therefore these augmented matrices are equivalent (even though they're not *equal*), which we denote with $\sim$:

$$\left[\begin{array}{cc|c} 3 & -2 & 1 \\ 1 & 4 & 5 \end{array}\right] \neq \left[\begin{array}{cc|c} 3 & -2 & 1 \\ 4 & 2 & 6 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 3 & -2 & 1 \\ 1 & 4 & 5 \end{array}\right] \sim \left[\begin{array}{cc|c} 3 & -2 & 1 \\ 4 & 2 & 6 \end{array}\right]$$

$\Diamond$

Row Reduction of Matrices (LE2)

Activity 1.2.3 Consider whether these matrix manipulations (A) must keep the *same* solution set, or (B) might result in a *different* solution set for the corresponding linear system.

(a) Swapping two rows, for example:

$$\left[\begin{array}{cc|c} 1 & 2 & 4 \\ 1 & 3 & 5 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 3 & 5 \\ 1 & 2 & 4 \end{array}\right] \qquad \begin{aligned} x+2y&=4 \\ x+3y&=5 \end{aligned} \qquad \begin{aligned} x+3y&=5 \\ x+2y&=4 \end{aligned}$$

A. Solutions must be the *same*. B. Solutions might be *different*.

(b) Swapping two columns, for example:

$$\left[\begin{array}{cc|c} 1 & 2 & 4 \\ 1 & 3 & 5 \end{array}\right] \sim \left[\begin{array}{cc|c} 2 & 1 & 4 \\ 3 & 1 & 5 \end{array}\right] \qquad \begin{aligned} x+2y&=4 \\ x+3y&=5 \end{aligned} \qquad \begin{aligned} 2x+y&=4 \\ 3x+y&=5 \end{aligned}$$

A. Solutions must be the *same*. B. Solutions might be *different*.

(c) Add a constant to every term of a row, for example:

$$\left[\begin{array}{cc|c} 1 & 2 & 4 \\ 1 & 3 & 5 \end{array}\right] \sim \left[\begin{array}{cc|c} 1+3 & 2+3 & 4+3 \\ 1 & 3 & 5 \end{array}\right] \qquad \begin{aligned} x+2y&=4 \\ x+3y&=5 \end{aligned} \qquad \begin{aligned} 4x+5y&=7 \\ x+3y&=5 \end{aligned}$$

A. Solutions must be the *same*. B. Solutions might be *different*.

(d) Multiply a row by a nonzero constant, for example:

$$\left[\begin{array}{cc|c} 1 & 2 & 4 \\ 1 & 3 & 5 \end{array}\right] \sim \left[\begin{array}{cc|c} 3(1) & 3(2) & 3(4) \\ 1 & 3 & 5 \end{array}\right] \qquad \begin{aligned} x+2y&=4 \\ x+3y&=5 \end{aligned} \qquad \begin{aligned} 3x+6y&=12 \\ x+3y&=5 \end{aligned}$$

A. Solutions must be the *same*. B. Solutions might be *different*.

(e) Add one row to another row, for example:

$$\left[\begin{array}{cc|c} 1 & 2 & 4 \\ 1 & 3 & 5 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 2 & 4 \\ 1+1 & 3+2 & 5+4 \end{array}\right] \qquad \begin{aligned} x+2y&=4 \\ x+3y&=5 \end{aligned} \qquad \begin{aligned} ?\,x+\,?\,y&=\,? \\ ?\,x+\,?\,y&=\,? \end{aligned}$$

A. Solutions must be the *same*. B. Solutions might be *different*.

(f) Replace a column with zeros, for example:

$$\left[\begin{array}{cc|c} 1 & 2 & 4 \\ 1 & 3 & 5 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 4 \\ 1 & 0 & 5 \end{array}\right]$$

$$\begin{aligned} x + 2y &= 4 \\ x + 3y &= 5 \end{aligned} \qquad \begin{aligned} ?\,x + \,?\,y &= \,? \\ ?\,x + \,?\,y &= \,? \end{aligned}$$

A. Solutions must be the *same*.

B. Solutions might be *different*.

(g) Replace a row with zeros, for example:

$$\left[\begin{array}{cc|c} 1 & 2 & 4 \\ 1 & 3 & 5 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & 0 & 0 \end{array}\right]$$

$$\begin{aligned} x + 2y &= 4 \\ x + 3y &= 5 \end{aligned} \qquad \begin{aligned} ?\,x + \,?\,y &= \,? \\ ?\,x + \,?\,y &= \,? \end{aligned}$$

A. Solutions must be the *same*.

B. Solutions might be *different*.

Row Reduction of Matrices (LE2)

Activity 1.2.4

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How does adding row multiples to other rows affect a linear system's solution set?

A. Solutions must be the *same*.

B. Solutions might be *different*.

Row Reduction of Matrices (LE2)

Definition 1.2.5 The following three **row operations** produce equivalent augmented matrices.

1. Swap two rows, for example, $R_1 \leftrightarrow R_2$:

$$\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 4 & 5 & 6 \end{array}\right] \sim \left[\begin{array}{cc|c} 4 & 5 & 6 \\ 1 & 2 & 3 \end{array}\right]$$

2. Multiply a row by a nonzero constant, for example, $2R_1 \to R_1$:

$$\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 4 & 5 & 6 \end{array}\right] \sim \left[\begin{array}{cc|c} 2(1) & 2(2) & 2(3) \\ 4 & 5 & 6 \end{array}\right]$$

3. Add a constant multiple of one row to another row, for example, $R_2 - 4R_1 \to R_2$:

$$\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 4 & 5 & 6 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 2 & 3 \\ 4-4(1) & 5-4(2) & 6-4(3) \end{array}\right]$$

Observe that we will use the following notation: (Combination of old rows) $\to$ (New row).

$\Diamond$

Row Reduction of Matrices (LE2)

Activity 1.2.6 Each of the following linear systems has the same solution set.

A)

$$\begin{alignedat}{4} x &+ 2y &+ z &= 3 \\ -x &- y &+ z &= 1 \\ 2x &+ 5y &+ 3z &= 7 \end{alignedat}$$

B)

$$\begin{alignedat}{4} 2x &+ 5y &+ 3z &= 7 \\ -x &- y &+ z &= 1 \\ x &+ 2y &+ z &= 3 \end{alignedat}$$

C)

$$\begin{alignedat}{4} x & & - z &= 1 \\ & y &+ 2z &= 4 \\ & y &+ z &= 1 \end{alignedat}$$

D)

$$\begin{alignedat}{4} x &+ 2y &+ z &= 3 \\ & y &+ 2z &= 4 \\ 2x &+ 5y &+ 3z &= 7 \end{alignedat}$$

E)

$$\begin{alignedat}{4} x & & - z &= 1 \\ & y &+ 2z &= 4 \\ & & z &= 3 \end{alignedat}$$

F)

$$\begin{alignedat}{4} x &+ 2y &+ z &= 3 \\ & y &+ 2z &= 4 \\ & y &+ z &= 1 \end{alignedat}$$

Sort these six equivalent linear systems from most complicated to simplest (in your opinion).

Row Reduction of Matrices (LE2)

Activity 1.2.7 Here we've written the sorted linear systems from Activity 1.2.6 as augmented matrices.

$$\left[\begin{array}{ccc|c} 2 & 5 & 3 & 7 \\ -1 & -1 & 1 & 1 \\ 1 & 2 & 1 & 3 \end{array}\right] \sim \left[\begin{array}{ccc|c} \boxed{1} & 2 & 1 & 3 \\ -1 & -1 & 1 & 1 \\ 2 & 5 & 3 & 7 \end{array}\right] \sim \left[\begin{array}{ccc|c} \boxed{1} & 2 & 1 & 3 \\ 0 & 1 & 2 & 4 \\ 2 & 5 & 3 & 7 \end{array}\right] \sim$$

$$\sim \left[\begin{array}{ccc|c} \boxed{1} & 2 & 1 & 3 \\ 0 & \boxed{1} & 2 & 4 \\ 0 & 1 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} \boxed{1} & 0 & -1 & 1 \\ 0 & \boxed{1} & 2 & 4 \\ 0 & 1 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} \boxed{1} & 0 & -1 & 1 \\ 0 & \boxed{1} & 2 & 4 \\ 0 & 0 & -1 & -3 \end{array}\right]$$

Assign the following row operations to each step used to manipulate each matrix to the next:

$R_1 \leftrightarrow R_3$

$R_3 - 1R_2 \to R_3$ $\qquad$ $R_2 + 1R_1 \to R_2$

$R_3 - 2R_1 \to R_3$ $\qquad$ $R_1 - 2R_3 \to R_1$

Row Reduction of Matrices (LE2)

Definition 1.2.8 A matrix is in **reduced row echelon form** (**RREF**) if

1. The leftmost nonzero term of each row is 1. We call these terms **pivots**.
2. Each pivot is to the right of every higher pivot.
3. Each term that is either above or below a pivot is 0.
4. All zero rows (rows whose terms are all 0) are at the bottom of the matrix.

Every matrix has a unique reduced row echelon form. If A is a matrix, we write RREF(A) for the reduced row echelon form of that matrix. $\Diamond$

Row Reduction of Matrices (LE2)

Activity 1.2.9 Recall that a matrix is in **reduced row echelon form** (**RREF**) if

1. The leftmost nonzero term of each row is 1. We call these terms **pivots**.
2. Each pivot is to the right of every higher pivot.
3. Each term that is either above or below a pivot is 0.
4. All zero rows (rows whose terms are all 0) are at the bottom of the matrix.

For each matrix, mark the leading terms, and label it as RREF or not RREF. For the ones not in RREF, determine which rule is violated and how it might be fixed.

$$A = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad B = \left[\begin{array}{ccc|c} 1 & 0 & 4 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right] \qquad C = \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

Row Reduction of Matrices (LE2)

Activity 1.2.10 Recall that a matrix is in **reduced row echelon form** (**RREF**) if

1. The leftmost nonzero term of each row is 1. We call these terms **pivots**.
2. Each pivot is to the right of every higher pivot.
3. Each term that is either above or below a pivot is 0.
4. All zero rows (rows whose terms are all 0) are at the bottom of the matrix.

For each matrix, mark the leading terms, and label it as RREF or not RREF. For the ones not in RREF, determine which rule is violated and how it might be fixed.

$$D = \left[\begin{array}{ccc|c} 1 & 0 & 2 & -3 \\ 0 & 3 & 3 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad E = \left[\begin{array}{ccc|c} 0 & 1 & 0 & 7 \\ 1 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad F = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

Remark 1.2.11 In practice, if we simply need to convert a matrix into reduced row echelon form, we use technology to do so.

However, it is also important to understand the **Gauss-Jordan elimination** algorithm that a computer or calculator uses to convert a matrix (augmented or not) into reduced row echelon form. Understanding this algorithm will help us better understand how to interpret the results in many applications we use it for in Chapter 2.

Row Reduction of Matrices (LE2)

Activity 1.2.12

(a) Consider the matrix:

$$\begin{bmatrix} 2 & 6 & -1 & 6 \\ 1 & 3 & -1 & 2 \\ -1 & -3 & 2 & 0 \end{bmatrix}.$$

Which row operation is the best choice for the first move in converting to RREF?

A. Add row 3 to row 2 ($R_2 + R_3 \to R_2$)

B. Add row 2 to row 3 ($R_3 + R_2 \to R_3$)

C. Swap row 1 to row 2 ($R_1 \leftrightarrow R_2$)

D. Add -2 row 2 to row 1 ($R_1 - 2R_2 \to R_1$)

(b) Consider the matrix:

$$\begin{bmatrix} \boxed{1} & 3 & -1 & 2 \\ 2 & 6 & -1 & 6 \\ -1 & -3 & 2 & 0 \end{bmatrix}.$$

Which row operation is the best choice for the next move in converting to RREF?

A. Add row 1 to row 3 ($R_3 + R_1 \to R_3$)

B. Add -2 row 1 to row 2 ($R_2 - 2R_1 \to R_2$)

C. Add 2 row 2 to row 3 ($R_3 + 2R_2 \to R_3$)

D. Add 2 row 3 to row 2 ($R_2 + 2R_3 \to R_2$)

(c) Consider the matrix:

$$\begin{bmatrix} \boxed{1} & 3 & -1 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 \end{bmatrix}.$$

Which row operation is the best choice for the next move in converting to RREF?

A. Add row 1 to row 2 ($R_2 + R_1 \to R_2$)

B. Add -1 row 3 to row 2 ($R_2 - R_3 \to R_2$)

C. Add -1 row 2 to row 3 ($R_3 - R_2 \to R_3$)

D. Add row 2 to row 1 ($R_1 + R_2 \to R_1$)

(d) Consider the matrix:

$$\begin{bmatrix} 2 & 6 & -1 & 6 \\ 1 & 3 & -1 & 2 \\ -1 & -3 & 2 & 0 \end{bmatrix}.$$

Mark the position where the first pivot should be. Which row operation is the best choice for the first move in converting to RREF?

A. Add row 3 to row 2 ($R_2 + R_3 \to R_2$)

B. Add row 2 to row 3 ($R_3 + R_2 \to R_3$)

C. Swap row 1 to row 2 ($R_1 \leftrightarrow R_2$)

D. Add -2 row 2 to row 1 ($R_1 - 2R_2 \to R_1$)

Observation 1.2.13 The steps for the Gauss-Jordan elimination algorithm may be summarized as follows:

1. Ignoring any rows that already have marked pivots, identify the leftmost column with a nonzero entry.

2. Use row operations to obtain a pivot of value 1 in the topmost row that does not already have a marked pivot.

3. Mark this pivot, then use row operations to change all values above and below the marked pivot to 0.

4. Repeat these steps until the matrix is in RREF.

In particular, *once a pivot is marked, it should remain in the same position.* This will keep you from undoing your progress towards an RREF matrix.

Row Reduction of Matrices (LE2)

Activity 1.2.14 Complete the following RREF calculation (multiple row operations may be needed for certain steps):

$$A = \begin{bmatrix} 2 & 3 & 2 & 3 \\ -2 & 1 & 6 & 1 \\ -1 & -3 & -4 & 1 \end{bmatrix} \sim \begin{bmatrix} \boxed{1} & ? & ? & ? \\ -2 & 1 & 6 & 1 \\ -1 & -3 & -4 & 1 \end{bmatrix} \sim \begin{bmatrix} \boxed{1} & ? & ? & ? \\ 0 & ? & ? & ? \\ 0 & ? & ? & ? \end{bmatrix}$$

$$\sim \begin{bmatrix} \boxed{1} & ? & ? & ? \\ 0 & \boxed{1} & ? & ? \\ 0 & ? & ? & ? \end{bmatrix} \sim \begin{bmatrix} \boxed{1} & 0 & ? & ? \\ 0 & \boxed{1} & ? & ? \\ 0 & 0 & ? & ? \end{bmatrix} \sim \cdots \sim \begin{bmatrix} \boxed{1} & 0 & -2 & 0 \\ 0 & \boxed{1} & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Row Reduction of Matrices (LE2)

Activity 1.2.15 Consider the matrix

$$A = \begin{bmatrix} 2 & 4 & 2 & -4 \\ -2 & -4 & 1 & 1 \\ 3 & 6 & -1 & -4 \end{bmatrix}.$$

Compute $\operatorname{RREF}(A)$.

Row Reduction of Matrices (LE2)

Activity 1.2.16 Consider the non-augmented and augmented matrices

$$A = \left[\begin{array}{cccc} 2 & 4 & 2 & -4 \\ -2 & -4 & 1 & 1 \\ 3 & 6 & -1 & -4 \end{array}\right] \qquad B = \left[\begin{array}{ccc|c} 2 & 4 & 2 & -4 \\ -2 & -4 & 1 & 1 \\ 3 & 6 & -1 & -4 \end{array}\right].$$

Can RREF(A) be used to find RREF(B)?

A. Yes, RREF(A) and RREF(B) are exactly the same.

B. Yes, RREF(A) may be slightly modified to find RREF(B).

C. No, a new calculation is required.

Row Reduction of Matrices (LE2)

Activity 1.2.17 Free browser-based technologies for mathematical computation are available online.

(a) Go to https://sagecell.sagemath.org/.

(b) In the dropdown on the right, you can select a number of different languages. Select "Octave" for the Matlab-compatible syntax used by this text.

(c) Type `rref([1,4,6;2,5,7])` and then press the Evaluate button to compute the RREF of $\begin{bmatrix} 1 & 4 & 6 \\ 2 & 5 & 7 \end{bmatrix}$.

(d) Now try using whitespace to write out the matrix and compute RREF instead:

```
A = [1 3 2
     2 5 7]

rref(A)
```

Row Reduction of Matrices (LE2)

Activity 1.2.18 Find three examples of linear systems for which the RREF of their augmented matrices is equal to

$$\left[\begin{array}{ccc|c} 1 & 4 & 2 & -4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Row Reduction of Matrices (LE2)

Activity 1.2.19 Which of the following matrices are not in RREF?

$$A = \left[\begin{array}{ccc|c} 1 & 0 & 2 & -3 \\ 0 & 3 & 3 & -3 \\ 0 & 0 & 0 & 1 \end{array}\right] \qquad B = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right] \qquad C = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 0 & 4 \end{array}\right]$$

1.3 Counting Solutions for Linear Systems (LE3)

Learning Outcomes

- Determine the number of solutions for a system of linear equations or a vector equation.

Counting Solutions for Linear Systems (LE3)

Activity 1.3.1

(a) Without referring to your Activity Book, which of the four criteria for a matrix to be in Reduced Row Echelon Form (RREF) can you recall?

(b) Which, if any, of the following matrices are in RREF? You may refer to the Activity Book now for criteria that you may have forgotten.

$$P = \left[\begin{array}{ccc|c} 1 & 0 & \frac{2}{3} & -3 \\ 0 & 3 & 3 & -\frac{3}{5} \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad Q = \left[\begin{array}{ccc|c} 0 & 1 & 0 & 7 \\ 1 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad R = \left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{2} & 4 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

Remark 1.3.2 We will frequently need to know the reduced row echelon form of matrices during the remainder of this course, so unless you're told otherwise, feel free to use technology (see Activity 1.2.17) to compute RREFs efficiently.

Counting Solutions for Linear Systems (LE3)

Activity 1.3.3 Consider the following system of equations.

$$\begin{aligned} 3x_1 - 2x_2 + 13x_3 &= 6 \\ 2x_1 - 2x_2 + 10x_3 &= 2 \\ -x_1 + 3x_2 - 6x_3 &= 11 \\ 4x_1 + x_2 + x_3 &= 1. \end{aligned}$$

(a) Convert this to an augmented matrix and use technology to compute its reduced row echelon form:

$$\text{RREF}\left[\begin{array}{ccc|c} ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \end{array}\right] = \left[\begin{array}{ccc|c} ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \end{array}\right]$$

(b) Use the RREF matrix to write a linear system equivalent to the original system.

(c) How many solutions must this system have?

A. Zero   B. Only one   C. Infinitely-many

Counting Solutions for Linear Systems (LE3)

Activity 1.3.4 Consider the vector equation

$$x_1 \begin{bmatrix} 3 \\ 2 \\ -1 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ -2 \\ 0 \\ 7 \end{bmatrix} + x_3 \begin{bmatrix} 13 \\ 10 \\ -3 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \\ 1 \\ -2 \end{bmatrix}$$

(a) Convert this to an augmented matrix and use technology to compute its reduced row echelon form:

$$\text{RREF} \left[\begin{array}{ccc|c} ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \end{array}\right] = \left[\begin{array}{ccc|c} ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \end{array}\right]$$

(b) Use the RREF matrix to write a linear system equivalent to the original system.

(c) How many solutions must this system have?

A. Zero

B. Only one

C. Infinitely-many

Counting Solutions for Linear Systems (LE3)

Activity 1.3.5 What contradictory equations besides $0 = 1$ may be obtained from the RREF of an augmented matrix?

A. $x = 0$ is an obtainable contradiction

B. $x = y$ is an obtainable contradiction

C. $0 = 17$ is an obtainable contradiction

D. $0 = 1$ is the only obtainable contradiction

Counting Solutions for Linear Systems (LE3)

Activity 1.3.6 Consider the following linear system.

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 1 \\ 2x_1 + 4x_2 + 8x_3 &= 0 \\ 3x_1 + 6x_2 + 11x_3 &= 1 \\ x_1 + 2x_2 + 5x_3 &= -1 \end{aligned}$$

(a) Find its corresponding augmented matrix A and find RREF(A).

(b) Use the RREF matrix to write a linear system equivalent to the original system.

(c) How many solutions must this system have?

A. Zero B. One C. Infinitely-many

Counting Solutions for Linear Systems (LE3)

Fact 1.3.7 *By finding* RREF(A) *from a linear system's corresponding augmented matrix A, we can immediately tell how many solutions the system has.*

- *If the linear system given by* RREF(A) *includes the contradiction*

  $$0 = 1,$$

  that is, the RREF *matrix includes the row*

  $$\left[\begin{array}{ccc|c} 0 & \cdots & 0 & 1 \end{array}\right],$$

  then the system is inconsistent*, which means it has* zero *solutions and we may write*

  $$\textit{Solution set } = \{\} \qquad \textit{or} \qquad \textit{Solution set } = \emptyset.$$

- *If the linear system given by* RREF(A) *sets every variable of the system to a specific value; that is we have:*

  $$\begin{aligned} x_1 &= s_1 \\ x_2 &= s_2 \\ &\vdots \\ x_n &= s_n \end{aligned}$$

  (with some possible extra $0 = 0$ equations), then the system is consistent *with exactly* one *solution, and we may write*

  $$\textit{Solution } = \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix} \qquad \textit{but} \qquad \textit{Solution set } = \left\{ \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix} \right\}.$$

- *Otherwise, the system given by the* RREF *matrix must* not include a $0 = 1$ contradiction *while* at least one variable is not required to equal a specific value. *This means it is* consistent *with* infinitely-many *different solutions. We'll learn how to find such solution sets in* *Section 1.4*.

Counting Solutions for Linear Systems (LE3)

Activity 1.3.8 Consider each of the following systems of linear equations or vector equations.

(a)

$$\begin{alignedat}{4} x_1 &\; - \;& x_2 &\; - \;& 3\,x_3 &\; = \;& 8 \\ 3\,x_1 &\; - \;& 2\,x_2 &\; - \;& 5\,x_3 &\; = \;& 17 \\ x_1 &\; - \;& x_2 &\; - \;& 2\,x_3 &\; = \;& 7 \\ 10\,x_1 &\; - \;& 8\,x_2 &\; - \;& 21\,x_3 &\; = \;& 65 \end{alignedat}$$

(i) Explain and demonstrate how to find a simpler linear system that has the same solution set.

(ii) Explain whether this solution set has no solutions, one solution, or infinitely-many solutions. If the set is finite, describe it using set notation.

(b)

$$\begin{alignedat}{4} x_1 &\; - \;& 5\,x_2 &\; - \;& 15\,x_3 &\; = \;& -8 \\ & & x_2 &\; + \;& 3\,x_3 &\; = \;& 1 \\ x_1 & & & & &\; = \;& 2 \\ 5\,x_1 &\; - \;& 7\,x_2 &\; - \;& 21\,x_3 &\; = \;& -10 \end{alignedat}$$

(i) Explain and demonstrate how to find a simpler linear system that has the same solution set.

(ii) Explain whether this solution set has no solutions, one solution, or infinitely-many solutions. If the set is finite, describe it using set notation.

(c)

$$\begin{alignedat}{4} -2\,x_1 &\; + \;& 2\,x_2 &\; + \;& 5\,x_3 &\; = \;& 1 \\ -x_1 &\; + \;& x_2 &\; + \;& 2\,x_3 &\; = \;& 1 \\ 2\,x_1 &\; - \;& 2\,x_2 &\; + \;& x_3 &\; = \;& -7 \\ -2\,x_1 &\; + \;& 2\,x_2 &\; + \;& 16\,x_3 &\; = \;& -10 \end{alignedat}$$

(i) Explain and demonstrate how to find a simpler linear system that has the same solution set.

(ii) Explain whether this solution set has no solutions, one solution, or infinitely-many solutions. If the set is finite, describe it using set notation.

Counting Solutions for Linear Systems (LE3)

Activity 1.3.9

(a) In Fact 1.1.11, we stated, but did not prove the assertion that all linear systems are one of the following:

(a) *Consistent with one solution:* its solution set contains a single vector, e.g. $\left\{\begin{bmatrix}1\\2\\3\end{bmatrix}\right\}$

(b) *Consistent with infinitely-many solutions*: its solution set contains infinitely many vectors, e.g. $\left\{\begin{bmatrix}1\\2-3a\\a\end{bmatrix} \middle| a\in\mathbb{R}\right\}$

(c) *Inconsistent*: its solution set is the empty set, denoted by either $\{\}$ or $\emptyset$.

(b) Explain why this fact is a consequence of Fact 1.3.7 above.

1.4 Linear Systems with Infinitely-Many Solutions (LE4)

Learning Outcomes

- Compute the solution set for a system of linear equations or a vector equation with infinitely many solutions.

Linear Systems with Infinitely-Many Solutions (LE4)

Activity 1.4.1 Write down any three linear systems and determine if they are consistent, have a single solution, or have infinitely many solutions.

Activity 1.4.2 Consider this simplified linear system found to be equivalent to the system from Activity 1.3.6:

$$\begin{aligned} x_1 + 2x_2 \quad\quad &= 4 \\ x_3 &= -1 \\ 0 &= 0 \\ 0 &= 0 \end{aligned}$$

Earlier, we determined this system has infinitely-many solutions, since x_1 and x_2 are not required by the RREF matrix to equal specific values (even though x_3 is).

(a) Let $x_1 = a$ and write the solution set in the form $\left\{ \left[\begin{array}{c} a \\ ? \\ ? \end{array}\right] \middle| \, a \in \mathbb{R} \right\}$.

(b) Let $x_2 = b$ and write the solution set in the form $\left\{ \left[\begin{array}{c} ? \\ b \\ ? \end{array}\right] \middle| \, b \in \mathbb{R} \right\}$.

(c) Which of these was easier? What features of the RREF matrix $\left[\begin{array}{ccc|c} \boxed{1} & 2 & 0 & 4 \\ 0 & 0 & \boxed{1} & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ caused this?

Linear Systems with Infinitely-Many Solutions (LE4)

Definition 1.4.3 Recall that the pivots of a matrix in RREF form are the leading 1s in each non-zero row.

The pivot columns in an augmented matrix correspond to the **bound variables** in the system of equations (x_1, x_3 below). The remaining variables are called **free variables** (x_2 below).

$$\left[\begin{array}{ccc|c} \boxed{1} & 2 & 0 & 4 \\ 0 & 0 & \boxed{1} & -1 \end{array}\right]$$

To efficiently solve a system in RREF form, assign letters to the free variables, and then solve for the bound variables. $\Diamond$

Linear Systems with Infinitely-Many Solutions (LE4)

Activity 1.4.4 Find the solution set for the system

$$\begin{alignedat}{6} 2x_1 &-{}& 2x_2 &-{}& 6x_3 &+{}& x_4 &-{}& x_5 &={}& 3 \\ -x_1 &+{}& x_2 &+{}& 3x_3 &-{}& x_4 &+{}& 2x_5 &={}& -3 \\ x_1 &-{}& 2x_2 &-{}& x_3 &+{}& x_4 &+{}& x_5 &={}& 2 \end{alignedat}$$

by doing the following.

(a) Row-reduce its augmented matrix.

(b) Assign letters to the free variables (given by the non-pivot columns):

$$? = a$$

$$? = b$$

(c) Solve for the bound variables (given by the pivot columns) to show that

$$? = 1 + 5a + 2b$$

$$? = 1 + 2a + 3b$$

$$? = 3 + 3b$$

(d) Replace x_1 through x_5 with the appropriate expressions of a, b in the following set-builder notation.

$$\left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} \middle| \, a, b \in \mathbb{R} \right\}$$

Linear Systems with Infinitely-Many Solutions (LE4)

Remark 1.4.5 Don't forget to correctly express the solution set of a linear system. Systems with zero or one solutions may be written by listing their elements, while systems with infinitely-many solutions may be written using set-builder notation.

- *Inconsistent*: $\emptyset$ or $\{\}$
  - (not 0 or $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$)
- *Consistent* with *one solution*: e.g. $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\}$
  - (not just $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$)
- *Consistent* with *infinitely-many solutions*: e.g. $\left\{ \begin{bmatrix} 1 \\ 2-3a \\ a \end{bmatrix} \middle| a \in \mathbb{R} \right\}$
  - (not just $\begin{bmatrix} 1 \\ 2-3a \\ a \end{bmatrix}$)

Linear Systems with Infinitely-Many Solutions (LE4)

Activity 1.4.6 Consider the following system of linear equations.

$$x_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 5 \\ -5 \end{bmatrix} + x_4 \begin{bmatrix} -3 \\ 13 \\ -13 \end{bmatrix} = \begin{bmatrix} -3 \\ 12 \\ -12 \end{bmatrix}.$$

(a) Explain how to find a simpler system or vector equation that has the same solution set.

(b) Explain how to describe this solution set using set notation.

Linear Systems with Infinitely-Many Solutions (LE4)

Activity 1.4.7 Consider the following system of linear equations.

$$\begin{alignedat}{4} x_1 & & & {}-{} & 2\,x_3 & {}={} & -3 \\ 5\,x_1 & {}+{} & x_2 & {}-{} & 7\,x_3 & {}={} & -18 \\ 5\,x_1 & {}-{} & x_2 & {}-{} & 13\,x_3 & {}={} & -12 \\ x_1 & {}+{} & 3\,x_2 & {}+{} & 7\,x_3 & {}={} & -12 \end{alignedat}$$

(a) Explain how to find a simpler system or vector equation that has the same solution set.

(b) Explain how to describe this solution set using set notation.

Linear Systems with Infinitely-Many Solutions (LE4)

Activity 1.4.8 Consider the following linear system, its augmented matrix A, and $\operatorname{RREF}(A)$:

$$\begin{alignedat}{4} x_1 & {}-{} & x_2 & {}+{} & x_3 & {}={} & 4 \\ & & x_2 & {}-{} & 2\,x_3 & {}={} & -1 \\ & & x_2 & {}-{} & 2\,x_3 & {}={} & -3 \\ x_1 & {}+{} & 2\,x_2 & {}-{} & 5\,x_3 & {}={} & 0 \end{alignedat}$$

$$A = \left[\begin{array}{ccc|c} 1 & -1 & 1 & 4 \\ 0 & 1 & -2 & -1 \\ 0 & 1 & -2 & -3 \\ 1 & 2 & -5 & 0 \end{array}\right], \quad \operatorname{RREF}(A) = \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

All of the following statements are not accurate or otherwise incorrect; identify what is problematic about the statements and correct them.

(a) The matrix A is inconsistent.

(b) The linear system has two bound variables and one free variable.

(c) The solution set to the given linear system is $\{\emptyset\}$.

Activity 1.4.9 Consider the following linear system, its augmented matrix B, and RREF(B):

$$\begin{alignedat}{6} 2\,x_1 &\;-\;& 2\,x_2 &\;-\;& 8\,x_3 &\;+\;& 3\,x_4 &\;-\;& 9\,x_5 &\;=\;& -17 \\ -x_1 && &\;+\;& x_3 &\;-\;& x_4 &\;+\;& 2\,x_5 &\;=\;& 6 \\ 2\,x_1 &\;-\;& x_2 &\;-\;& 5\,x_3 &\;+\;& x_4 &\;-\;& 5\,x_5 &\;=\;& -10 \\ -x_1 &\;+\;& 3\,x_2 &\;+\;& 10\,x_3 && &\;+\;& 7\,x_5 &\;=\;& 6 \end{alignedat}$$

$$B = \left[\begin{array}{ccccc|c} 2 & -2 & -8 & 3 & -9 & -17 \\ -1 & 0 & 1 & -1 & 2 & 6 \\ 2 & -1 & -5 & 1 & -5 & -10 \\ -1 & 3 & 10 & 0 & 7 & 6 \end{array}\right]$$

$$\text{RREF}(B) = \left[\begin{array}{ccccc|c} 1 & 0 & -1 & 0 & -1 & -3 \\ 0 & 1 & 3 & 0 & 2 & 1 \\ 0 & 0 & 0 & 1 & -1 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

All of the following statements are not accurate or otherwise incorrect; identify what is problematic about the statements and correct them.

(a) The matrix B is consistent with infinitely many solutions.

(b) The solution set is given by $\left[\begin{array}{c} a+b-3 \\ -3\,a-2\,b+1 \\ a \\ b-3 \\ b \end{array}\right]$.

(c) The variables x_3, x_5 are free. Setting them equal to a, b respectively and solving for the bound variables, the solution set to the linear system is given by $\left\{ \left[\begin{array}{c} a+b-3 \\ -3\,a-2\,b+1 \\ b-3 \end{array}\right] \middle|\, a, b \in \mathbb{R} \right\}$.

Chapter 2

Euclidean Vectors (EV)

Learning Outcomes

What is a space of Euclidean vectors?
By the end of this chapter, you should be able to...

1. Determine if a Euclidean vector can be written as a linear combination of a given set of Euclidean vectors by solving an appropriate vector equation.

2. Determine if a set of Euclidean vectors spans $\mathbb{R}^m$ by solving appropriate vector equations.

3. Determine if a subset of $\mathbb{R}^n$ is a subspace or not.

4. Determine if a set of Euclidean vectors is linearly dependent or independent by solving an appropriate vector equation.

5. Explain why a set of Euclidean vectors is or is not a basis of $\mathbb{R}^n$.

6. Compute a basis for the subspace spanned by a given set of Euclidean vectors, and determine the dimension of the subspace.

7. Find a basis for the solution set of a homogeneous system of equations.

Readiness Assurance.

Before beginning this chapter, you should be able to...

1. Use set builder notation to describe sets of vectors.
   - Review: YouTube[1]
2. Add Euclidean vectors and multiply Euclidean vectors by scalars.
   - Review: Khan Academy (1)[2] (2)[3]
3. Perform basic manipulations of augmented matrices and linear systems.
   - Review: Section 1.1, Section 1.2, Section 1.3

2.1 Linear Combinations (EV1)

Learning Outcomes

- Determine if a Euclidean vector can be written as a linear combination of a given set of Euclidean vectors by solving an appropriate vector equation.

[1] youtu.be/xnfUZ-NTsCE

[2] www.khanacademy.org/math/linear-algebra/vectors-and-spaces/vectors/v/adding-vectors

[3] www.khanacademy.org/math/linear-algebra/vectors-and-spaces/vectors/v/multiplying-vector-by-scalar

Linear Combinations (EV1)

Activity 2.1.1 Discuss which of the vectors $\vec{u} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 0 \\ 3 \\ -1 \end{bmatrix}$ is a solution to the given vector equation:

$$x_1 \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 5 \end{bmatrix}$$

Note 2.1.2 We've been working with **Euclidean vector spaces** of the form

$$\mathbb{R}^n = \left\{ \left[\begin{array}{c} x_1 \\ x_2 \\ \vdots \\ x_n \end{array}\right] \middle|\, x_1, x_2, \ldots, x_n \in \mathbb{R} \right\}.$$

There are other kinds of **vector spaces** as well (e.g. polynomials, matrices), which we will investigate in Section 3.5. But understanding the structure of *Euclidean* vectors on their own will be beneficial, even when we turn our attention to other kinds of vectors.

We will use the phrase **vector space** freely from this point on, even while delaying a formal definition. Readers can choose to interpret this to mean *Euclidean vector space*, i.e $\mathbb{R}^n$ for some n, if they wish; we do this as all of the statements we make using the term **vector space** are also true for all vector spaces as defined in Definition 3.5.7.

Likewise, when we multiply a vector by a real number, as in $-3\left[\begin{array}{c} 1 \\ -1 \\ 2 \end{array}\right] = \left[\begin{array}{c} -3 \\ 3 \\ -6 \end{array}\right]$, we refer to this real number as a **scalar**.

We often use letters like V and W to refer to vector spaces (Euclidean or otherwise)

Linear Combinations (EV1)

Definition 2.1.3 A **linear combination** of a set of vectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is given by $c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n$ for any choice of scalar multiples $c_1, c_2, \ldots, c_n$.

For example, we can say $\begin{bmatrix} 3 \\ 0 \\ 5 \end{bmatrix}$ is a linear combination of the vectors $\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ since

$$\begin{bmatrix} 3 \\ 0 \\ 5 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + 1 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}.$$

$\Diamond$

Definition 2.1.4 The **span** of a set of vectors is the collection of all linear combinations of that set:

$$\text{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\} = \{c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n \,|\, c_i \in \mathbb{R}\}\,.$$

For example:

$$\text{span}\left\{\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}\right\} = \left\{a\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + b\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \,\middle|\, a, b \in \mathbb{R}\right\}.$$

$\Diamond$

Activity 2.1.5 Consider $\operatorname{span}\left\{\begin{bmatrix}1\\2\end{bmatrix}\right\}$.

(a) Sketch the four Euclidean vectors

$$1\begin{bmatrix}1\\2\end{bmatrix}=\begin{bmatrix}1\\2\end{bmatrix},\quad 3\begin{bmatrix}1\\2\end{bmatrix}=\begin{bmatrix}3\\6\end{bmatrix},\quad 0\begin{bmatrix}1\\2\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix},\quad -2\begin{bmatrix}1\\2\end{bmatrix}=\begin{bmatrix}-2\\-4\end{bmatrix}$$

in the same xy plane by drawing an arrow to the (x, y) coordinate associated with each vector.

(b) Sketch a representation of all the vectors belonging to

$$\operatorname{span}\left\{\begin{bmatrix}1\\2\end{bmatrix}\right\}=\left\{a\begin{bmatrix}1\\2\end{bmatrix}\middle|\, a\in\mathbb{R}\right\}$$

in the xy plane. Which of the following geometrical objects best describes this sketch?

A. A line

B. A plane

C. A parabola

D. A circle

Linear Combinations (EV1)

Activity 2.1.6 Consider $\operatorname{span}\left\{\begin{bmatrix}1\\2\end{bmatrix},\begin{bmatrix}-1\\1\end{bmatrix}\right\}$.

(a) Sketch the following five Euclidean vectors in the same xy plane.

$$1\begin{bmatrix}1\\2\end{bmatrix}+0\begin{bmatrix}-1\\1\end{bmatrix}=? \qquad 0\begin{bmatrix}1\\2\end{bmatrix}+1\begin{bmatrix}-1\\1\end{bmatrix}=? \qquad 1\begin{bmatrix}1\\2\end{bmatrix}+1\begin{bmatrix}-1\\1\end{bmatrix}=?$$

$$-2\begin{bmatrix}1\\2\end{bmatrix}+1\begin{bmatrix}-1\\1\end{bmatrix}=? \qquad -1\begin{bmatrix}1\\2\end{bmatrix}+-2\begin{bmatrix}-1\\1\end{bmatrix}=?$$

(b) Correct the SageMath code cell below to generate an illustration of several vectors belonging to

$$\operatorname{span}\left\{\begin{bmatrix}1\\2\end{bmatrix},\begin{bmatrix}-1\\1\end{bmatrix}\right\}=\left\{a\begin{bmatrix}1\\2\end{bmatrix}+b\begin{bmatrix}-1\\1\end{bmatrix}\middle|\, a,b\in\mathbb{R}\right\}$$

in the xy plane.

```
# create empty plot
p = plot([])

# do this 100 times
for _ in range(100):
    # pick random a value from -99 to 99
    a = randrange(-99,100)
    # pick random b value from -99 to 99
    b = randrange(-99,100)
    # plot random linear combination of two vectors based on a,b
    p += plot(a*vector([1,2])+b*vector([FIXME]))

# display plot
show(p)
```

Based on this illustration, which of these geometrical objects best describes the span of these two vectors?

A. A line B. A plane C. A parabola D. A circle

Linear Combinations (EV1)

Activity 2.1.7 Sketch a representation of all the vectors belonging to $\operatorname{span}\left\{\begin{bmatrix} 6 \\ -4 \end{bmatrix}, \begin{bmatrix} -3 \\ 2 \end{bmatrix}\right\}$ in the xy plane, or adapt the code in the previous activity to illustrate this span.

Which of these geometrical objects best describes the span of these two vectors?

A. A line

B. A plane

C. A parabola

D. A cube

Linear Combinations (EV1)

Activity 2.1.8 Consider the following questions to discover whether a Euclidean vector belongs to a span.

(a) The Euclidean vector $\begin{bmatrix} -1 \\ -6 \\ 1 \end{bmatrix}$ belongs to span $\left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} \right\}$ exactly when there exists a solution to which of these vector equations?

A. $x_1 \begin{bmatrix} -1 \\ -6 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix} = \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix}$

B. $x_1 \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ -6 \\ 1 \end{bmatrix}$

C. $x_1 \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ -6 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix} = 0$

(b) Use technology to find RREF of the corresponding augmented matrix, and then use that matrix to find the solution set of the vector equation.

(c) Given this solution set, does $\begin{bmatrix} -1 \\ -6 \\ 1 \end{bmatrix}$ belong to span $\left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} \right\}$?

Linear Combinations (EV1)

Observation 2.1.9 The following are all equivalent statements:

- The vector $\vec{b}$ belongs to $\operatorname{span}\{\vec{v}_1, \dots, \vec{v}_n\}$.
- The vector $\vec{b}$ is a linear combination of the vectors $\vec{v}_1, \dots, \vec{v}_n$.
- The vector equation $x_1\vec{v}_1 + \dots + x_n\vec{v}_n = \vec{b}$ is consistent.
- The linear system corresponding to $\left[\vec{v}_1 \ \dots \ \vec{v}_n \,|\, \vec{b}\right]$ is consistent.
- RREF $\left[\vec{v}_1 \ \dots \ \vec{v}_n \,|\, \vec{b}\right]$ doesn't have a row $[0 \cdots 0 \,|\, 1]$ representing the contradiction $0 = 1$.

Linear Combinations (EV1)

Activity 2.1.10 Consider the following claim:

$\left[\begin{array}{c}-6\\2\\-6\end{array}\right]$ is a linear combination of the vectors $\left[\begin{array}{c}1\\0\\2\end{array}\right]$, $\left[\begin{array}{c}3\\0\\6\end{array}\right]$, $\left[\begin{array}{c}2\\0\\4\end{array}\right]$, and $\left[\begin{array}{c}-4\\1\\-5\end{array}\right]$.

(a) Write a statement involving the solutions of a vector equation that's equivalent to this claim.

(b) Explain why the statement you wrote is true.

(c) Since your statement was true, use the solution set to describe a linear combination of $\left[\begin{array}{c}1\\0\\2\end{array}\right]$, $\left[\begin{array}{c}3\\0\\6\end{array}\right]$, $\left[\begin{array}{c}2\\0\\4\end{array}\right]$, and $\left[\begin{array}{c}-4\\1\\-5\end{array}\right]$ that equals $\left[\begin{array}{c}-6\\2\\-6\end{array}\right]$.

Linear Combinations (EV1)

Activity 2.1.11 Consider the following claim:

$$\begin{bmatrix} -5 \\ -1 \\ -7 \end{bmatrix} \text{ belongs to span } \left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 6 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix}, \begin{bmatrix} -4 \\ 1 \\ -5 \end{bmatrix} \right\}.$$

(a) Write a statement involving the solutions of a vector equation that's equivalent to this claim.

(b) Explain why the statement you wrote is false, to conclude that the vector does not belong to the span.

Activity 2.1.12 Are the sets

$$\left\{\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}\right\}$$

and

$$\operatorname{span}\left\{\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}\right\}$$

equal or nonequal to each other?

A. Equal

B. Non-equal

Remark 2.1.13 It is important to remember that

$$\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\} \neq \operatorname{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}.$$

For example,

$$\left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \right\}$$

is a set containing exactly two vectors, while

$$\operatorname{span}\left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \right\} = \left\{ a \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + b \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \middle| a, b \in \mathbb{R} \right\}$$

is a set containing infinitely-many vectors. See the below Sage cell for an illustration.

```
v1 = vector([1,-1,2])
v2 = vector([1,2,1])

# illustrate set of two vectors

p = plot(v1)
p += plot(v2)
show(p)

# illustrate the *span* that set

p = plot([])
for _ in range(1000):
    a = randrange(-99,100)
    b = randrange(-99,100)
    p += plot(a*v1+b*v2)
show(p)
```

Linear Combinations (EV1)

Activity 2.1.14 Before next class, find some time to do the following:

(a) Without referring to your activity book, write down the definition of a linear combination of vectors.

(b) Let $\vec{u} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix}$. Write down an example $\vec{w_1} = \begin{bmatrix} ? \\ ? \\ ? \end{bmatrix}$ of a linear combination of $\vec{u}, \vec{v}$. Then write down an example $\vec{w_2} = \begin{bmatrix} ? \\ ? \\ ? \end{bmatrix}$ that is *not* a linear combination of $\vec{u}, \vec{v}$.

(c) Draw a rough sketch of the vectors $\vec{u} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix}$, $\vec{w_1} = \begin{bmatrix} ? \\ ? \\ ? \end{bmatrix}$, and $\vec{w_2} = \begin{bmatrix} ? \\ ? \\ ? \end{bmatrix}$ in $\mathbb{R}^3$.

2.2 Spanning Sets (EV2)

Learning Outcomes

- Determine if a set of Euclidean vectors spans $\mathbb{R}^m$ by solving appropriate vector equations.

Spanning Sets (EV2)

Activity 2.2.1 Given a set of ingredients and a meal, a recipe is a list of amounts of each ingredient required to prepare the given meal.

(a) Use the words *vector* and *linear combination* to create a new statement that is analogous to one above.

(b) Building on your analogy, what role might the word *span* play?

Observation 2.2.2 Any single non-zero vector/number x in $\mathbb{R}^1$ spans $\mathbb{R}^1$, since $\mathbb{R}^1 = \{cx \mid c \in \mathbb{R}\}$.

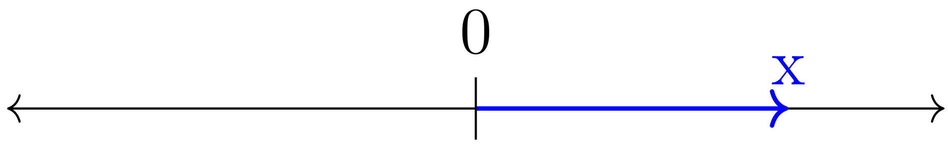

Figure 1 An $\mathbb{R}^1$ vector

Spanning Sets (EV2)

Activity 2.2.3 How many vectors are required to span $\mathbb{R}^2$? Sketch a drawing in the xy plane to support your answer.

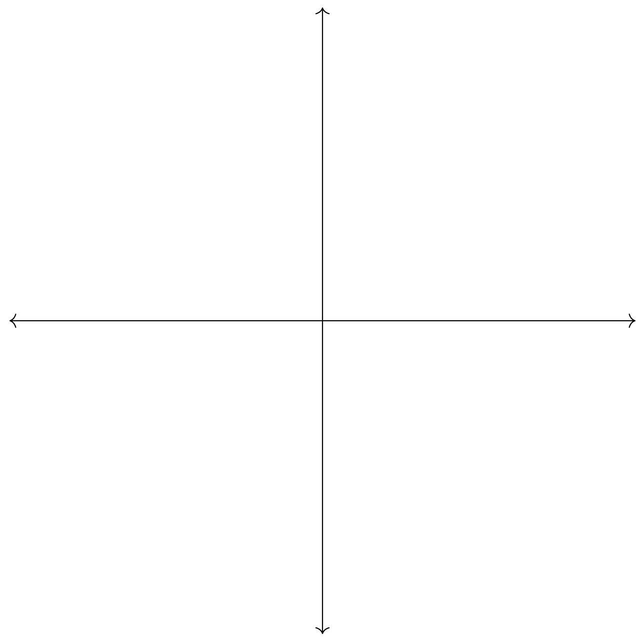

Figure 2 The xy plane $\mathbb{R}^2$

A. 1

B. 2

C. 3

D. 4

E. Infinitely Many

Spanning Sets (EV2)

Activity 2.2.4 How many vectors are required to span $\mathbb{R}^3$?

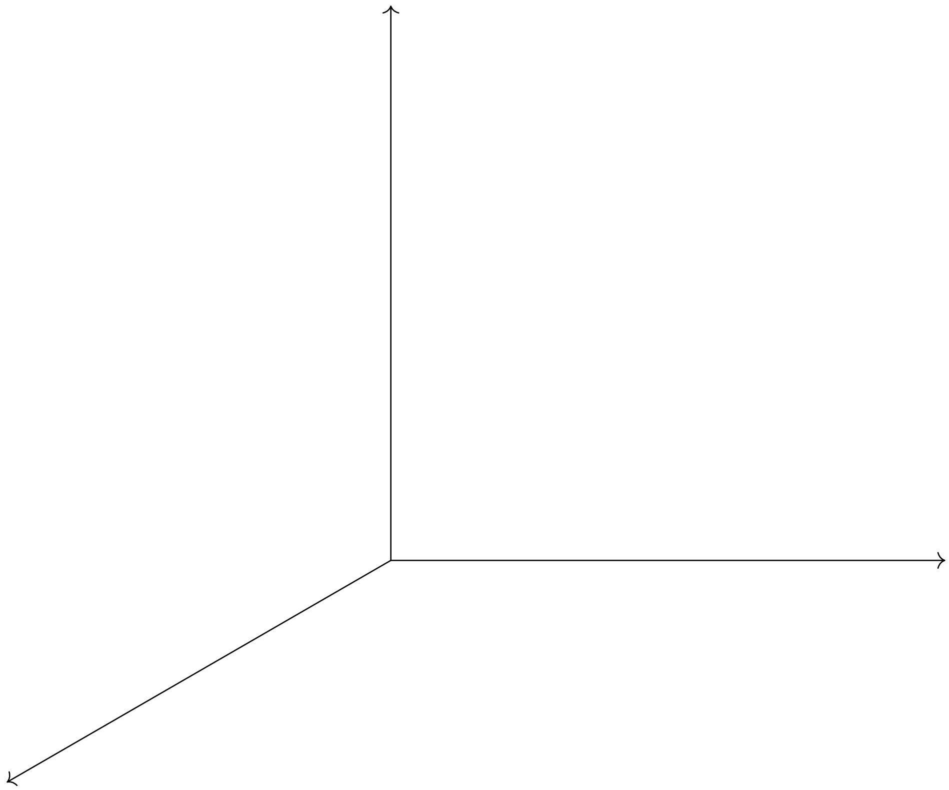

Figure 3 $\mathbb{R}^3$ space

A. 1

B. 2

C. 3

D. 4

E. Infinitely Many

Fact 2.2.5 *At least m vectors are required to span $\mathbb{R}^m$.*

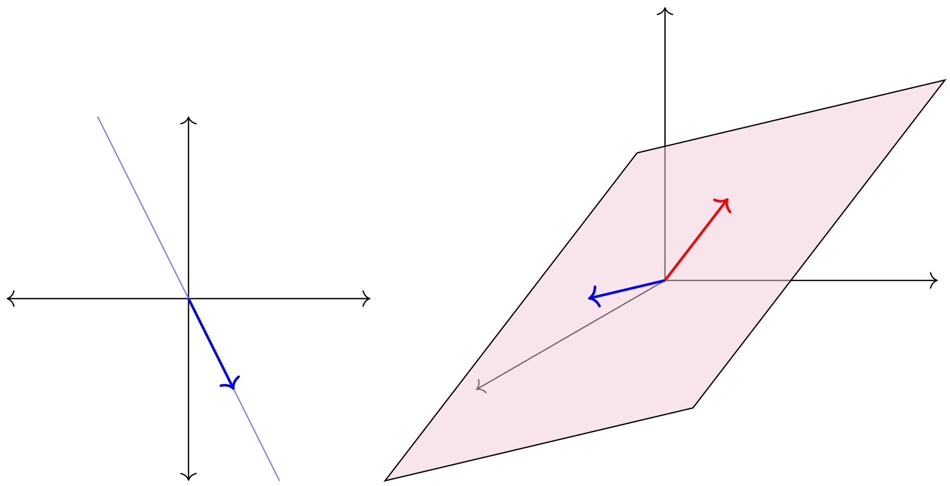

Figure 4 *Failed attempts to span $\mathbb{R}^m$ by $< m$ vectors*

Spanning Sets (EV2)

Activity 2.2.6 Consider the question: Does every vector in $\mathbb{R}^3$ belong to $\operatorname{span}\left\{\begin{bmatrix}1\\-1\\0\end{bmatrix}, \begin{bmatrix}-2\\0\\1\end{bmatrix}, \begin{bmatrix}-2\\-2\\2\end{bmatrix}\right\}$?

(a) Determine if $\begin{bmatrix}7\\-3\\-2\end{bmatrix}$ belongs to $\operatorname{span}\left\{\begin{bmatrix}1\\-1\\0\end{bmatrix}, \begin{bmatrix}-2\\0\\1\end{bmatrix}, \begin{bmatrix}-2\\-2\\2\end{bmatrix}\right\}$.

(b) Given this result, do we now know whether every vector in $\mathbb{R}^3$ belongs to $\operatorname{span}\left\{\begin{bmatrix}1\\-1\\0\end{bmatrix}, \begin{bmatrix}-2\\0\\1\end{bmatrix}, \begin{bmatrix}-2\\-2\\2\end{bmatrix}\right\}$?

(c) Determine if $\begin{bmatrix}0\\-4\\3\end{bmatrix}$ belongs to $\operatorname{span}\left\{\begin{bmatrix}1\\-1\\0\end{bmatrix}, \begin{bmatrix}-2\\0\\1\end{bmatrix}, \begin{bmatrix}-2\\-2\\2\end{bmatrix}\right\}$.

(d) Determine if $\begin{bmatrix}2\\5\\7\end{bmatrix}$ belongs to $\operatorname{span}\left\{\begin{bmatrix}1\\-1\\0\end{bmatrix}, \begin{bmatrix}-2\\0\\1\end{bmatrix}, \begin{bmatrix}-2\\-2\\2\end{bmatrix}\right\}$.

(e) Fix the SageMath code below to visualize $\operatorname{span}\left\{\begin{bmatrix}1\\-1\\0\end{bmatrix}, \begin{bmatrix}-2\\0\\1\end{bmatrix}, \begin{bmatrix}-2\\-2\\2\end{bmatrix}\right\}$.

Spanning Sets (EV2)

```
# create empty plot
p = plot([])

# define three vectors
v1 = vector([1,-1,0])
v2 = vector(FIXME)
v3 = vector(FIXME)

# do this 100 times
for _ in range(100):
    # choose random coefficients
    a = randrange(-9,10)
    b = randrange(-9,10)
    c = randrange(-9,10)
    # create linear combination
    linear_combo = a*v1 + b*v2 + FIXME
    # add it to the plot
    p += plot(linear_combo)

# show the plot
show(p)
```

Spanning Sets (EV2)

Activity 2.2.7 We'd prefer a more methodical method to decide if every vector in $\mathbb{R}^n$ belongs to some spanning set, compared to the guess-and-check methods we used in Activity 2.2.6.

(a) An arbitrary vector $\begin{bmatrix} ? \\ ? \\ ? \end{bmatrix}$ belongs to span $\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ -2 \\ 2 \end{bmatrix} \right\}$ provided the equation

$$x_1 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ -2 \\ 2 \end{bmatrix} = \begin{bmatrix} ? \\ ? \\ ? \end{bmatrix}$$

has...

A. no solutions.

B. exactly one solution.

C. at least one solution.

D. infinitely-many solutions.

(b) We're guaranteed at least one solution if the RREF of the corresponding augmented matrix has no contradictions; likewise, we have no solutions if the RREF corresponds to the contradiction $0 = 1$. Given

$$\left[\begin{array}{ccc|c} 1 & -2 & -2 & ? \\ -1 & 0 & -2 & ? \\ 0 & 1 & 2 & ? \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 2 & ? \\ 0 & 1 & 2 & ? \\ 0 & 0 & 0 & ? \end{array}\right]$$

we may conclude that the set does not span all of $\mathbb{R}^3$ because...

A. the row $[0\,1\,2\,|\,?]$ prevents a contradiction.

B. the row $[0\,1\,2\,|\,?]$ allows a contradiction.

C. the row $[0\,0\,0\,|\,?]$ prevents a contradiction.

D. the row $[0\,0\,0\,|\,?]$ allows a contradiction.

Fact 2.2.8 *The set* $\{\vec{v}_1, \dots, \vec{v}_n\}$ *spans all of* $\mathbb{R}^m$ *exactly when the vector equation*

$$x_1\vec{v}_1 + \cdots + x_n\vec{v}_n = \vec{w}$$

is consistent for every *vector* $\vec{w} \in \mathbb{R}^m$.

Likewise, the set $\{\vec{v}_1, \dots, \vec{v}_n\}$ *fails to span all of* $\mathbb{R}^m$ *exactly when the vector equation*

$$x_1\vec{v}_1 + \cdots + x_n\vec{v}_n = \vec{w}$$

is inconsistent for some *vector* $\vec{w} \in \mathbb{R}^m$.

Note these two possibilities are decided based on whether or not the RREF of the vector equation's coefficient matrix (that is, $\mathrm{RREF}[\vec{v}_1 \,\dots\, \vec{v}_n]$*) has either all pivot rows, or at least one non-pivot row (a row of zeroes):*

$$\left[\begin{array}{ccc|} 1 & -2 & -2 \\ -1 & 0 & -2 \\ 0 & 1 & 2 \end{array}\right] \sim \left[\begin{array}{ccc|} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right].$$

Spanning Sets (EV2)

Activity 2.2.9 Consider the set of vectors $S = \left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -4 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 7 \\ -3 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 5 \\ 7 \end{bmatrix}, \begin{bmatrix} 3 \\ 13 \\ 7 \\ 16 \end{bmatrix} \right\}$ and the question "Does $\mathbb{R}^4 = \operatorname{span} S$?"

(a) Rewrite this question in terms of the solutions to a vector equation.

(b) Answer your new question, and use this to answer the original question.

Spanning Sets (EV2)

Activity 2.2.10 Let $\vec{v}_1, \vec{v}_2, \vec{v}_3 \in \mathbb{R}^7$ be three Euclidean vectors, and suppose $\vec{w}$ is another vector with $\vec{w} \in \operatorname{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$. What can you conclude about $\operatorname{span}\{\vec{w}, \vec{v}_1, \vec{v}_2, \vec{v}_3\}$?

A. $\operatorname{span}\{\vec{w}, \vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is larger than $\operatorname{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$.

B. $\operatorname{span}\{\vec{w}, \vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is the same as $\operatorname{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$.

C. $\operatorname{span}\{\vec{w}, \vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is smaller than $\operatorname{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$.

Spanning Sets (EV2)

Activity 2.2.11 One of our important results in this lesson is Fact 2.2.5, which states that a set of n vectors is required to span $\mathbb{R}^n$. While we developed some geometric intuition for why this true, we did not prove it in class. Before coming to class next time, follow the steps outlined below to convince yourself of this fact using the concepts we learned in this lesson.

(a) Let $\{\vec{v}_1, \ldots, \vec{v}_m\}$ be a set of vectors living in $\mathbb{R}^n$ and assume that $m < n$. How many rows and how many columns will the matrix $[\vec{v}_1 \cdots \vec{v}_m]$ have?

(b) Given no additional information about the vectors $\vec{v}_1, \ldots, \vec{v}_m$, what is the maximum possible number of pivots in $\text{RREF}[\vec{v}_1 \ \ldots \ \vec{v}_m]$?

(c) Conclude that our given set of vector cannot span all of $\mathbb{R}^n$.

2.3 Subspaces (EV3)

Learning Outcomes

- Determine if a subset of $\mathbb{R}^n$ is a subspace or not.

Subspaces (EV3)

Activity 2.3.1 Consider the linear equation

$$x + 2y + z = 0.$$

(a) Verify that both $\vec{v} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ are solutions.

(b) Is the vector $2\vec{v} - 3\vec{w}$ also a solution?

Observation 2.3.2 Recall that if $S = \{\vec{v}_1, \ldots, \vec{v}_n\}$ is subset of vectors in $\mathbb{R}^n$, then $\text{span}(S)$ is the set of all linear combinations of vectors in S. In EV2 (Section 2.2), we learned how to decide whether $\text{span}(S)$ was equal to all of $\mathbb{R}^n$ or something strictly smaller.

Subspaces (EV3)

Activity 2.3.3 Let's consider the relationship between vectors within a spanning set.

(a) Let S denote a set of vectors in $\mathbb{R}^3$ and suppose that $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \in \operatorname{span}(S)$. Which of the following vectors might *not* belong to $\operatorname{span}(S)$?

A. $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

B. $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$

C. $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

D. $-2 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

(b) More generally, let S denote a set of vectors in $\mathbb{R}^n$ and suppose that $\vec{v}, \vec{w} \in \operatorname{span}(S)$ and $c \in \mathbb{R}$. Which of the following vectors *must* belong to $\operatorname{span}(S)$?

A. $\vec{0}$

B. $\vec{v} + \vec{w}$

C. $c\vec{v}$

D. All of these

Definition 2.3.4 A **homogeneous** system of linear equations is one of the form:

$$\begin{array}{rrrrr} a_{11}x_1 + & a_{12}x_2 + \ldots + & a_{1n}x_n & = 0 \\ a_{21}x_1 + & a_{22}x_2 + \ldots + & a_{2n}x_n & = 0 \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1}x_1 + & a_{m2}x_2 + \ldots + & a_{mn}x_n & = 0 \end{array}$$

This system is equivalent to the vector equation:

$$x_1\vec{v}_1 + \cdots + x_n\vec{v}_n = \vec{0}$$

and the augmented matrix:

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & 0 \\ a_{21} & a_{22} & \cdots & a_{2n} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & 0 \end{array}\right]$$

$\Diamond$

Activity 2.3.5 Consider an arbitrary homogeneous vector equation $x_1\vec{v}_1 + \cdots + x_n\vec{v}_n = \vec{0}$.

(a) Is this equation consistent?

A. No, it has no solutions.

B. Yes, it is guaranteed to have at least one solution.

C. More information is required.

(b) Suppose that $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$ are both solutions to the homogeneous vector equation $x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3 = \vec{0}$. This means that

$$1\vec{v}_1 + 2\vec{v}_2 + 3\vec{v}_3 = \vec{0} \text{ and } 4\vec{v}_1 + 5\vec{v}_2 + 6\vec{v}_3 = \vec{0}.$$

Therefore by adding these equations:

$$(1+4)\vec{v}_1 + (2+5)\vec{v}_2 + (3+6)\vec{v}_3 = \vec{0},$$

we may conclude that the vector $\begin{bmatrix} 1+4 \\ 2+5 \\ 3+6 \end{bmatrix}$ is...

A. another solution.

B. not a solution.

C. is equal to $\vec{0}$.

(c) More generally, if $\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$ and $\begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$ are both solutions to $x_1\vec{v}_1 + \cdots + x_n\vec{v}_n = \vec{0}$, we know that

$$a_1\vec{v}_1 + \cdots + a_n\vec{v}_n = \vec{0} \text{ and } b_1\vec{v}_1 + \cdots + b_n\vec{v}_n = \vec{0}.$$

Therefore by adding these equations:

$$(a_1 + b_1)\vec{v}_1 + \cdots + (a_n + b_n)\vec{v}_n = \vec{0},$$

we may conclude that the vector $\begin{bmatrix} a_1 + b_1 \\ \vdots \\ a_n + b_n \end{bmatrix}$ is...

A. another solution.

B. not a solution.

C. is equal to $\vec{0}$.

(d) Similarly, if $\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$ is a solution and $c \in \mathbb{R}$, we know first that

$$a_1\vec{v}_1 + \cdots + a_n\vec{v}_n = \vec{0}$$

and by multiplying both sides by c we also know

$$(ca_1)\vec{v}_1 + \cdots + (ca_n)\vec{v}_n = \vec{0}.$$

Thus we may conclude that the vector $\begin{bmatrix} ca_1 \\ \vdots \\ ca_n \end{bmatrix}$ is...

A. another solution.

B. not a solution.

C. is equal to $\vec{0}$.

Subspaces (EV3)

Observation 2.3.6 If S is any set of vectors in $\mathbb{R}^n$, then the set $\text{span}(S)$ has the following properties:

- the set $\text{span}(S)$ is non-empty.
- the set $\text{span}(S)$ is "closed under addition": for any $\vec{u}, \vec{v} \in \text{span}(S)$, the sum $\vec{u} + \vec{v}$ is also in $\text{span}(S)$.
- the set $\text{span}(S)$ is "closed under scalar multiplication": for any $\vec{u} \in \text{span}(S)$ and scalar $c \in \mathbb{R}$, the product $c\vec{u}$ is also in $\text{span}(S)$.

Likewise, if W is the solution set to a homogenous vector equation, it too satisfies:

- the set W is non-empty.
- the set W is "closed under addition": for any $\vec{u}, \vec{v} \in W$, the sum $\vec{u} + \vec{v}$ is also in W.
- the set W is "closed under scalar multiplication" : for any $\vec{u} \in W$ and scalar $c \in \mathbb{R}$, the product $c\vec{u}$ is also in W.

Subspaces (EV3)

Definition 2.3.7 A subset W of a vector space is called a **subspace** provided that it satisfies the following properties:

- the subset is non-empty.
- the subset is **closed under addition**: for any $\vec{u}, \vec{v} \in W$, the sum $\vec{u} + \vec{v}$ is also in W.
- the subset is **closed under scalar multiplication**: for any $\vec{u} \in W$ and scalar $c \in \mathbb{R}$, the product $c\vec{u}$ is also in W.

$\Diamond$

Observation 2.3.8 Note the similarities between a planar subspace spanned by two non-colinear vectors in $\mathbb{R}^3$, and the Euclidean plane $\mathbb{R}^2$. While they are not the same thing (and shouldn't be referred to interchangeably), algebraists call such similar spaces **isomorphic**; we'll learn what this means more carefully in a later chapter.

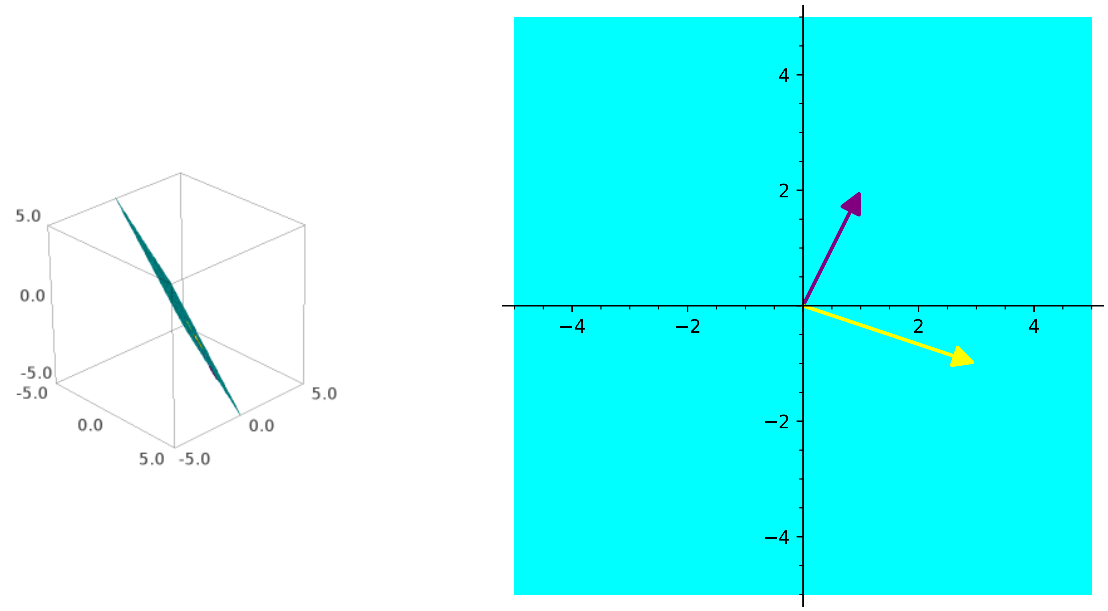

Figure 5 A planar subset of $\mathbb{R}^3$ compared with the plane $\mathbb{R}^2$.

Activity 2.3.9 To show that sets of Euclidean vectors form subspaces, we will need to prove that certain equalities hold.

(a) Consider the following argument that $5 = 7$:

$$\begin{aligned} & 5 = 7 \\ \Rightarrow\quad & 5 - 6 = 7 - 6 \\ \Rightarrow\quad & -1 = 1 \\ \Rightarrow\quad & (-1)^2 = (1)^2 \\ \Rightarrow\quad & 1 = 1 \end{aligned}$$

Is this reasoning valid?

A. Yes

B. No

(b) Consider the following argument that $5 = 7$:

$$\begin{aligned} & 1 = 1 \\ \Rightarrow\quad & (-1)^2 = (1)^2 \\ \Rightarrow\quad & -1 = 1 \\ \Rightarrow\quad & 5 - 6 = 7 - 6 \\ \Rightarrow\quad & 5 = 7 \end{aligned}$$

Is this reasoning valid?

A. Yes

B. No

Remark 2.3.10 Proofs of an equality LEFT = RIGHT should generally be of one of these forms:

1. Using a chain of equalities:

$$\begin{aligned} \text{LEFT} &= \cdots \\ &= \cdots \\ &= \cdots \\ &= \text{RIGHT} \end{aligned}$$

2. Using two chains of equalities:

$$\begin{aligned} \text{LEFT} &= \cdots \\ &= \cdots \\ &= \cdots \\ &= \text{SAME} \end{aligned} \qquad\qquad \begin{aligned} \text{RIGHT} &= \cdots \\ &= \cdots \\ &= \cdots \\ &= \text{SAME} \end{aligned}$$

3. Manipulating a known fact THIS = THAT into the desired equation:

$$\begin{aligned} & & \text{THIS} &= \text{THAT} \\ \Rightarrow & & \cdots &= \cdots \\ \Rightarrow & & \cdots &= \cdots \\ \Rightarrow & & \text{LEFT} &= \text{RIGHT} \end{aligned}$$

Subspaces (EV3)

Activity 2.3.11 Let $W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| \, x + 2y + z = 0 \right\}$. Consider the following questions to prove that W is a subspace.

(a) Is W non-empty?

A. Yes.

B. No.

(b) Let's assume that $\vec{u} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ are in W. What equations are we assuming to be true?

A. $x + 2y + z = 0$.

B. $a + 2b + c = 0$.

C. Both of these.

D. Neither of these.

(c) Which equation must be verified to show that $\vec{u} + \vec{v} = \begin{bmatrix} x + a \\ y + b \\ z + c \end{bmatrix}$ also belongs to W?

A. $(x + a) + 2(y + b) + (z + c) = 0$.

B. $x + a + 2y + b + z + c = 0$.

C. $x + 2y + z = a + 2b + c$.

(d) Use your assumptions to complete the following proof of $(x+a)+2(y+b)+(z+c) = 0$.

$$\begin{aligned}(x + a) + 2(y + b) + (z + c) &= \; ? \\ &= (\,?\,) + (\,?\,) \\ &= 0 + 0 \\ &= 0\end{aligned}$$

(e) Have we proven W is a subspace of $\mathbb{R}^3$?

A. Yes

B. Not yet

(f) Assume that $\vec{u} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ belongs to W, and $c \in \mathbb{R}$. Which equation must be verified to show that $c\vec{u} = \begin{bmatrix} cx \\ cy \\ cz \end{bmatrix}$ also belongs to W?

A. $(cx) + 2(cy) + (cz) = 0$.

B. $x + 2y + z = c$.

C. $x + 2y + z + c = 0$.

(g) Complete the following proof of $(cx)+2(cy)+(cz) = 0$ from the assumption $x+2y+z = 0$.

$$\begin{aligned} & & x + 2y + z &= 0 \\ \Rightarrow & & ?\,[x + 2y + z] &= \,?\,[0] \\ \Rightarrow & & ? &= \,? \\ \Rightarrow & & (cx) + 2(cy) + (cz) &= 0 \end{aligned}$$

(h) Have we proven W is a subspace of $\mathbb{R}^3$?

A. Yes

B. Not yet

Subspaces (EV3)

Activity 2.3.12 Let $W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| x + 2y + z = 4 \right\}$.

(a) Is W non-empty?

A. Yes.

B. No.

(b) Which of these statements is valid?

A. $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \in W$, *and* $\begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} \in W$, so W *is* a subspace.

B. $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \in W$, *and* $\begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} \in W$, so W *is not* a subspace.

C. $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \in W$, *but* $\begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} \notin W$, so W *is* a subspace.

D. $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \in W$, *but* $\begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} \notin W$, so W *is not* a subspace.

(c) Which of these statements is valid?

(a) $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \in W$, *and* $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \in W$, so W *is* a subspace.

(b) $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \in W$, *and* $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \in W$, so W *is not* a subspace.

(c) $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \in W$, *but* $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \notin W$, so W *is* a subspace.

(d) $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \in W$, *but* $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \notin W$, so W *is not* a subspace.

Remark 2.3.13 In summary, *any one* of the following is enough to prove that a nonempty subset W is *not* a subspace:

- Show that W is empty. (Or even just show $\vec{0} \notin W$.)
- Find specific values for $\vec{u}, \vec{v} \in W$ such that $\vec{u} + \vec{v} \notin W$.
- Find specific values for $c \in \mathbb{R}, \vec{v} \in W$ such that $c\vec{v} \notin W$.

If you cannot do any of these, then W can be proven to be a subspace by doing *all* of the following:

1. Show that W is non-empty. (Usually by showing $\vec{0} \in W$.)
2. For all $\vec{u}, \vec{v} \in W$ (not just specific values), $\vec{u} + \vec{v} \in W$.
3. For all $\vec{v} \in W$ and $c \in \mathbb{R}$ (not just specific values), $c\vec{v} \in W$.

Subspaces (EV3)

Activity 2.3.14 Consider these subsets of $\mathbb{R}^3$:

$$R = \left\{ \left[\begin{array}{c} x \\ y \\ z \end{array}\right] \middle|\, y = z + 1 \right\} \qquad S = \left\{ \left[\begin{array}{c} x \\ y \\ z \end{array}\right] \middle|\, y = |z| \right\} \qquad T = \left\{ \left[\begin{array}{c} x \\ y \\ z \end{array}\right] \middle|\, z = xy \right\}.$$

(a) Show R isn't a subspace by showing that $\vec{0} \notin R$.

(b) Show S isn't a subspace by finding two vectors $\vec{u}, \vec{v} \in S$ such that $\vec{u} + \vec{v} \notin S$.

(c) Show T isn't a subspace by finding a vector $\vec{v} \in T$ such that $2\vec{v} \notin T$.

Subspaces (EV3)

Activity 2.3.15 Consider the following two sets of Euclidean vectors:

$$U = \left\{ \left[\begin{array}{c} x \\ y \end{array}\right] \middle| \, 7\,x + 4\,y = 0 \right\} \qquad W = \left\{ \left[\begin{array}{c} x \\ y \end{array}\right] \middle| \, 3\,xy^2 = 0 \right\}$$

Explain why one of these sets is a subspace of $\mathbb{R}^2$ and one is not.

Activity 2.3.16

(a) Consider the following attempted proof that

$$U = \left\{ \left[\begin{array}{c} x \\ y \end{array} \right] \middle| x + y = xy \right\}$$

is closed under scalar multiplication.

> Let $\left[\begin{array}{c} x \\ y \end{array} \right] \in U$, so we know that $x + y = xy$. We want to show $k \left[\begin{array}{c} x \\ y \end{array} \right] = \left[\begin{array}{c} kx \\ ky \end{array} \right] \in U$, that is, $(kx)+(ky) = (kx)(ky)$. This is verified by the following calculation:
>
> $$\begin{aligned} & (kx) + (ky) = (kx)(ky) \\ \Rightarrow \quad & k(x + y) = k^2xy \\ \Rightarrow \quad & 0[k(x + y)] = 0[k^2xy] \\ \Rightarrow \quad & 0 = 0 \end{aligned}$$

Is this reasoning valid?

A. Yes

B. No

(b) Does this fix the proof?

$$\begin{aligned} & x + y = xy \\ \Rightarrow \quad & k(x + y) = k(xy) \\ \Rightarrow \quad & (kx) + (ky) = (kx)(ky) \end{aligned}$$

A. Yes

B. No

Subspaces (EV3)

Remark 2.3.17 Recall that in Activity 2.2.1 we used the words *vector*, *linear combination*, and *span* to make an analogy with recipes, ingredients, and meals. In this analogy, a *recipe* was defined to be a list of amounts of each ingredient to build a particular meal.

Subspaces (EV3)

Activity 2.3.18

(a) Given the set of ingredients $S = \{\text{flour}, \text{yeast}, \text{salt}, \text{water}, \text{sugar}, \text{milk}\}$, how should we think of the subspace $\text{span}(S)$?

(b) What is one meal that lives in the subspace $\text{span}(S)$?

(c) What is one meal that does not live in the subspace $\text{span}(S)$?

Activity 2.3.19 Let

$$W = \left\{ \left[\begin{array}{c} x \\ y \\ z \\ w \end{array}\right] \middle| x+y=3z+2w \right\}.$$

The set W is a subspace. Below are two attempted proofs of the fact that W is closed under vector addition. Both of them are invalid; explain why.

(a) Let $\vec{u} = \left[\begin{array}{c} 1 \\ 4 \\ 1 \\ 1 \end{array}\right], \vec{v} = \left[\begin{array}{c} 2 \\ -1 \\ 1 \\ -1 \end{array}\right]$. Then both $\vec{u}, \vec{v}$ are elements of W. Their sum is

$$\vec{w} = \left[\begin{array}{c} 3 \\ 3 \\ 2 \\ 0 \end{array}\right]$$

and since

$$3+3 = 3\cdot(2) + 2\cdot(0),$$

it follows that $\vec{w}$ is also in W and so W is closed under vector addition.

(b) If $\left[\begin{array}{c} x \\ y \\ z \\ w \end{array}\right], \left[\begin{array}{c} a \\ b \\ c \\ d \end{array}\right]$ are in W, we need to show that $\left[\begin{array}{c} x+a \\ y+b \\ z+c \\ w+d \end{array}\right]$ is also in W. To be in W, we need

$$(x+a)+(y+b) = 3(z+c)+2(w+d).$$

Well, if

$$(x+a)+(y+b) = 3(z+c)+2(w+d),$$

then we know that

$$x+y-3z-2w+a+b-3c-2d = 0$$

by moving everything over to the left hand side. Since we are assuming that $x+y-3z-2w=0$ and $a+b-3c-2d=0$, it follows that $0=0$, which is true, which proves that vector addition is closed.

2.4 Linear Independence (EV4)

Learning Outcomes

- Determine if a set of Euclidean vectors is linearly dependent or independent by solving an appropriate vector equation.

Linear Independence (EV4)

Activity 2.4.1 Consider the vector equation

$$x_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 7 \\ 4 \end{bmatrix}.$$

(a) Decide which of $\begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$ or $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is a solution vector.

(b) Consider now the following vector equation:

$$y_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + y_2 \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} + y_3 \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} + y_4 \begin{bmatrix} -1 \\ 7 \\ 4 \end{bmatrix} = \vec{0}.$$

How is this vector equation related to the original one?

(c) Use the solution vector you found in part (a) to construct a solution vector to this new equation.

Linear Independence (EV4)

Activity 2.4.2 Consider the two sets

$$S = \left\{ \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} \right\} \qquad T = \left\{ \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -11 \end{bmatrix} \right\}$$

where T contains a vector missing from S. Which of the following is true?

A. $\operatorname{span} S$ contains a vector missing from $\operatorname{span} T$.

B. $\operatorname{span} T$ contains a vector missing from $\operatorname{span} S$.

C. $\operatorname{span} S$ and $\operatorname{span} T$ contain the same vectors.

Definition 2.4.3 We say that a set of vectors is **linearly dependent** if one vector in the set belongs to the span of the others. Otherwise, we say the set is **linearly independent**.

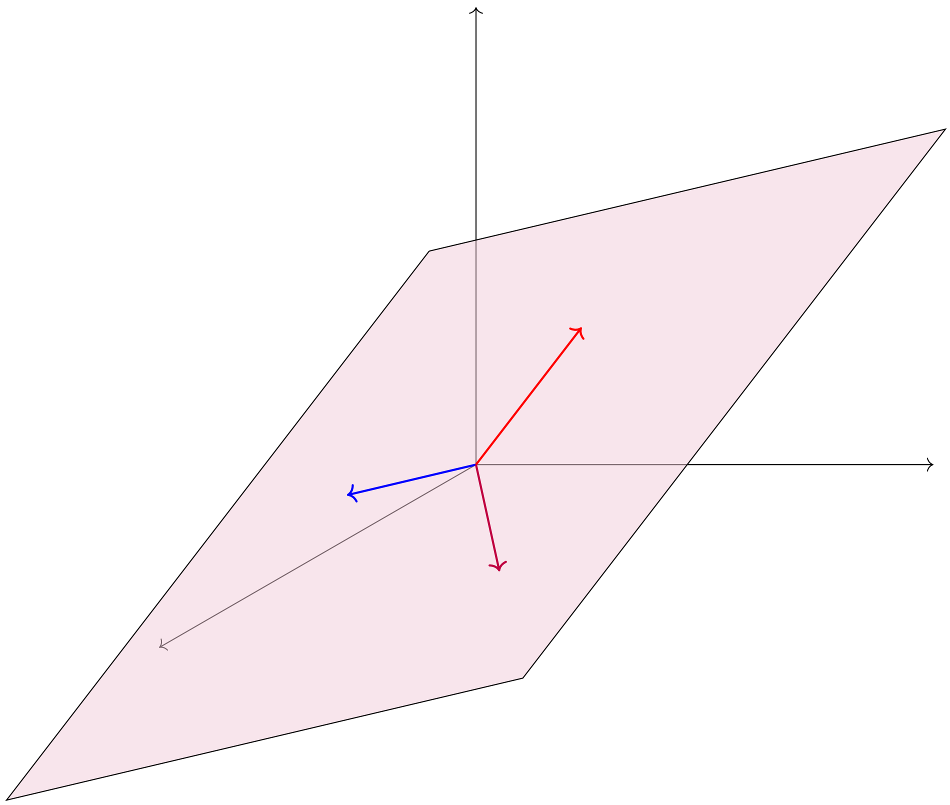

Figure 6 A linearly dependent set of three vectors

You can think of linearly dependent sets as containing a redundant vector, in the sense that you can drop a vector out without reducing the span of the set. In the above image, all three vectors lay in the same planar subspace, but only two vectors are needed to span the plane, so the set is linearly dependent. ◊

Remark 2.4.4 In Activity 2.4.2 we had

$$S = \left\{ \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} \right\} \quad \neq \quad T = \left\{ \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -11 \end{bmatrix} \right\}$$

different, while

$$\operatorname{span} S = \left\{ a \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} + b \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} \middle| a, b \in \mathbb{R} \right\} =$$

$$\operatorname{span} T = \left\{ a \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} + b \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} + c \begin{bmatrix} -1 \\ 0 \\ -11 \end{bmatrix} \middle| a, b, c \in \mathbb{R} \right\}$$

were the same. This is possible because while S is linearly independent, T's third vector made it linearly dependent:

$$1 \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} - 3 \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ -11 \end{bmatrix}$$

Linear Independence (EV4)

Activity 2.4.5 Consider the following three vectors in $\mathbb{R}^3$:

$$\vec{v}_1 = \begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}, \text{ and } \vec{v}_3 = \begin{bmatrix} -2 \\ 5 \\ 4 \end{bmatrix}.$$

(a) Let $\vec{w} = 3\vec{v}_1 - \vec{v}_2 - 5\vec{v}_3 = \begin{bmatrix} ? \\ ? \\ ? \end{bmatrix}$. The set $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{w}\}$ is...

A. linearly dependent: at least one vector is a linear combination of others

B. linearly independent: no vector is a linear combination of others

(b) Find

$$\text{RREF}\left[\begin{array}{ccc|c} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \vec{w} \end{array}\right] = \text{RREF}\left[\begin{array}{ccc|c} -2 & 1 & -2 & ? \\ 0 & 3 & 5 & ? \\ 0 & 0 & 4 & ? \end{array}\right] = ?.$$

What does this tell you about solution set for the vector equation $x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3 = \vec{w}$?

A. It is inconsistent.

B. It is consistent with one solution.

C. It is consistent with infinitely many solutions.

(c) Find

$$\text{RREF}\left[\begin{array}{cccc|c} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \vec{w} & \vec{0} \end{array}\right] = \text{RREF}\left[\begin{array}{cccc|c} -2 & 1 & -2 & ? & 0 \\ 0 & 3 & 5 & ? & 0 \\ 0 & 0 & 4 & ? & 0 \end{array}\right] = ?.$$

What does this tell you about solution set for the vector equation $x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3 + x_4\vec{w} = \vec{0}$?

A. It is inconsistent.

B. It is consistent with one solution.

C. It is consistent with infinitely many solutions.

(d) Which of the following is the best conclusion obtained when we solved $x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3 + x_4\vec{w} = \vec{0}$?

A. A pivot column in the *augmented* matrix $\text{RREF}\left[\begin{array}{cccc|c} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \vec{w} & \vec{0} \end{array}\right]$ guarantees the linear independence of $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{w}\}$ by preventing contradictions.

B. A pivot column in the *coefficient* matrix $\text{RREF}\left[\begin{array}{cccc} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \vec{w} \end{array}\right]$ guarantees the linear independence of $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{w}\}$ by preventing contradictions.

C. A non-pivot column in the *augmented* matrix $\text{RREF}\left[\begin{array}{cccc|c} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \vec{w} & \vec{0} \end{array}\right]$ guarantees the linear dependence of $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{w}\}$ by describing a linear combination of one vector in terms of the others.

D. A non-pivot column in the *coefficient* matrix $\text{RREF}\left[\begin{array}{cccc} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \vec{w} \end{array}\right]$ guarantees the linear dependence of $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{w}\}$ by describing a linear combination of one vector in terms of the others.

Linear Independence (EV4)

Fact 2.4.6 *For any vector space, the set* $\{\vec{v}_1, \dots \vec{v}_n\}$ *is linearly dependent if and only if the vector equation* $x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_n\vec{v}_n = \vec{0}$ *is consistent with infinitely many solutions.*

Likewise, the set of vectors $\{\vec{v}_1, \dots \vec{v}_n\}$ *is linearly independent if and only the vector equation*

$$x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_n\vec{v}_n = \vec{0}$$

has exactly one solution: $\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$.

Activity 2.4.7 Find

$$\operatorname{RREF}\left[\begin{array}{ccccc|c} 2 & 2 & 3 & -1 & 4 & 0 \\ 3 & 0 & 13 & 10 & 3 & 0 \\ 0 & 0 & 7 & 7 & 0 & 0 \\ -1 & 3 & 16 & 14 & 1 & 0 \end{array}\right]$$

and mark the part of the matrix that demonstrates that

$$S = \left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 13 \\ 7 \\ 16 \end{bmatrix}, \begin{bmatrix} -1 \\ 10 \\ 7 \\ 14 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 0 \\ 1 \end{bmatrix} \right\}$$

is linearly dependent (the part that shows its linear system has infinitely many solutions).

Linear Independence (EV4)

Activity 2.4.8

(a) Write a statement involving the solutions of a vector equation that's equivalent to each claim:

(i) "The set of vectors $\left\{\begin{bmatrix}1\\-1\\0\\-1\end{bmatrix},\begin{bmatrix}5\\5\\3\\1\end{bmatrix},\begin{bmatrix}9\\11\\6\\2\end{bmatrix}\right\}$ is linearly *independent*."

(ii) "The set of vectors $\left\{\begin{bmatrix}1\\-1\\0\\-1\end{bmatrix},\begin{bmatrix}5\\5\\3\\1\end{bmatrix},\begin{bmatrix}9\\11\\6\\2\end{bmatrix}\right\}$ is linearly *dependent*."

(b) Explain how to determine which of these statements is true.

Linear Independence (EV4)

Observation 2.4.9 Compare the following results:

- A set of $\mathbb{R}^m$ vectors $\{\vec{v}_1, \dots \vec{v}_n\}$ is linearly independent if and only if RREF $\begin{bmatrix} \vec{v}_1 & \dots & \vec{v}_n \end{bmatrix}$ has all pivot *columns*.
- A set of $\mathbb{R}^m$ vectors $\{\vec{v}_1, \dots \vec{v}_n\}$ is linearly dependent if and only if RREF $\begin{bmatrix} \vec{v}_1 & \dots & \vec{v}_n \end{bmatrix}$ has at least one non-pivot *column*.
- A set of $\mathbb{R}^m$ vectors $\{\vec{v}_1, \dots \vec{v}_n\}$ spans $\mathbb{R}^m$ if and only if RREF $\begin{bmatrix} \vec{v}_1 & \dots & \vec{v}_n \end{bmatrix}$ has all pivot *rows*.
- A set of $\mathbb{R}^m$ vectors $\{\vec{v}_1, \dots \vec{v}_n\}$ fails to span $\mathbb{R}^m$ if and only if RREF $\begin{bmatrix} \vec{v}_1 & \dots & \vec{v}_n \end{bmatrix}$ has at least one non-pivot *row*.

Linear Independence (EV4)

Activity 2.4.10 What is the largest number of $\mathbb{R}^4$ vectors that can form a linearly independent set?

A. 3

B. 4

C. 5

D. You can have infinitely many vectors and still be linearly independent.

Linear Independence (EV4)

Activity 2.4.11 Is it possible for the set of Euclidean vectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n, \vec{0}\}$ to be linearly independent?

A. Yes

B. No

Linear Independence (EV4)

Remark 2.4.12 Recall that in Activity 2.2.1 we used the words *vector*, *linear combination*, and *span* to make an analogy with recipes, ingredients, and meals. In this analogy, a *recipe* was defined to be a list of amounts of each ingredient to build a particular meal.

Linear Independence (EV4)

Activity 2.4.13 Consider the statement: The set of vectors $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linearly dependent because the vector $\vec{v}_3$ is a linear combination of $\vec{v_1}$ and $\vec{v}_2$. Construct an analogous statement involving ingredients, meals, and recipes, using the terms *linearly (in)dependent* and *linear combination*.

Linear Independence (EV4)

Activity 2.4.14 The following exercises are designed to help develop your geometric intuition around linear dependence.

(a) Draw sketches that depict the following:

- Three linearly independent vectors in $\mathbb{R}^3$.
- Three linearly dependent vectors in $\mathbb{R}^3$.

(b) If you have three linearly dependent vectors, is it necessarily the case that one of the vectors is a multiple of the other?

2.5 Identifying a Basis (EV5)

Learning Outcomes

- Explain why a set of Euclidean vectors is or is not a basis of $\mathbb{R}^n$.

Identifying a Basis (EV5)

Remark 2.5.1 Recall that in Activity 2.2.1 we used the words *vector*, *linear combination*, and *span* to make an analogy with recipes, ingredients, and meals. In this analogy, a *recipe* was defined to be a list of amounts of each ingredient to build a particular meal.

Identifying a Basis (EV5)

Activity 2.5.2 Consider the following set of ingredients:

$$S = \{\text{tomato}, \text{olive oil}, \text{dough}, \text{cheese}, \text{pizza sauce}, \text{garlic}\}$$

(a) Does ”pizza” live inside of $\text{span}(S)$?

(b) Identify which ingredients in S make the set linearly dependent.

(c) Can you think of a subset S' of S that is linearly independent and for which ”pizza” is still in $\text{span}\, S'$?

Identifying a Basis (EV5)

Activity 2.5.3 Consider the set of vectors

$$S = \left\{ \begin{bmatrix} 3 \\ -2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -16 \\ -5 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

(a) Given

$$\left[\begin{array}{ccccc|c} 3 & 2 & 0 & 1 & 3 & 5 \\ -2 & 4 & -16 & 2 & 3 & 2 \\ -1 & 1 & -5 & 3 & 0 & 0 \\ 0 & 1 & -3 & 0 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{ccccc|c} 1 & 0 & 2 & 0 & 0 & 1 \\ 0 & 1 & -3 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

Express the vector $\begin{bmatrix} 5 \\ 2 \\ 0 \\ 1 \end{bmatrix}$ as a linear combination of the vectors in S, i.e. find scalars such that

$$? \begin{bmatrix} 3 \\ -2 \\ -1 \\ 0 \end{bmatrix} + ? \begin{bmatrix} 2 \\ 4 \\ 1 \\ 1 \end{bmatrix} + ? \begin{bmatrix} 0 \\ -16 \\ -5 \\ -3 \end{bmatrix} + ? \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix} + ? \begin{bmatrix} 3 \\ 3 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ 0 \\ 1 \end{bmatrix}.$$

(b) Find a *different* way to express the vector $\begin{bmatrix} 5 \\ 2 \\ 0 \\ 1 \end{bmatrix}$ as a linear combination of the vectors in S:

$$? \begin{bmatrix} 3 \\ -2 \\ -1 \\ 0 \end{bmatrix} + ? \begin{bmatrix} 2 \\ 4 \\ 1 \\ 1 \end{bmatrix} + ? \begin{bmatrix} 0 \\ -16 \\ -5 \\ -3 \end{bmatrix} + ? \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix} + ? \begin{bmatrix} 3 \\ 3 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ 0 \\ 1 \end{bmatrix}.$$

(c) Consider another vector $\begin{bmatrix} 8 \\ 6 \\ 7 \\ 5 \end{bmatrix}$. Without computing the RREF of another matrix, do we already know how many ways can this vector be written as a linear combination of the vectors in S?

A. Yes, zero.

B. Yes, one.

C. Yes, infinitely-many.

D. No, computing a new matrix RREF is necessary.

Identifying a Basis (EV5)

Activity 2.5.4 Let's review some of the terminology we've been dealing with...

(a) If every vector in a vector space can be constructed as one or more linear combinations of vectors in a set S, we can say...

A. the set S spans the vector space.

B. the set S fails to span the vector space.

C. the set S is linearly independent.

D. the set S is linearly dependent.

(b) If the zero vector $\vec{0}$ can be constructed as a *unique* linear combination of vectors in a set S (the combination multiplying every vector by the scalar value 0), we can say...

A. the set S spans the vector space.

B. the set S fails to span the vector space.

C. the set S is linearly independent.

D. the set S is linearly dependent.

(c) If every vector of a vector space can either be constructed as a *unique* linear combination of vectors in a set S, or not at all, we can say...

A. the set S spans the vector space.

B. the set S fails to span the vector space.

C. the set S is linearly independent.

D. the set S is linearly dependent.

Identifying a Basis (EV5)

Definition 2.5.5 A **basis** of a vector space V is a set of vectors S contained in V for which

1. *Every* vector in the vector space can be expressed as a linear combination of the vectors in S.

2. For each vector $\vec{v}$ in the vector space, there is only *one* way to write it as a linear combination of the vectors in S.

These two properties may be expressed more succinctly as the statement "Every vector in V can be expressed *uniquely* as a linear combination of the vectors in S". $\Diamond$

Identifying a Basis (EV5)

Observation 2.5.6 In terms of a vector equation, a set $S = \{\vec{v}_1, \dots, \vec{v}_n\}$ is a basis of a vector space if the vector equation

$$x_1\vec{v_1} + \cdots + x_n\vec{v_n} = \vec{w}$$

has a *unique* solution for *every* vector $\vec{w}$ in the vector space.

Put another way, a basis may be thought of as a minimal set of "building blocks" that can be used to construct any other vector of the vector space.

Identifying a Basis (EV5)

Activity 2.5.7 Let S be a basis (Definition 2.5.5) for a vector space. Then...

A. the set S must both span the vector space and be linearly independent.

B. the set S must span the vector space but could be linearly dependent.

C. the set S must be linearly independent but could fail to span the vector space.

D. the set S could fail to span the vector space and could be linearly dependent.

Identifying a Basis (EV5)

Activity 2.5.8 The vectors

$$\hat{\imath} = (1,0,0) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad \hat{\jmath} = (0,1,0) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad \hat{k} = (0,0,1) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

form a basis $\{\hat{\imath}, \hat{\jmath}, \hat{k}\}$ used frequently in multivariable calculus.

Find the unique linear combination of these vectors

$$?\,\hat{\imath} + \,?\,\hat{\jmath} + \,?\,\hat{k}$$

that equals the vector

$$(3,-2,4) = \begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix}$$

in xyz space.

Identifying a Basis (EV5)

Definition 2.5.9 The **standard basis** of $\mathbb{R}^n$ is the set $\{\vec{e}_1, \ldots, \vec{e}_n\}$ where

$$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} \qquad \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} \qquad \cdots \qquad \vec{e}_n = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

In particular, the standard basis for $\mathbb{R}^3$ is $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\} = \{\hat{\imath}, \hat{\jmath}, \hat{k}\}$. $\Diamond$

Identifying a Basis (EV5)

Activity 2.5.10 Use technology to find the RREF of an appropriate matrix and determine if each of the following sets is a basis for $\mathbb{R}^4$. (Don't forget to use `format rat` to nicely format Octave's output.)

(a)

$$\left\{\begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\end{bmatrix}\right\}$$

A. A basis, because it both spans $\mathbb{R}^4$ and is linearly independent.

B. Not a basis, because while it spans $\mathbb{R}^4$, it is linearly dependent.

C. Not a basis, because while it is linearly independent, it fails to span $\mathbb{R}^4$.

D. Not a basis, because not only does it fail to span $\mathbb{R}^4$, it's also linearly dependent.

(b)

$$\left\{\begin{bmatrix}2\\3\\0\\-1\end{bmatrix}, \begin{bmatrix}2\\0\\0\\3\end{bmatrix}, \begin{bmatrix}4\\3\\0\\2\end{bmatrix}, \begin{bmatrix}-3\\0\\1\\3\end{bmatrix}\right\}$$

A. A basis, because it both spans $\mathbb{R}^4$ and is linearly independent.

B. Not a basis, because while it spans $\mathbb{R}^4$, it is linearly dependent.

C. Not a basis, because while it is linearly independent, it fails to span $\mathbb{R}^4$.

D. Not a basis, because not only does it fail to span $\mathbb{R}^4$, it's also linearly dependent.

(c)

$$\left\{\begin{bmatrix}2\\3\\0\\-1\end{bmatrix}, \begin{bmatrix}2\\0\\0\\3\end{bmatrix}, \begin{bmatrix}3\\13\\7\\16\end{bmatrix}, \begin{bmatrix}-1\\10\\7\\14\end{bmatrix}, \begin{bmatrix}4\\3\\0\\2\end{bmatrix}\right\}$$

A. A basis, because it both spans $\mathbb{R}^4$ and is linearly independent.

B. Not a basis, because while it spans $\mathbb{R}^4$, it is linearly dependent.

C. Not a basis, because while it is linearly independent, it fails to span $\mathbb{R}^4$.

D. Not a basis, because not only does it fail to span $\mathbb{R}^4$, it's also linearly dependent.

(d)

$$\left\{\begin{bmatrix}2\\3\\0\\-1\end{bmatrix}, \begin{bmatrix}4\\3\\0\\2\end{bmatrix}, \begin{bmatrix}-3\\0\\1\\3\end{bmatrix}, \begin{bmatrix}3\\6\\1\\5\end{bmatrix}\right\}$$

A. A basis, because it both spans $\mathbb{R}^4$ and is linearly independent.

B. Not a basis, because while it spans $\mathbb{R}^4$, it is linearly dependent.

C. Not a basis, because while it is linearly independent, it fails to span $\mathbb{R}^4$.

D. Not a basis, because not only does it fail to span $\mathbb{R}^4$, it's also linearly dependent.

(e)

$$\left\{ \begin{bmatrix} 5 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 1 \\ 3 \end{bmatrix} \right\}$$

A. A basis, because it both spans $\mathbb{R}^4$ and is linearly independent.

B. Not a basis, because while it spans $\mathbb{R}^4$, it is linearly dependent.

C. Not a basis, because while it is linearly independent, it fails to span $\mathbb{R}^4$.

D. Not a basis, because not only does it fail to span $\mathbb{R}^4$, it's also linearly dependent.

Identifying a Basis (EV5)

Activity 2.5.11 If $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ is a basis for $\mathbb{R}^4$, that means $\text{RREF}[\vec{v}_1\,\vec{v}_2\,\vec{v}_3\,\vec{v}_4]$ has a pivot in every row (because it spans), and has a pivot in every column (because it's linearly independent).

What is $\text{RREF}[\vec{v}_1\,\vec{v}_2\,\vec{v}_3\,\vec{v}_4]$?

$$\text{RREF}[\vec{v}_1\,\vec{v}_2\,\vec{v}_3\,\vec{v}_4] = \begin{bmatrix} ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \end{bmatrix}$$

Identifying a Basis (EV5)

Fact 2.5.12 *The set* $\{\vec{v}_1, \dots, \vec{v}_m\}$ *is a basis for* $\mathbb{R}^n$ *if and only if* $m = n$ *and* $\mathrm{RREF}[\vec{v}_1 \; \dots \; \vec{v}_n] = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}$.

That is, a basis for $\mathbb{R}^n$ *must have exactly* n *vectors and its square matrix must row-reduce to the so-called* ***identity matrix*** *containing all zeros except for a downward diagonal of ones. (We will learn where the identity matrix gets its name in a later module.)*

Identifying a Basis (EV5)

Activity 2.5.13 Let S denote a set of vectors in $\mathbb{R}^n$. Without referring to your Activity Book, write down:

(a) The definition of what it means for S to be linearly independent.

(b) The definition of what it means for S to span $\mathbb{R}^n$.

(c) The definition of what it means for S to be a basis for $\mathbb{R}^n$.

Identifying a Basis (EV5)

Activity 2.5.14 You are going on a trip and need to pack. Let S denote the set of items that you are packing in your suitcase.

(a) Give an example of such a set of items S that you would say ”spans” everything you need, but is linearly dependent.

(b) Give an example of such a set of items S that is linearly independent, but does not ”span” everything you need.

(c) Give an example of such a set S that you might reasonably consider to be a ”basis” for what you need?

2.6 Subspace Basis and Dimension (EV6)

Learning Outcomes

- Compute a basis for the subspace spanned by a given set of Euclidean vectors, and determine the dimension of the subspace.

Subspace Basis and Dimension (EV6)

Activity 2.6.1 Consider the set S of vectors in $\mathbb{R}^4$ given by

$$S = \left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \\ -1 \end{bmatrix} \right\}$$

(a) Is the set S linearly independent or linearly dependent?

(b) How would you describe the subspace $\operatorname{span} S$ geometrically?

(c) What do the spaces $\operatorname{span} S$ and $\mathbb{R}^2$ have in common? In what ways do they differ?

Subspace Basis and Dimension (EV6)

Observation 2.6.2 Recall from section Section 2.3 that a **subspace** of a vector space is the result of spanning a set of vectors from that vector space.

Recall also that a linearly dependent set contains "redundant" vectors. For example, only two of the three vectors in Figure 14 are needed to span the planar subspace.

Subspace Basis and Dimension (EV6)

Activity 2.6.3 Consider the subspace of $\mathbb{R}^4$ given by $W = \operatorname{span}\left\{\begin{bmatrix}2\\3\\0\\1\end{bmatrix}, \begin{bmatrix}2\\0\\1\\-1\end{bmatrix}, \begin{bmatrix}2\\-3\\2\\-3\end{bmatrix}, \begin{bmatrix}1\\5\\-1\\0\end{bmatrix}\right\}$.

(a) Mark the column of RREF $\begin{bmatrix}2 & 2 & 2 & 1\\3 & 0 & -3 & 5\\0 & 1 & 2 & -1\\1 & -1 & -3 & 0\end{bmatrix}$ that shows that W's spanning set is linearly dependent.

(b) What would be the result of removing the vector that gave us this column?

A. The set still spans W, and remains linearly dependent.

B. The set still spans W, but is now also linearly independent.

C. The set no longer spans W, and remains linearly dependent.

D. The set no longer spans W, but is now linearly independent.

Subspace Basis and Dimension (EV6)

Definition 2.6.4 Let W be a subspace of a vector space. A **basis** for W is a linearly independent set of vectors that spans W (but not necessarily the entire vector space). $\Diamond$

Observation 2.6.5 So given a set $S = \{\vec{v}_1, \ldots, \vec{v}_m\}$, to compute a basis for the subspace $\operatorname{span} S$, simply remove the vectors corresponding to the non-pivot columns of $\operatorname{RREF}[\vec{v}_1 \; \ldots \; \vec{v}_m]$. For example, since

$$\operatorname{RREF}\begin{bmatrix} 1 & 2 & 0 & 1 \\ 2 & 4 & -2 & 0 \\ 3 & 6 & -2 & 1 \end{bmatrix} = \begin{bmatrix} \boxed{1} & 2 & 0 & 1 \\ 0 & 0 & \boxed{1} & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

the subspace $W = \operatorname{span}\left\{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right\}$ has $\left\{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ -2 \end{bmatrix}\right\}$ as a basis.

Subspace Basis and Dimension (EV6)

Activity 2.6.6

(a) Find a basis for $\operatorname{span} S$ where

$$S = \left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ -1 \\ 0 \end{bmatrix} \right\}.$$

(b) Find a basis for $\operatorname{span} T$ where

$$T = \left\{ \begin{bmatrix} 2 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Observation 2.6.7 Even though we found different bases for them, span S and span T are exactly the same subspace of $\mathbb{R}^4$, since

$$S = \left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ -1 \\ 0 \end{bmatrix} \right\} = \left\{ \begin{bmatrix} 2 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 0 \\ 1 \end{bmatrix} \right\} = T.$$

Thus the basis for a subspace is not unique in general.

Subspace Basis and Dimension (EV6)

Fact 2.6.8 *Any non-trivial real vector space has infinitely-many different bases, but all the bases for a given vector space are exactly the same size.*

For example,

$$\{\vec{e}_1, \vec{e}_2, \vec{e}_3\} \text{ and } \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\} \text{ and } \left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ 5 \end{bmatrix} \right\}$$

are all valid bases for $\mathbb{R}^3$*, and they all contain three vectors.*

Definition 2.6.9 The **dimension** of a vector space or subspace is equal to the size of any basis for the vector space.

As you'd expect, $\mathbb{R}^n$ has dimension n. For example, $\mathbb{R}^3$ has dimension 3 because any basis for $\mathbb{R}^3$ such as

$$\{\vec{e}_1, \vec{e}_2, \vec{e}_3\} \text{ and } \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\} \text{ and } \left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ 5 \end{bmatrix} \right\}$$

contains exactly three vectors. $\Diamond$

Subspace Basis and Dimension (EV6)

Activity 2.6.10 Consider the following subspace W of $\mathbb{R}^4$:

$$W = \operatorname{span}\left\{\begin{bmatrix}1\\0\\0\\-1\end{bmatrix}, \begin{bmatrix}-2\\0\\0\\2\end{bmatrix}, \begin{bmatrix}-3\\1\\-5\\5\end{bmatrix}, \begin{bmatrix}12\\-3\\15\\-18\end{bmatrix}\right\}.$$

(a) Explain and demonstrate how to find a basis of W.

(b) Explain and demonstrate how to find the dimension of W.

Subspace Basis and Dimension (EV6)

Activity 2.6.11 The dimension of a subspace may be found by doing what with an appropriate RREF matrix?

A. Count the rows.

B. Count the non-pivot columns.

C. Count the pivots.

D. Add the number of pivot rows and pivot columns.

Activity 2.6.12 In Observation 2.6.5, we found a basis for the subspace

$$W = \operatorname{span}\left\{\begin{bmatrix}1\\2\\3\end{bmatrix},\begin{bmatrix}2\\4\\6\end{bmatrix},\begin{bmatrix}0\\-2\\-2\end{bmatrix},\begin{bmatrix}1\\0\\1\end{bmatrix}\right\}.$$

To do so, we use the results of the calculation:

$$\operatorname{RREF}\begin{bmatrix}1 & 2 & 0 & 1\\2 & 4 & -2 & 0\\3 & 6 & -2 & 1\end{bmatrix} = \begin{bmatrix}\boxed{1} & 2 & 0 & 1\\0 & 0 & \boxed{1} & 1\\0 & 0 & 0 & 0\end{bmatrix}$$

to conclude that the set $\left\{\begin{bmatrix}1\\2\\3\end{bmatrix},\begin{bmatrix}0\\-2\\-2\end{bmatrix}\right\}$, the set of vectors *corresponding* to the pivot columns of the RREF, is a basis for W.

(a) Explain why neither of the vectors $\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}0\\1\\0\end{bmatrix}$ are elements of W.

(b) Explain why this shows that, in general, when we calculate a basis for $W = \operatorname{span}\{\vec{v}_1, \ldots, \vec{v}_n\}$, the pivot columns of $\operatorname{RREF}[\vec{v}_1 \ldots \vec{v}_n]$ themselves do not form a basis for W.

2.7 Homogeneous Linear Systems (EV7)

Learning Outcomes

- Find a basis for the solution set of a homogeneous system of equations.

Homogeneous Linear Systems (EV7)

Remark 2.7.1 Recall from Section 2.3 that a **homogeneous** system of linear equations is one of the form:

$$\begin{array}{rrrrl} a_{11}x_1 + & a_{12}x_2 + & \ldots + & a_{1n}x_n & = 0 \\ a_{21}x_1 + & a_{22}x_2 + & \ldots + & a_{2n}x_n & = 0 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1}x_1 + & a_{m2}x_2 + & \ldots + & a_{mn}x_n & = 0 \end{array}$$

This system is equivalent to the vector equation:

$$x_1\vec{v}_1 + \cdots + x_n\vec{v}_n = \vec{0}$$

and the augmented matrix:

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & 0 \\ a_{21} & a_{22} & \cdots & a_{2n} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & 0 \end{array}\right].$$

Homogeneous Linear Systems (EV7)

Activity 2.7.2

(a) In Section 2.3, we observed that if

$$x_1\vec{v}_1 + \cdots + x_n\vec{v}_n = \vec{0}$$

is a homogeneous vector equation, then:

- The zero vector $\vec{0}$ is a solution;
- The sum of any two solutions is again a solution;
- Multiplying a solution by a scalar produces another solution.

(b) Based on this recollection, which of the following best describes the solution set to the homogeneous equation?

A. A basis for $\mathbb{R}^n$.

B. A subspace of $\mathbb{R}^n$.

C. All of $\mathbb{R}^n$.

D. The empty set.

Homogeneous Linear Systems (EV7)

Activity 2.7.3 Consider the homogeneous system of equations

$$\begin{aligned} x_1 + 2x_2 \qquad + \ x_4 &= 0 \\ 2x_1 + 4x_2 - x_3 - 2x_4 &= 0 \\ 3x_1 + 6x_2 - x_3 - \ x_4 &= 0 \end{aligned}$$

(a) Find its solution set (a subspace of $\mathbb{R}^4$).

(b) Rewrite this solution space in the form

$$\left\{ a \begin{bmatrix} ? \\ ? \\ ? \\ ? \end{bmatrix} + b \begin{bmatrix} ? \\ ? \\ ? \\ ? \end{bmatrix} \middle| a, b \in \mathbb{R} \right\}.$$

(c) Which of these choices best describes the set of two vectors $\left\{ \begin{bmatrix} ? \\ ? \\ ? \\ ? \end{bmatrix}, \begin{bmatrix} ? \\ ? \\ ? \\ ? \end{bmatrix} \right\}$ used in this solution space?

A. The set is linearly dependent.

B. The set is linearly independent.

C. The set spans the solution space.

D. The set is a basis of the solution space.

Homogeneous Linear Systems (EV7)

Activity 2.7.4 Consider the homogeneous system of equations

$$\begin{aligned} 2x_1 + 4x_2 + 2x_3 - 3x_4 + 31x_5 + 2x_6 - 16x_7 &= 0 \\ -1x_1 - 2x_2 + 4x_3 - x_4 + 2x_5 + 9x_6 + 3x_7 &= 0 \\ x_1 + 2x_2 + x_3 + x_4 + 3x_5 + 6x_6 + 7x_7 &= 0 \end{aligned}$$

(a) Find its solution set (a subspace of $\mathbb{R}^7$).

(b) Rewrite this solution space in the form

$$\left\{ a \begin{bmatrix} ? \\ ? \\ ? \\ ? \\ ? \\ ? \\ ? \end{bmatrix} + b \begin{bmatrix} ? \\ ? \\ ? \\ ? \\ ? \\ ? \\ ? \end{bmatrix} + c \begin{bmatrix} ? \\ ? \\ ? \\ ? \\ ? \\ ? \\ ? \end{bmatrix} + d \begin{bmatrix} ? \\ ? \\ ? \\ ? \\ ? \\ ? \\ ? \end{bmatrix} \middle| \, a, b, c, d \in \mathbb{R} \right\}.$$

(c) Which of these choices best describes the set of vectors $\left\{ \begin{bmatrix} ? \\ ? \\ ? \\ ? \\ ? \\ ? \\ ? \end{bmatrix}, \begin{bmatrix} ? \\ ? \\ ? \\ ? \\ ? \\ ? \\ ? \end{bmatrix}, \begin{bmatrix} ? \\ ? \\ ? \\ ? \\ ? \\ ? \\ ? \end{bmatrix}, \begin{bmatrix} ? \\ ? \\ ? \\ ? \\ ? \\ ? \\ ? \end{bmatrix} \right\}$ used in this solution space?

A. The set is linearly dependent.

B. The set is linearly independent.

C. The set spans the solution space.

D. The set is a basis for the solution space.

Fact 2.7.5 *The coefficients of the free variables in the solution space of a linear system always yield linearly independent vectors that span the solution space.*

Thus if

$$\left\{ a\begin{bmatrix} -2\\1\\0\\0\\0\\0\\0 \end{bmatrix} + b\begin{bmatrix} -7\\0\\-1\\5\\1\\0\\0 \end{bmatrix} + c\begin{bmatrix} -1\\0\\-3\\-2\\0\\1\\0 \end{bmatrix} + d\begin{bmatrix} 1\\0\\-2\\-6\\0\\0\\1 \end{bmatrix} \;\middle|\; a,b,c,d\in\mathbb{R} \right\} = \operatorname{span}\left\{ \begin{bmatrix} -2\\1\\0\\0\\0\\0\\0 \end{bmatrix}, \begin{bmatrix} -7\\0\\-1\\5\\1\\0\\0 \end{bmatrix}, \begin{bmatrix} -1\\0\\-3\\-2\\0\\1\\0 \end{bmatrix}, \right.$$

is the solution space for a homogeneous system, then

$$\left\{ \begin{bmatrix} -2\\1\\0\\0\\0\\0\\0 \end{bmatrix}, \begin{bmatrix} -7\\0\\-1\\5\\1\\0\\0 \end{bmatrix}, \begin{bmatrix} -1\\0\\-3\\-2\\0\\1\\0 \end{bmatrix}, \begin{bmatrix} 1\\0\\-2\\-6\\0\\0\\1 \end{bmatrix} \right\}$$

is a basis for the solution space.

Homogeneous Linear Systems (EV7)

Activity 2.7.6 Consider the homogeneous system of equations

$$\begin{aligned} x_1 - 3x_2 + 2x_3 &= 0 \\ 2x_1 + 6x_2 + 4x_3 &= 0 \\ x_1 + 6x_2 - 4x_3 &= 0 \end{aligned}$$

(a) Find its solution space.

(b) Which of these is the best choice of basis for this solution space?

A $\{\}$

B $\{\vec{0}\}$

C The basis does not exist

Homogeneous Linear Systems (EV7)

Activity 2.7.7 To create a computer-animated film, an animator first models a scene as a subset of $\mathbb{R}^3$. Then to transform this three-dimensional visual data for display on a two-dimensional movie screen or television set, the computer could apply a linear transformation that maps visual information at the point $(x, y, z) \in \mathbb{R}^3$ onto the pixel located at $(x + y, y - z) \in \mathbb{R}^2$.

(a) What homogeneous linear system describes the positions (x, y, z) within the original scene that would be aligned with the pixel $(0, 0)$ on the screen?

(b) Solve this system to describe these locations.

Activity 2.7.8 Let $S = \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -4 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -2 \\ 3 \end{bmatrix} \right\}$ and $A = \begin{bmatrix} -2 & -1 & 1 \\ 1 & 0 & 0 \\ 0 & -4 & -2 \\ 0 & 1 & 3 \end{bmatrix}$; note that

$$\operatorname{RREF}(A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

The following statements are all *invalid* for at least one reason. Determine what makes them invalid and, suggest alternative *valid* statements that the author may have meant instead.

(a) The matrix A is linearly independent because $\operatorname{RREF}(A)$ has a pivot in each column.

(b) The matrix A does not span $\mathbb{R}^4$ because $\operatorname{RREF}(A)$ has a row of zeroes.

(c) The set of vectors S spans.

(d) The set of vectors S is a basis.

Chapter 3

Algebraic Properties of Linear Maps (AT)

Learning Outcomes

How can we understand linear maps algebraically?
By the end of this chapter, you should be able to...

1. Determine if a map between Euclidean vector spaces is linear or not.
2. Translate back and forth between a linear transformation of Euclidean spaces and its standard matrix, and perform related computations.
3. Compute a basis for the kernel and a basis for the image of a linear map, and verify that the rank-nullity theorem holds for a given linear map.
4. Determine if a given linear map is injective and/or surjective.
5. Explain why a given set with defined addition and scalar multiplication does satisfy a given vector space property, but nonetheless isn't a vector space.
6. Answer questions about vector spaces of polynomials or matrices.

Readiness Assurance.

Before beginning this chapter, you should be able to...

1. State the definition of a spanning set, and determine if a set of Euclidean vectors spans $\mathbb{R}^n$.
   - Review: Section 2.2
2. State the definition of linear independence, and determine if a set of Euclidean vectors is linearly dependent or independent.
   - Review: Section 2.4
3. State the definition of a basis, and determine if a set of Euclidean vectors is a basis.
   - Review: Section 2.5, Section 2.6
4. Find a basis of the solution space to a homogeneous system of linear equations.
   - Review: Section 2.7

3.1 Linear Transformations (AT1)

Learning Outcomes

- Determine if a map between Euclidean vector spaces is linear or not.

Linear Transformations (AT1)

Activity 3.1.1

(a) What is our definition for a set S of vectors to be linearly independent?

(b) What specific calculation would you perform to test is a set S of Euclidean vectors is linearly independent?

Linear Transformations (AT1)

Activity 3.1.2

(a) What is our definition for a set S of vectors in $\mathbb{R}^n$ to span $\mathbb{R}^n$?

(b) What specific calculation would you perform to test is a set S of Euclidean vectors spans all of $\mathbb{R}^n$?

Linear Transformations (AT1)

Definition 3.1.3 A **linear transformation** (also called a **linear map**) is a map between vector spaces that preserves the vector space operations. More precisely, if V and W are vector spaces, a map $T : V \to W$ is called a linear transformation if

1. $T(\vec{v} + \vec{w}) = T(\vec{v}) + T(\vec{w})$ for any $\vec{v}, \vec{w} \in V$, and
2. $T(c\vec{v}) = cT(\vec{v})$ for any $c \in \mathbb{R}$, and $\vec{v} \in V$.

In other words, a map is linear when vector space operations can be applied before or after the transformation without affecting the result. $\Diamond$

Definition 3.1.4 Given a linear transformation $T : V \to W$, V is called the **domain** of T and W is called the **co-domain** of T.

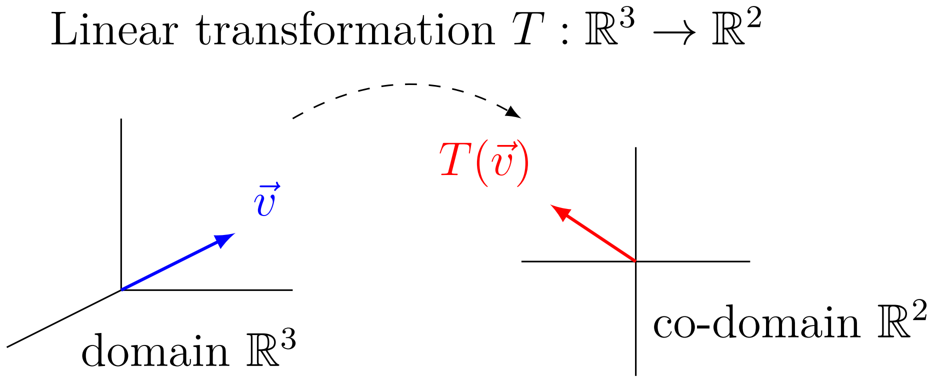

Figure 7 A linear transformation with a domain of $\mathbb{R}^3$ and a co-domain of $\mathbb{R}^2$

$\Diamond$

Observation 3.1.5 One example of a linear transformation $\mathbb{R}^3 \to \mathbb{R}^2$ is the projection of three-dimensional data onto a two-dimensional screen, as is necessary for computer animation in film or video games.

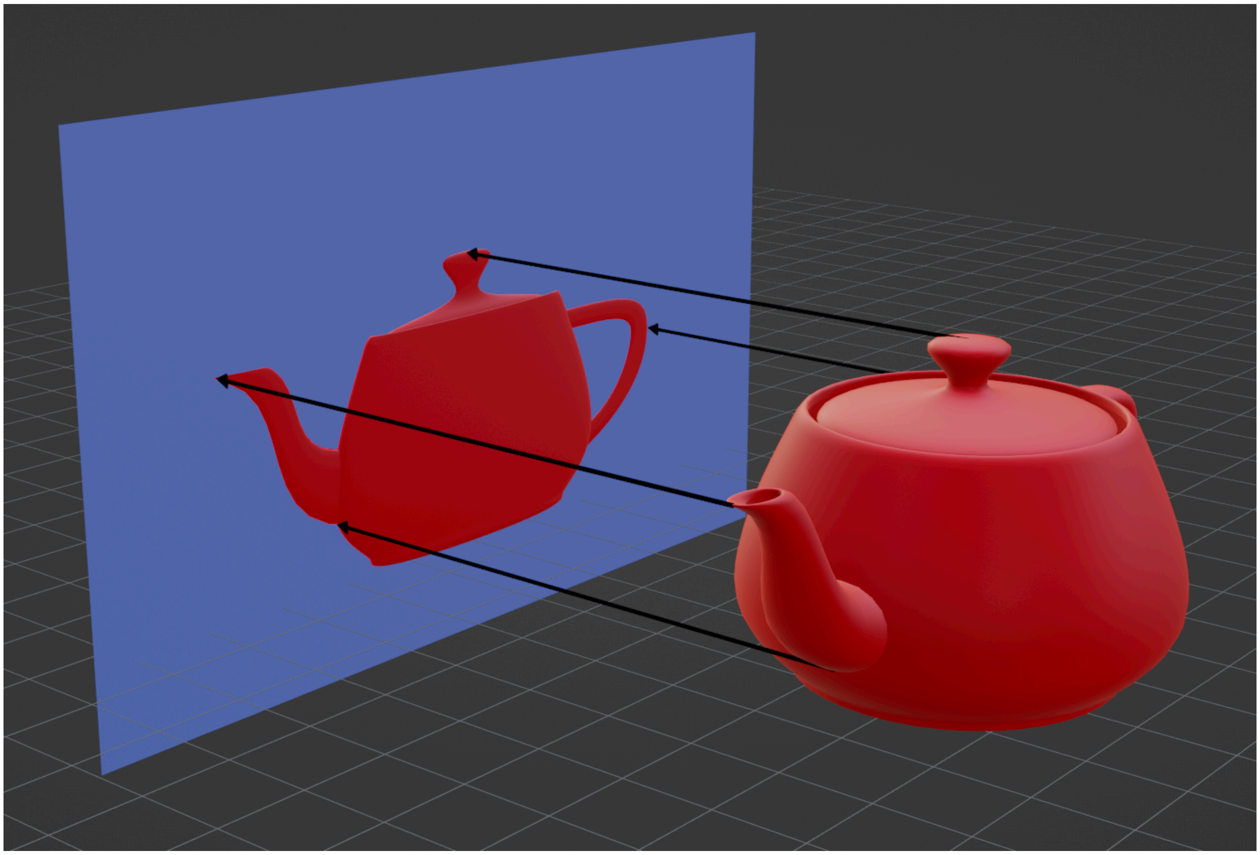

Figure 8 A projection of a $3D$ teapot onto a $2D$ screen

Linear Transformations (AT1)

Activity 3.1.6 Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x - z \\ 3y \end{bmatrix}.$$

(a) Compute the result of adding vectors before a T transformation:

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} u \\ v \\ w \end{bmatrix}\right) = T\left(\begin{bmatrix} x+u \\ y+v \\ z+w \end{bmatrix}\right)$$

A. $\begin{bmatrix} x - u + z - w \\ 3y - 3v \end{bmatrix}$

B. $\begin{bmatrix} x + u - z - w \\ 3y + 3v \end{bmatrix}$

C. $\begin{bmatrix} x + u \\ 3y + 3v \\ z + w \end{bmatrix}$

D. $\begin{bmatrix} x - u \\ 3y - 3v \\ z - w \end{bmatrix}$

(b) Compute the result of adding vectors after a T transformation:

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) + T\left(\begin{bmatrix} u \\ v \\ w \end{bmatrix}\right) = \begin{bmatrix} x - z \\ 3y \end{bmatrix} + \begin{bmatrix} u - w \\ 3v \end{bmatrix}$$

A. $\begin{bmatrix} x - u + z - w \\ 3y - 3v \end{bmatrix}$

B. $\begin{bmatrix} x + u - z - w \\ 3y + 3v \end{bmatrix}$

C. $\begin{bmatrix} x + u \\ 3y + 3v \\ z + w \end{bmatrix}$

D. $\begin{bmatrix} x - u \\ 3y - 3v \\ z - w \end{bmatrix}$

(c) Is T a linear transformation?

A. Yes.

B. No.

C. More work is necessary to know.

(d) Compute the result of scalar multiplication before a T transformation:

$$T\left(c\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = T\left(\begin{bmatrix} cx \\ cy \\ cz \end{bmatrix}\right)$$

A. $\begin{bmatrix} cx - cz \\ 3cy \end{bmatrix}$

B. $\begin{bmatrix} cx + cz \\ -3cy \end{bmatrix}$

C. $\begin{bmatrix} x + c \\ 3y + c \\ z + c \end{bmatrix}$

D. $\begin{bmatrix} x - c \\ 3y - c \\ z - c \end{bmatrix}$

(e) Compute the result of scalar multiplication after a T transformation:

$$cT\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = c\begin{bmatrix} x - z \\ 3y \end{bmatrix}$$

A. $\begin{bmatrix} cx - cz \\ 3cy \end{bmatrix}$

B. $\begin{bmatrix} cx + cz \\ -3cy \end{bmatrix}$

C. $\begin{bmatrix} x + c \\ 3y + c \\ z + c \end{bmatrix}$

D. $\begin{bmatrix} x - c \\ 3y - c \\ z - c \end{bmatrix}$

(f) Is T a linear transformation?

A. Yes.

B. No.

C. More work is necessary to know.

Linear Transformations (AT1)

Activity 3.1.7 Let $S : \mathbb{R}^2 \to \mathbb{R}^4$ be given by

$$S\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x+y \\ x^2 \\ y+3 \\ y-2^x \end{bmatrix}$$

(a) Compute

$$S\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \end{bmatrix}\right) = S\left(\begin{bmatrix} 2 \\ 4 \end{bmatrix}\right)$$

A. $\begin{bmatrix} 6 \\ 4 \\ 7 \\ 0 \end{bmatrix}$ B. $\begin{bmatrix} -3 \\ 0 \\ 1 \\ 5 \end{bmatrix}$ C. $\begin{bmatrix} -3 \\ -1 \\ 7 \\ 5 \end{bmatrix}$ D. $\begin{bmatrix} 6 \\ 4 \\ 10 \\ -1 \end{bmatrix}$

(b) Compute

$$S\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) + S\left(\begin{bmatrix} 2 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 0+1 \\ 0^2 \\ 1+3 \\ 1-2^0 \end{bmatrix} + \begin{bmatrix} 2+3 \\ 2^2 \\ 3+3 \\ 3-2^2 \end{bmatrix}$$

A. $\begin{bmatrix} 6 \\ 4 \\ 7 \\ 0 \end{bmatrix}$ B. $\begin{bmatrix} -3 \\ 0 \\ 1 \\ 5 \end{bmatrix}$ C. $\begin{bmatrix} -3 \\ -1 \\ 7 \\ 5 \end{bmatrix}$ D. $\begin{bmatrix} 6 \\ 4 \\ 10 \\ -1 \end{bmatrix}$

(c) Is S a linear transformation?

A. Yes.

B. No.

C. More work is necessary to know.

Linear Transformations (AT1)

Activity 3.1.8 Fill in the ?s, assuming $T : \mathbb{R}^3 \to \mathbb{R}^3$ is linear:

$$T\left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right) = T\left(\,?\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right) = \,?\,T\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} ? \\ ? \\ ? \end{bmatrix}$$

Remark 3.1.9 In summary, *any one* of the following is enough to prove that $T : V \to W$ is *not* a linear transformation:

- Find specific values for $\vec{v}, \vec{w} \in V$ such that $T(\vec{v} + \vec{w}) \neq T(\vec{v}) + T(\vec{w})$.
- Find specific values for $\vec{v} \in V$ and $c \in \mathbb{R}$ such that $T(c\vec{v}) \neq cT(\vec{v})$.
- Show $T(\vec{0}) \neq \vec{0}$.

If you cannot do any of these, then T can be proven to be a linear transformation by doing *both* of the following:

1. For all $\vec{v}, \vec{w} \in V$ (not just specific values), $T(\vec{v} + \vec{w}) = T(\vec{v}) + T(\vec{w})$.
2. For all $\vec{v} \in V$ and $c \in \mathbb{R}$ (not just specific values), $T(c\vec{v}) = cT(\vec{v})$.

(Note the similarities between this process and showing that a subset of a vector space is or is not a subspace: Remark 2.3.14.)

Linear Transformations (AT1)

Activity 3.1.10

(a) Consider the following maps of Euclidean vectors $P : \mathbb{R}^3 \to \mathbb{R}^3$ and $Q : \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$P\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} -2\,x - 3\,y - 3\,z \\ 3\,x + 4\,y + 4\,z \\ 3\,x + 4\,y + 5\,z \end{bmatrix} \quad \text{and} \quad Q\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x - 4\,y + 9\,z \\ y - 2\,z \\ 8\,y^2 - 3\,xz \end{bmatrix}.$$

Which do you *suspect*?

A. P is linear, but Q is not.

B. Q is linear, but P is not.

C. Both maps are linear.

D. Neither map is linear.

(b) Consider the following map of Euclidean vectors $S : \mathbb{R}^2 \to \mathbb{R}^2$

$$S\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x + 2\,y \\ 9\,xy \end{bmatrix}.$$

Prove that S *is not* a linear transformation.

(c) Consider the following map of Euclidean vectors $T : \mathbb{R}^2 \to \mathbb{R}^2$

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 8\,x - 6\,y \\ 6\,x - 4\,y \end{bmatrix}.$$

Prove that T *is* a linear transformation.

Linear Transformations (AT1)

Activity 3.1.11 Let $f(x) = x^3 - 1$. Then, $f\colon \mathbb{R} \to \mathbb{R}$ is a function with domain and codomain equal to $\mathbb{R}$. Is $f(x)$ is a linear transformation?

Linear Transformations (AT1)

Activity 3.1.12 Consider two vectors $\vec{u}, \vec{v}$ and their sum $\vec{u} + \vec{v}$.

(a) Is it the case that rotating $\vec{u} + \vec{v}$ about the origin by $\frac{\pi}{2} = 90°$ is the same as first rotating each of $\vec{u}, \vec{v}$ and then adding them together?

(b) Is it the case that rotating $5\vec{u}$ about the origin by $\frac{\pi}{2} = 90°$ is the same as first rotating $\vec{u}$ by $\frac{\pi}{2} = 90°$ and then scaling by 5?

(c) Based on this, do you suspect that the transformation $R\colon \mathbb{R}^2 \to \mathbb{R}^2$ given by rotating vectors about the origin through an angle of $\frac{\pi}{2} = 90°$ is linear? Do you think there is anything special about the angle $\frac{\pi}{2} = 90°$?

Linear Transformations (AT1)

Activity 3.1.13 In Activity 2.2.1, we made an analogy between vectors and linear combinations with ingredients and recipes. Let us think of *cooking* as a transformation of ingredients. In this analogy, would it be appropriate for us to consider "cooking" to be a linear transformation or not? Describe your reasoning.

3.2 Standard Matrices (AT2)

Learning Outcomes

- Translate back and forth between a linear transformation of Euclidean spaces and its standard matrix, and perform related computations.

Standard Matrices (AT2)

Remark 3.2.1 Recall that a linear map $T : V \to W$ satisfies

1. $T(\vec{v} + \vec{w}) = T(\vec{v}) + T(\vec{w})$ for any $\vec{v}, \vec{w} \in V$.

2. $T(c\vec{v}) = cT(\vec{v})$ for any $c \in \mathbb{R}, \vec{v} \in V$.

In other words, a map is linear when vector space operations can be applied before or after the transformation without affecting the result.

Standard Matrices (AT2)

Activity 3.2.2 Can you recall the following?

(a) Given a transformation, what do the terms *domain* and *codomain* mean?

(b) What does the notation $T\colon V \to W$ mean?

Standard Matrices (AT2)

Activity 3.2.3 Suppose $T : \mathbb{R}^3 \to \mathbb{R}^2$ is a linear map, and you know $T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$. What is $T\left(\begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}\right)$?

A. $\begin{bmatrix} 6 \\ 3 \end{bmatrix}$

B. $\begin{bmatrix} -9 \\ 6 \end{bmatrix}$

C. $\begin{bmatrix} -4 \\ -2 \end{bmatrix}$

D. $\begin{bmatrix} 6 \\ -4 \end{bmatrix}$

Standard Matrices (AT2)

Activity 3.2.4 Suppose $T : \mathbb{R}^3 \to \mathbb{R}^2$ is a linear map, and you know $T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$. What is $T\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right)$?

A. $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

B. $\begin{bmatrix} 3 \\ -1 \end{bmatrix}$

C. $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$

D. $\begin{bmatrix} 5 \\ -8 \end{bmatrix}$

Standard Matrices (AT2)

Activity 3.2.5 Suppose $T : \mathbb{R}^3 \to \mathbb{R}^2$ is a linear map, and you know $T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$. What is $T\left(\begin{bmatrix} -2 \\ 0 \\ -3 \end{bmatrix}\right)$?

A. $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

B. $\begin{bmatrix} 3 \\ -1 \end{bmatrix}$

C. $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$

D. $\begin{bmatrix} 5 \\ -8 \end{bmatrix}$

Standard Matrices (AT2)

Activity 3.2.6 Suppose $T:\mathbb{R}^3\to\mathbb{R}^2$ is a linear map, and you know $T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right)=\begin{bmatrix}2\\1\end{bmatrix}$ and $T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right)=\begin{bmatrix}-3\\2\end{bmatrix}$. What piece of information would help you compute $T\left(\begin{bmatrix}0\\4\\-1\end{bmatrix}\right)$?

A. The value of $T\left(\begin{bmatrix}0\\4\\0\end{bmatrix}\right)=\begin{bmatrix}-4\\16\end{bmatrix}$.

B. The value of $T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right)=\begin{bmatrix}-1\\4\end{bmatrix}$.

C. The value of $T\left(\begin{bmatrix}1\\1\\1\end{bmatrix}\right)=\begin{bmatrix}-2\\7\end{bmatrix}$.

D. Any of the above.

Observation 3.2.7 Since all three choices in Activity 3.2.6 create a spanning and linearly independent set along with $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, they each may be used to compute $T\left(\begin{bmatrix} 0 \\ 4 \\ -1 \end{bmatrix}\right)$:

$$T\left(\begin{bmatrix} 0 \\ 4 \\ -1 \end{bmatrix}\right) = T\left(\begin{bmatrix} 0 \\ 4 \\ 0 \end{bmatrix}\right) - T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -4 \\ 16 \end{bmatrix} - \begin{bmatrix} -3 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 14 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 0 \\ 4 \\ -1 \end{bmatrix}\right) = 4T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) - T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = 4\begin{bmatrix} -1 \\ 4 \end{bmatrix} - \begin{bmatrix} -3 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 14 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 0 \\ 4 \\ -1 \end{bmatrix}\right) = 4T\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right) - 5T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) - 4T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right)$$

$$= 4\begin{bmatrix} -2 \\ 7 \end{bmatrix} - 5\begin{bmatrix} -3 \\ 2 \end{bmatrix} - 4\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -8+15-8 \\ 28-10-4 \end{bmatrix} = \begin{bmatrix} -1 \\ 14 \end{bmatrix}$$

Standard Matrices (AT2)

Fact 3.2.8 *Consider any basis* $\{\vec{b}_1, \ldots, \vec{b}_n\}$ *for* V*. Since every vector* $\vec{v}$ *can be written as a linear combination of basis vectors,* $\vec{v} = x_1\vec{b}_1 + \cdots + x_n\vec{b}_n$*, we may compute* $T(\vec{v})$ *as follows:*

$$T(\vec{v}) = T(x_1\vec{b}_1 + \cdots + x_n\vec{b}_n) = x_1T(\vec{b}_1) + \cdots + x_nT(\vec{b}_n).$$

Therefore any linear transformation $T : V \to W$ *can be defined by just describing the values of* $T(\vec{b}_i)$*.*

Put another way, the images of the basis vectors completely ***determine*** *the transformation* T*.*

Standard Matrices (AT2)

Definition 3.2.9 Since a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is determined by its action on the standard basis $\{\vec{e}_1, \ldots, \vec{e}_n\}$, it is convenient to store this information in an $m \times n$ matrix, called the **standard matrix** of T, given by $[T(\vec{e}_1) \cdots T(\vec{e}_n)]$.

For example, let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear map determined by the following values for T applied to the standard basis of $\mathbb{R}^3$.

$$T\left(\vec{e}_1\right) = T\left(\left[\begin{smallmatrix} 1 \\ 0 \\ 0 \end{smallmatrix}\right]\right) = \left[\begin{smallmatrix} 2 \\ 1 \end{smallmatrix}\right] \quad T\left(\vec{e}_2\right) = T\left(\left[\begin{smallmatrix} 0 \\ 1 \\ 0 \end{smallmatrix}\right]\right) = \left[\begin{smallmatrix} -1 \\ 4 \end{smallmatrix}\right] \quad T\left(\vec{e}_3\right) = T\left(\left[\begin{smallmatrix} 0 \\ 0 \\ 1 \end{smallmatrix}\right]\right) = \left[\begin{smallmatrix} -3 \\ 2 \end{smallmatrix}\right]$$

Then the standard matrix corresponding to T is

$$\left[\begin{array}{ccc} T(\vec{e}_1) & T(\vec{e}_2) & T(\vec{e}_3) \end{array}\right] = \left[\begin{array}{ccc} 2 & -1 & -3 \\ 1 & 4 & 2 \end{array}\right].$$

$\Diamond$

Standard Matrices (AT2)

Activity 3.2.10 Let $T : \mathbb{R}^4 \to \mathbb{R}^3$ be the linear transformation given by

$$T(\vec{e}_1) = \begin{bmatrix} 0 \\ 3 \\ -2 \end{bmatrix} \qquad T(\vec{e}_2) = \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \qquad T(\vec{e}_3) = \begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix} \qquad T(\vec{e}_4) = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$$

Write the standard matrix $[T(\vec{e}_1) \cdots T(\vec{e}_n)]$ for T.

Standard Matrices (AT2)

Activity 3.2.11 Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x + 3z \\ 2x - y - 4z \end{bmatrix}$$

(a) Compute $T(\vec{e}_1)$, $T(\vec{e}_2)$, and $T(\vec{e}_3)$.

(b) Find the standard matrix for T.

Standard Matrices (AT2)

Fact 3.2.12 *Because every linear map $T : \mathbb{R}^n \to \mathbb{R}^m$ has a linear combination of the variables in each component, and thus $T(\vec{e}_i)$ yields exactly the coefficients of x_i, the standard matrix for T is simply an array of the coefficients of the x_i:*

$$T\left(\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}\right) = \begin{bmatrix} ax + by + cz + dw \\ ex + fy + gz + hw \end{bmatrix} \qquad A = \begin{bmatrix} a & b & c & d \\ e & f & g & h \end{bmatrix}$$

Since the formula for a linear transformation T and its standard matrix A may both be used to compute the transformation of a vector $\vec{x}$, we will often write $T(\vec{x})$ and $A\vec{x}$ interchangeably:

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\right) = \begin{bmatrix} ax_1 + bx_2 + cx_3 + dx_4 \\ ex_1 + fx_2 + gx_3 + hx_4 \end{bmatrix} = A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} a & b & c & d \\ e & f & g & h \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

Standard Matrices (AT2)

Activity 3.2.13

(a) Explain and demonstrate how to compute the standard matrix for the linear transformation $S : \mathbb{R}^2 \to \mathbb{R}^4$ given by

$$S\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 9\,x_1 - 2\,x_2 \\ -3\,x_1 \\ 5\,x_1 - x_2 \\ -6\,x_2 \end{bmatrix}$$

by computing transformations of the standard basic vectors:

$$S(\vec{e}_1) = \begin{bmatrix} ? \\ ? \\ ? \\ ? \end{bmatrix} \quad S(\vec{e}_2) = \begin{bmatrix} ? \\ ? \\ ? \\ ? \end{bmatrix} \quad \to \quad \begin{bmatrix} ? & ? \\ ? & ? \\ ? & ? \\ ? & ? \end{bmatrix}$$

(b) Let $T : \mathbb{R}^4 \to \mathbb{R}^3$ be the linear transformation given by the standard matrix

$$\begin{bmatrix} -2 & -4 & 2 & -2 \\ -4 & 3 & -3 & 2 \\ 5 & 0 & 2 & -6 \end{bmatrix}.$$

Explain and demonstrate how to compute $T\left(\begin{bmatrix} -5 \\ 0 \\ -3 \\ -2 \end{bmatrix}\right)$ by using the values of transformed standard basic vectors:

$$T\left(\begin{bmatrix} -5 \\ 0 \\ -3 \\ -2 \end{bmatrix}\right) = \,?\,T(\vec{e}_1) + \,?\,T(\vec{e}_2) + \,?\,T(\vec{e}_3) + \,?\,T(\vec{e}_4)$$

Standard Matrices (AT2)

Activity 3.2.14 Consider the linear transformation $R\colon \mathbb{R}^2 \to \mathbb{R}^2$ given by rotating vectors about the origin through an angle of $\frac{\pi}{4} = 45°$.

(a) If $\vec{e}_1, \vec{e}_2$ are the standard basis vectors of $\mathbb{R}^2$, calculate $R(\vec{e}_1), R(\vec{e}_2)$.

(b) What is the standard matrix representing R?

Standard Matrices (AT2)

Activity 3.2.15 Consider the linear transformation $S\colon \mathbb{R}^2 \to \mathbb{R}^2$ given by reflecting vectors across the line $x_1 = x_2$.

(a) If $\vec{e}_1, \vec{e}_2$ are the standard basis vectors of $\mathbb{R}^2$, calculate $S(\vec{e}_1), S(\vec{e}_2)$.

(b) What is the standard matrix representing S?

3.3 Image and Kernel (AT3)

Learning Outcomes

- Compute a basis for the kernel and a basis for the image of a linear map, and verify that the rank-nullity theorem holds for a given linear map.

Image and Kernel (AT3)

Activity 3.3.1 Consider the matrix $A = \begin{bmatrix} 3 & 4 & 7 & 1 \\ -1 & 1 & 0 & 2 \\ 2 & 1 & 3 & -1 \end{bmatrix}$.

(a) The matrix A is the standard matrix of a linear transformation T. What is the domain and the codomain of the transformation T?

(b) Describe how T transforms the standard basis vectors of the domain that you found above.

Image and Kernel (AT3)

Activity 3.3.2 Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be given by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} \qquad \text{with standard matrix } \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Which of these subspaces of $\mathbb{R}^2$ describes the set of all vectors that transform into $\vec{0}$?

A. $\left\{\begin{bmatrix} a \\ a \end{bmatrix} \middle| a \in \mathbb{R}\right\}$

B. $\left\{\begin{bmatrix} a \\ 0 \end{bmatrix} \middle| a \in \mathbb{R}\right\}$

C. $\left\{\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right\}$

D. $\left\{\begin{bmatrix} a \\ b \end{bmatrix} \middle| a, b \in \mathbb{R}\right\}$

Definition 3.3.3 Let $T : V \to W$ be a linear transformation, and let $\vec{z}$ be the additive identity (the "zero vector") of W. The **kernel** of T (also known as the **null space** of T) is an important subspace of V defined by

$$\ker T = \left\{ \vec{v} \in V \;\middle|\; T(\vec{v}) = \vec{z} \right\}$$

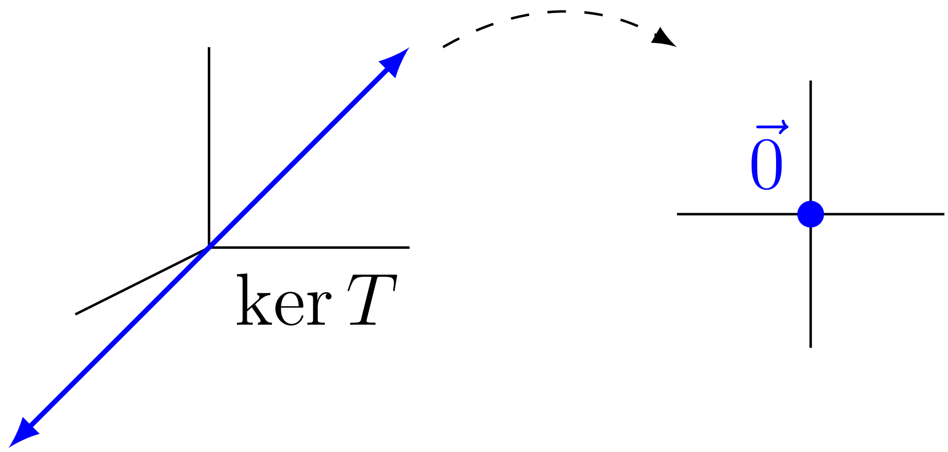

Figure 9 The kernel of a linear transformation

◊

Image and Kernel (AT3)

Activity 3.3.4 Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x \\ y \end{bmatrix} \qquad \text{with standard matrix } \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Which of these subspaces of $\mathbb{R}^3$ describes $\ker T$, the set of all vectors that transform into $\vec{0}$?

A. $\left\{ \begin{bmatrix} 0 \\ 0 \\ a \end{bmatrix} \middle| \, a \in \mathbb{R} \right\}$

B. $\left\{ \begin{bmatrix} a \\ a \\ 0 \end{bmatrix} \middle| \, a \in \mathbb{R} \right\}$

C. $\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$

D. $\left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} \middle| \, a, b, c \in \mathbb{R} \right\}$

Image and Kernel (AT3)

Activity 3.3.5 Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation given by the standard matrix

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 3x + 4y - z \\ x + 2y + z \end{bmatrix}$$

(a) Set $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ to find a linear system of equations whose solution set is the kernel.

(b) Use RREF(A) to solve this homogeneous system of equations and find a basis for the kernel of T.

Image and Kernel (AT3)

Activity 3.3.6 Let $T : \mathbb{R}^4 \to \mathbb{R}^3$ be the linear transformation given by

$$T\left(\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}\right) = \begin{bmatrix} 2x + 4y + 2z - 4w \\ -2x - 4y + z + w \\ 3x + 6y - z - 4w \end{bmatrix}.$$

Find a basis for the kernel of T.

Activity 3.3.7 Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be given by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} \qquad \text{with standard matrix } \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Which of these subspaces of $\mathbb{R}^3$ describes the set of all vectors that are the result of using T to transform $\mathbb{R}^2$ vectors?

A. $\left\{ \begin{bmatrix} 0 \\ 0 \\ a \end{bmatrix} \middle| \, a \in \mathbb{R} \right\}$

B. $\left\{ \begin{bmatrix} a \\ b \\ 0 \end{bmatrix} \middle| \, a, b \in \mathbb{R} \right\}$

C. $\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$

D. $\left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} \middle| \, a, b, c \in \mathbb{R} \right\}$

Definition 3.3.8 Let $T : V \to W$ be a linear transformation. The **image** of T is an important subspace of W defined by

$$\operatorname{Im} T = \left\{ \vec{w} \in W \;\middle|\; \text{there is some } \vec{v} \in V \text{ with } T(\vec{v}) = \vec{w} \right\}$$

In the examples below, the left example's image is all of $\mathbb{R}^2$, but the right example's image is a planar subspace of $\mathbb{R}^3$.

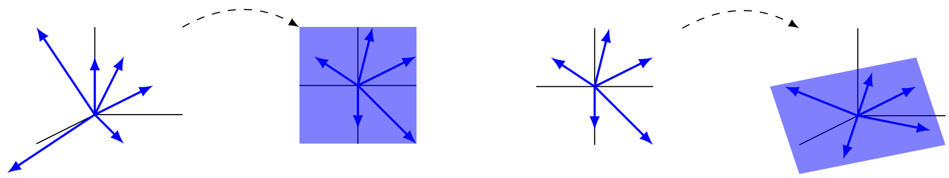

Figure 10 The image of a linear transformation

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Image and Kernel (AT3)

Activity 3.3.9 Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x \\ y \end{bmatrix} \qquad \text{with standard matrix } \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Which of these subspaces of $\mathbb{R}^2$ describes $\operatorname{Im} T$, the set of all vectors that are the result of using T to transform $\mathbb{R}^3$ vectors?

A. $\left\{\begin{bmatrix} a \\ a \end{bmatrix} \middle| a \in \mathbb{R}\right\}$

B. $\left\{\begin{bmatrix} a \\ 0 \end{bmatrix} \middle| a \in \mathbb{R}\right\}$

C. $\left\{\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right\}$

D. $\left\{\begin{bmatrix} a \\ b \end{bmatrix} \middle| a, b \in \mathbb{R}\right\}$

Image and Kernel (AT3)

Activity 3.3.10 Let $T : \mathbb{R}^4 \to \mathbb{R}^3$ be the linear transformation given by the standard matrix

$$A = \begin{bmatrix} 3 & 4 & 7 & 1 \\ -1 & 1 & 0 & 2 \\ 2 & 1 & 3 & -1 \end{bmatrix} = \begin{bmatrix} T(\vec{e}_1) & T(\vec{e}_2) & T(\vec{e}_3) & T(\vec{e}_4) \end{bmatrix}.$$

Consider the question: Which vectors $\vec{w}$ in $\mathbb{R}^3$ belong to $\operatorname{Im} T$?

(a) Determine if $\begin{bmatrix} 12 \\ 3 \\ 3 \end{bmatrix}$ belongs to $\operatorname{Im} T$.

(b) Determine if $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ belongs to $\operatorname{Im} T$.

(c) An arbitrary vector $\begin{bmatrix} ? \\ ? \\ ? \end{bmatrix}$ belongs to $\operatorname{Im} T$ provided the equation

$$x_1 T(\vec{e}_1) + x_2 T(\vec{e}_2) + x_3 T(\vec{e}_3) + x_4 T(\vec{e}_4) = \vec{w}$$

has...

A. no solutions.

B. exactly one solution.

C. at least one solution.

D. infinitely-many solutions.

(d) Based on this, how do $\operatorname{Im} T$ and $\operatorname{span}\{T(\vec{e}_1), T(\vec{e}_2), T(\vec{e}_3), T(\vec{e}_4)\}$ relate to each other?

A. The set $\operatorname{Im} T$ contains $\operatorname{span}\{T(\vec{e}_1), T(\vec{e}_2), T(\vec{e}_3), T(\vec{e}_4)\}$ but is not equal to it.

B. The set $\operatorname{span}\{T(\vec{e}_1), T(\vec{e}_2), T(\vec{e}_3), T(\vec{e}_4)\}$ contains $\operatorname{Im} T$ but is not equal to it.

C. The set $\operatorname{Im} T$ and $\operatorname{span}\{T(\vec{e}_1), T(\vec{e}_2), T(\vec{e}_3), T(\vec{e}_4)\}$ are equal to each other.

D. There is no relation between these two sets.

Observation 3.3.11 Let $T : \mathbb{R}^4 \to \mathbb{R}^3$ be the linear transformation given by the standard matrix

$$A = \begin{bmatrix} 3 & 4 & 7 & 1 \\ -1 & 1 & 0 & 2 \\ 2 & 1 & 3 & -1 \end{bmatrix}.$$

Since the set $\left\{ \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 7 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \right\}$ spans $\operatorname{Im} T$, we can obtain a basis for $\operatorname{Im} T$ by finding $\operatorname{RREF} A = \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ and only using the vectors corresponding to pivot columns:

$$\left\{ \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

In general, the **column space** of a matrix M refers to the subspace obtained by considering the span of its column vectors. Using this terminology, if the transformation T is represented by the matrix A, then the image of T is the **column space** of A.

Image and Kernel (AT3)

Fact 3.3.12 *Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation with standard matrix A.*

- *The kernel of T is the solution set of the homogeneous system given by the augmented matrix $\left[\begin{array}{c|c} A & \vec{0} \end{array}\right]$. Use the coefficients of its free variables to get a basis for the kernel (as in Fact 2.7.5).*
- *The image of T is the span of the columns of A. Remove the vectors creating non-pivot columns in* $\operatorname{RREF} A$ *to get a basis for the image (as in Observation 2.6.5).*

Image and Kernel (AT3)

Activity 3.3.13 Let $T : \mathbb{R}^3 \to \mathbb{R}^4$ be the linear transformation given by the standard matrix

$$A = \begin{bmatrix} 1 & -3 & 2 \\ 2 & -6 & 0 \\ 0 & 0 & 1 \\ -1 & 3 & 1 \end{bmatrix}.$$

Find a basis for the kernel and a basis for the image of T.

Image and Kernel (AT3)

Activity 3.3.14 Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation with standard matrix A. Which of the following is equal to the dimension of the kernel of T?

A. The number of pivot columns

B. The number of non-pivot columns

C. The number of pivot rows

D. The number of non-pivot rows

Image and Kernel (AT3)

Activity 3.3.15 Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation with standard matrix A. Which of the following is equal to the dimension of the image of T?

A. The number of pivot columns

B. The number of non-pivot columns

C. The number of pivot rows

D. The number of non-pivot rows

Image and Kernel (AT3)

Observation 3.3.16 Combining these with the observation that the number of columns is the dimension of the domain of T, we have the **rank-nullity theorem**:

> The dimension of the domain of T equals $\dim(\ker T) + \dim(\operatorname{Im} T)$.

The dimension of the image is called the **rank** of T (or A) and the dimension of the kernel is called the **nullity**.

Image and Kernel (AT3)

Activity 3.3.17 Let $T : \mathbb{R}^4 \to \mathbb{R}^3$ be the linear transformation given by

$$T\left(\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}\right) = \begin{bmatrix} x - y + 5\,z + 3\,w \\ -x - 4\,z - 2\,w \\ y - 2\,z - w \end{bmatrix}.$$

(a) Explain and demonstrate how to find the image of T and a basis for that image.

(b) Explain and demonstrate how to find the kernel of T and a basis for that kernel.

(c) Explain and demonstrate how to find the rank and nullity of T, and why the rank-nullity theorem holds for T.

Image and Kernel (AT3)

Activity 3.3.18 In this section, we've introduced two important subspaces that are associated with a linear transformation $T\colon V \to W$, namely: $\operatorname{Im} T$, the image of T, and $\ker T$, the kernel of T. The following sequence is designed to help you internalize these definitions. Try to complete them without referring to your Activity Book, and then check your answers.

(a) One of $\ker T$ and $\operatorname{Im} T$ is a subspace of the domain and the other is a subspace of the codomain. Which is which?

(b) Write down the precise definitions of these subspaces.

(c) How would you describe these definitions to a layperson?

(d) What picture, or other study strategy would be helpful to you in conceptualizing how these definitions fit together?

Activity 3.3.19 We can use our notation of span in relation to a matrix, not just in relation to a set of vectors. Given a matrix M

- the span of the set of all columns is the **column space**
- the span of the set of all rows is the **row space**

$$\text{Let } M = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 2 & 4 \\ -1 & 0 & -1 \end{bmatrix}$$

Is $\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$ in the column space of M? Is it in the row space of M?

A. Yes.

B. No.

Is $\begin{bmatrix} 1 \\ 10 \\ -3 \end{bmatrix}$ in the column space of M? Is it in the row space of M?

A. Yes.

B. No.

$$\text{Let } N = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 2 & -3 \\ -1 & 0 & -1 \end{bmatrix}$$

Are the row space and column space of N both equal to $\mathbb{R}^3$?

A. Yes.

B. No.

3.4 Injective and Surjective Linear Maps (AT4)

Learning Outcomes

- Determine if a given linear map is injective and/or surjective.

Injective and Surjective Linear Maps (AT4)

Activity 3.4.1 Consider the linear transformation $T\colon \mathbb{R}^4 \to \mathbb{R}^3$ that is represented by the standard matrix $A = \begin{bmatrix} 3 & 4 & 7 & 1 \\ -1 & 1 & 0 & 2 \\ 2 & 1 & 3 & -1 \end{bmatrix}$. Which of the following processes helps us compute a basis for $\operatorname{Im} T$ and which helps us compute a basis for $\ker T$?

A. Compute RREF(A) and consider the set of columns of A that correspond to columns in RREF(A) with pivots.

B. Calculate a basis for the solution space to the homogenous system of equations for which A is the coefficient matrix.

Injective and Surjective Linear Maps (AT4)

Definition 3.4.2 Let $T : V \to W$ be a linear transformation. T is called **injective** or **one-to-one** if T does not map two distinct vectors to the same place. More precisely, T is injective if $T(\vec{v}) \neq T(\vec{w})$ whenever $\vec{v} \neq \vec{w}$.

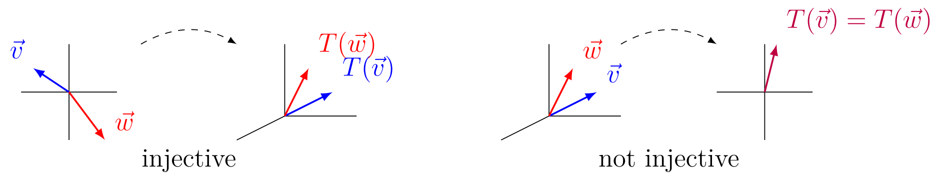


Figure 11 An injective transformation and a non-injective transformation

◊

Injective and Surjective Linear Maps (AT4)

Activity 3.4.3 Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x \\ y \end{bmatrix} \qquad \text{with standard matrix } \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Is T injective?

A. Yes, because $T(\vec{v}) = T(\vec{w})$ whenever $\vec{v} = \vec{w}$.

B. Yes, because $T(\vec{v}) \neq T(\vec{w})$ whenever $\vec{v} \neq \vec{w}$.

C. No, because $T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) \neq T\left(\begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}\right)$.

D. No, because $T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = T\left(\begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}\right)$.

Injective and Surjective Linear Maps (AT4)

Activity 3.4.4 Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be given by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} \qquad \text{with standard matrix } \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Is T injective?

A. Yes, because $T(\vec{v}) = T(\vec{w})$ whenever $\vec{v} = \vec{w}$.

B. Yes, because $T(\vec{v}) \neq T(\vec{w})$ whenever $\vec{v} \neq \vec{w}$.

C. No, because $T\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) \neq T\left(\begin{bmatrix} 3 \\ 4 \end{bmatrix}\right)$.

D. No, because $T\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = T\left(\begin{bmatrix} 3 \\ 4 \end{bmatrix}\right)$.

Definition 3.4.5 Let $T : V \to W$ be a linear transformation. T is called **surjective** or **onto** if every element of W is mapped to by an element of V. More precisely, for every $\vec{w} \in W$, there is some $\vec{v} \in V$ with $T(\vec{v}) = \vec{w}$.

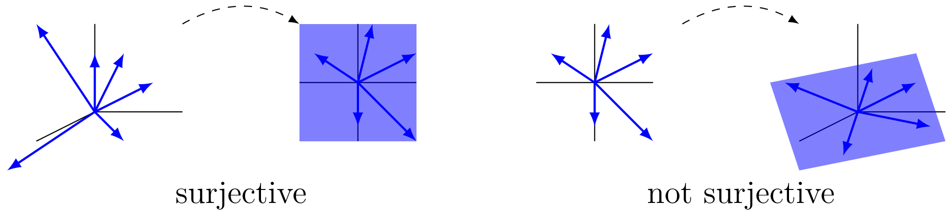


Figure 12 A surjective transformation and a non-surjective transformation

$\Diamond$

Injective and Surjective Linear Maps (AT4)

Activity 3.4.6 Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be given by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} \qquad \text{with standard matrix } \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Is T surjective?

A. Yes, because for every $\vec{w} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3$, there exists $\vec{v} = \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2$ such that $T(\vec{v}) = \vec{w}$.

B. No, because $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)$ can never equal $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

C. No, because $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)$ can never equal $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

Injective and Surjective Linear Maps (AT4)

Activity 3.4.7 Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x \\ y \end{bmatrix} \qquad \text{with standard matrix } \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Is T surjective?

A. Yes, because for every $\vec{w} = \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2$, there exists $\vec{v} = \begin{bmatrix} x \\ y \\ 42 \end{bmatrix} \in \mathbb{R}^3$ such that $T(\vec{v}) = \vec{w}$.

B. Yes, because for every $\vec{w} = \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2$, there exists $\vec{v} = \begin{bmatrix} 0 \\ 0 \\ z \end{bmatrix} \in \mathbb{R}^3$ such that $T(\vec{v}) = \vec{w}$.

C. No, because $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right)$ can never equal $\begin{bmatrix} 3 \\ -2 \end{bmatrix}$.

Injective and Surjective Linear Maps (AT4)

Activity 3.4.8 Let $T : V \to W$ be a linear transformation where $\ker T$ contains multiple vectors. What can you conclude?

A. T is injective

B. T is not injective

C. T is surjective

D. T is not surjective

Fact 3.4.9 *A linear transformation T is injective* if and only if $\ker T = \{\vec{0}\}$. *Put another way, an injective linear transformation may be recognized by its* ***trivial*** *kernel.*

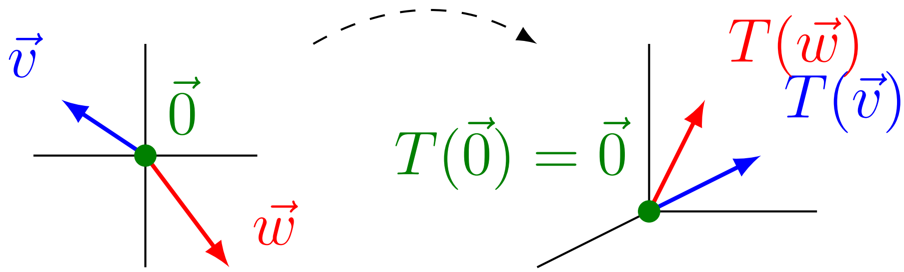


Figure 13 *A linear transformation with trivial kernel, which is therefore injective*

Injective and Surjective Linear Maps (AT4)

Activity 3.4.10 Let $T : V \to \mathbb{R}^3$ be a linear transformation where $\operatorname{Im} T$ may be spanned by only two vectors. What can you conclude?

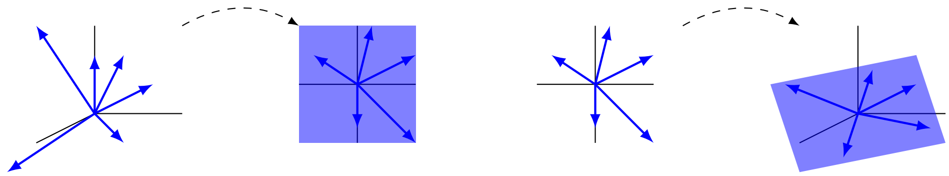
A. T is injective

B. T is not injective

C. T is surjective

D. T is not surjective

Fact 3.4.11 *A linear transformation $T : V \to W$ is surjective* if and only if $\operatorname{Im} T = W$. *Put another way, a surjective linear transformation may be recognized by its identical codomain and image.*



surjective, $\operatorname{Im} T = \mathbb{R}^2$ *not surjective*, $\operatorname{Im} T \neq \mathbb{R}^3$

Figure 14 *A linear transformation with identical codomain and image, which is therefore surjective; and a linear transformation with an image smaller than the codomain* $\mathbb{R}^3$*, which is therefore not surjective.*

Injective and Surjective Linear Maps (AT4)

Definition 3.4.12 A transformation that is both injective and surjective is said to be **bijective**. ◊

Injective and Surjective Linear Maps (AT4)

Activity 3.4.13 Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map with standard matrix A. Determine whether each of the following statements means T is (A) *injective*, (B) *surjective*, or (C) *bijective* (both).

1. The kernel of T is trivial, i.e. $\ker T = \{\vec{0}\}$.
2. The image of T equals its codomain, i.e. $\operatorname{Im} T = \mathbb{R}^m$.
3. For every $\vec{w} \in \mathbb{R}^m$, the set $\{\vec{v} \in \mathbb{R}^n | T(\vec{v}) = \vec{w}\}$ contains exactly one vector.

Injective and Surjective Linear Maps (AT4)

Activity 3.4.14 Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map with standard matrix A. Determine whether each of the following statements means T is (A) *injective*, (B) *surjective*, or (C) *bijective* (both).

1. The columns of A span $\mathbb{R}^m$.
2. The columns of A form a basis for $\mathbb{R}^m$.
3. The columns of A are linearly independent.

Injective and Surjective Linear Maps (AT4)

Activity 3.4.15 Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map with standard matrix A. Determine whether each of the following statements means T is (A) *injective*, (B) *surjective*, or (C) *bijective* (both).

1. RREF(A) is the identity matrix.
2. Every column of RREF(A) has a pivot.
3. Every row of RREF(A) has a pivot.

Injective and Surjective Linear Maps (AT4)

Activity 3.4.16 Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map with standard matrix A. Determine whether each of the following statements means T is (A) *injective*, (B) *surjective*, or (C) *bijective* (both).

1. The system of linear equations given by the augmented matrix $\left[\begin{array}{c|c} A & \vec{b} \end{array}\right]$ has a solution for all $\vec{b} \in \mathbb{R}^m$.
2. The system of linear equations given by the augmented matrix $\left[\begin{array}{c|c} A & \vec{b} \end{array}\right]$ has exactly one solution for all $\vec{b} \in \mathbb{R}^m$.
3. The system of linear equations given by the augmented matrix $\left[\begin{array}{c|c} A & \vec{0} \end{array}\right]$ has exactly one solution.

Injective and Surjective Linear Maps (AT4)

Observation 3.4.17 The easiest way to determine if the linear map with standard matrix A is injective is to see if $\mathrm{RREF}(A)$ has a pivot in each column.

The easiest way to determine if the linear map with standard matrix A is surjective is to see if $\mathrm{RREF}(A)$ has a pivot in each row.

Injective and Surjective Linear Maps (AT4)

Activity 3.4.18 What can you conclude about the linear map $T : \mathbb{R}^2 \to \mathbb{R}^3$ with standard matrix $\begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}$?

A. Its standard matrix has more columns than rows, so T is not injective.

B. Its standard matrix has more columns than rows, so T is injective.

C. Its standard matrix has more rows than columns, so T is not surjective.

D. Its standard matrix has more rows than columns, so T is surjective.

Injective and Surjective Linear Maps (AT4)

Activity 3.4.19 What can you conclude about the linear map $T : \mathbb{R}^3 \to \mathbb{R}^2$ with standard matrix $\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$?

A. Its standard matrix has more columns than rows, so T is not injective.

B. Its standard matrix has more columns than rows, so T is injective.

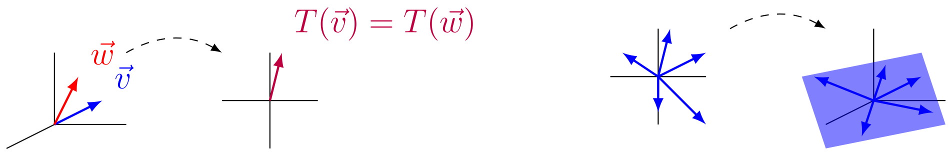
C. Its standard matrix has more rows than columns, so T is not surjective.

D. Its standard matrix has more rows than columns, so T is surjective.

Fact 3.4.20 *The following are true for any linear map $T : V \to W$:*

- *If* $\dim(V) > \dim(W)$*, then T is not injective.*
- *If* $\dim(V) < \dim(W)$*, then T is not surjective.*

Basically, a linear transformation cannot reduce dimension without collapsing vectors into each other, and a linear transformation cannot increase dimension from its domain to its image.



not injective, $3 > 2$ *not surjective,* $2 < 3$

Figure 15 *A linear transformation whose domain has a larger dimension than its codomain, and is therefore not injective; and a linear transformation whose domain has a smaller dimension than its codomain, and is therefore not surjective.*

But dimension arguments cannot *be used to prove a map* is *injective or surjective.*

Activity 3.4.21 Suppose $T : \mathbb{R}^n \to \mathbb{R}^4$ with standard matrix $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ a_{31} & a_{32} & \cdots & a_{3n} \\ a_{41} & a_{42} & \cdots & a_{4n} \end{bmatrix}$ is bijective.

(a) How many pivot rows must RREF A have?

(b) How many pivot columns must RREF A have?

(c) What is RREF A?

Injective and Surjective Linear Maps (AT4)

Activity 3.4.22 Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a bijective linear map with standard matrix A. Label each of the following as true or false.

A. RREF(A) is the identity matrix.

B. The columns of A form a basis for $\mathbb{R}^n$

C. The system of linear equations given by the augmented matrix $\left[\begin{array}{c|c} A & \vec{b} \end{array}\right]$ has exactly one solution for each $\vec{b} \in \mathbb{R}^n$.

Injective and Surjective Linear Maps (AT4)

Observation 3.4.23 The easiest way to show that the linear map with standard matrix A is bijective is to show that $\text{RREF}(A)$ is the identity matrix.

Injective and Surjective Linear Maps (AT4)

Activity 3.4.24 Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be given by the standard matrix

$$A = \begin{bmatrix} 2 & 1 & -1 \\ 4 & 1 & 1 \\ 6 & 2 & 1 \end{bmatrix}.$$

Which of the following must be true?

A. T is neither injective nor surjective

B. T is injective but not surjective

C. T is surjective but not injective

D. T is bijective.

Injective and Surjective Linear Maps (AT4)

Activity 3.4.25 Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 2x + y - z \\ 4x + y + z \\ 6x + 2y \end{bmatrix}.$$

Which of the following must be true?

A. T is neither injective nor surjective

B. T is injective but not surjective

C. T is surjective but not injective

D. T is bijective.

Injective and Surjective Linear Maps (AT4)

Activity 3.4.26 Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be given by

$$T\left(\left[\begin{array}{c} x \\ y \end{array}\right]\right) = \left[\begin{array}{c} 2x+3y \\ x-y \\ x+3y \end{array}\right].$$

Which of the following must be true?

A. T is neither injective nor surjective

B. T is injective but not surjective

C. T is surjective but not injective

D. T is bijective.

Injective and Surjective Linear Maps (AT4)

Activity 3.4.27 Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 2x + y - z \\ 4x + y + z \end{bmatrix}.$$

Which of the following must be true?

A. T is neither injective nor surjective

B. T is injective but not surjective

C. T is surjective but not injective

D. T is bijective.

Injective and Surjective Linear Maps (AT4)

Activity 3.4.28 Let $T\colon \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation with standard matrix A. We reasoned during class that the following statements are logically equivalent:

1. The columns of A are linearly independent.
2. RREF(A) has a pivot in each column.
3. The transformation T is injective.
4. The system of equations given by $[A|\vec{0}]$ has a unique solution.

While they are all logically equivalent, they are different statements that offer varied perspectives on our growing conceptual knowledge of linear algebra.

(a) If you are asked to decide if a transformation T is injective, which of the above statements do you think is the most useful?

(b) Can you think of some situations in which translating between these four statements might be useful to you?

Injective and Surjective Linear Maps (AT4)

Activity 3.4.29 Let $T\colon \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation with standard matrix A. We reasoned during class that the following statements are logically equivalent:

1. The columns of A span all of $\mathbb{R}^m$.
2. RREF(A) has a pivot in each row.
3. The transformation T is surjective.
4. The system of equations given by $[A|\vec{b}]$ is always consistent.

While they are all logically equivalent, they are different statements that offer varied perspectives on our growing conceptual knowledge of linear algebra.

(a) If you are asked to decide if a transformation T is surjective, which of the above statements do you think is the most useful?

(b) Can you think of some situations in which translating between these four statements might be useful to you?

3.5 Vector Spaces (AT5)

Learning Outcomes

- Explain why a given set with defined addition and scalar multiplication does satisfy a given vector space property, but nonetheless isn't a vector space.

Vector Spaces (AT5)

Activity 3.5.1

(a) How would you describe a sandwich to someone who has never seen a sandwich before?

(b) How would you describe to someone what a vector is?

Observation 3.5.2 Consider the following applications of properties of the real numbers $\mathbb{R}$:

1. $1 + (2 + 3) = (1 + 2) + 3$.
2. $7 + 4 = 4 + 7$.
3. There exists some $?$ where $5 + ? = 5$.
4. There exists some $?$ where $9 + ? = 0$.
5. $\frac{1}{2}(1 + 7)$ is the only number that is equally distant from 1 and 7.

Activity 3.5.3 Which of the following properties of $\mathbb{R}^2$ Euclidean vectors is NOT true?

A. $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \left(\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \right) = \left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \right) + \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$.

B. $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

C. There exists some $\begin{bmatrix} ? \\ ? \end{bmatrix}$ where $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} ? \\ ? \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

D. There exists some $\begin{bmatrix} ? \\ ? \end{bmatrix}$ where $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} ? \\ ? \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

E. $\frac{1}{2} \left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \right)$ is the only vector whose endpoint is equally distant from the endpoints of $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$.

Observation 3.5.4 Consider the following applications of properties of the real numbers $\mathbb{R}$:

1. $3(2(7)) = (3 \cdot 2)(7)$.
2. $1(19) = 19$.
3. There exists some ? such that $? \cdot 4 = 9$.
4. $3 \cdot (2 + 8) = 3 \cdot 2 + 3 \cdot 8$.
5. $(2 + 7) \cdot 4 = 2 \cdot 4 + 7 \cdot 4$.

Vector Spaces (AT5)

Activity 3.5.5 Which of the following properties of $\mathbb{R}^2$ Euclidean vectors is NOT true?

A. $a\left(b\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = ab\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

B. $1\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

C. There exists some ? such that ? $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$.

D. $a(\vec{u} + \vec{v}) = a\vec{u} + a\vec{v}$.

E. $(a + b)\vec{v} = a\vec{v} + b\vec{v}$.

Vector Spaces (AT5)

Fact 3.5.6 *Every Euclidean vector space* $\mathbb{R}^n$ *satisfies the following properties, where* $\vec{u}, \vec{v}, \vec{w}$ *are Euclidean vectors and* a, b *are scalars.*

1. *Vector addition is associative:* $\vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}$.
2. *Vector addition is commutative:* $\vec{u} + \vec{v} = \vec{v} + \vec{u}$.
3. *An additive identity exists: There exists some* $\vec{z}$ *where* $\vec{v} + \vec{z} = \vec{v}$.
4. *Additive inverses exist: There exists some* $-\vec{v}$ *where* $\vec{v} + (-\vec{v}) = \vec{z}$.
5. *Scalar multiplication is associative:* $a(b\vec{v}) = (ab)\vec{v}$.
6. *1 is a multiplicative identity:* $1\vec{v} = \vec{v}$.
7. *Scalar multiplication distributes over vector addition:* $a(\vec{u} + \vec{v}) = (a\vec{u}) + (a\vec{v})$.
8. *Scalar multiplication distributes over scalar addition:* $(a + b)\vec{v} = (a\vec{v}) + (b\vec{v})$.

Vector Spaces (AT5)

Definition 3.5.7 A **vector space** V is any set of mathematical objects, called **vectors**, and a set of numbers, called **scalars**, with associated addition $\oplus$ and scalar multiplication $\odot$ operations that satisfy the following properties. Let $\vec{u}, \vec{v}, \vec{w}$ be vectors belonging to V, and let a, b be scalars.

We always assume the codomain of our operations is V, i.e. that addition is a map $V \times V \to V$ and that scalar multiplication is a map $\mathbb{R} \times V \to V$.

Likewise, we only consider "real" vector spaces, i.e. those whose scalars come from $\mathbb{R}$. However, one can similarly define vector spaces with scalars from other fields like the complex or rational numbers.

1. Vector addition is associative: $\vec{u} \oplus (\vec{v} \oplus \vec{w}) = (\vec{u} \oplus \vec{v}) \oplus \vec{w}$.
2. Vector addition is commutative: $\vec{u} \oplus \vec{v} = \vec{v} \oplus \vec{u}$.
3. An additive identity exists: There exists some $\vec{z}$ where $\vec{v} \oplus \vec{z} = \vec{v}$.
4. Additive inverses exist: There exists some $-\vec{v}$ where $\vec{v} \oplus (-\vec{v}) = \vec{z}$.
5. Scalar multiplication is associative: $a \odot (b \odot \vec{v}) = (ab) \odot \vec{v}$.
6. 1 is a multiplicative identity: $1 \odot \vec{v} = \vec{v}$.
7. Scalar multiplication distributes over vector addition: $a \odot (\vec{u} \oplus \vec{v}) = (a \odot \vec{u}) \oplus (a \odot \vec{v})$.
8. Scalar multiplication distributes over scalar addition: $(a + b) \odot \vec{v} = (a \odot \vec{v}) \oplus (b \odot \vec{v})$.

$\Diamond$

Remark 3.5.8 Consider the set $\mathbb{C}$ of complex numbers with the usual definition for addition: $(a + b\mathbf{i}) \oplus (c + d\mathbf{i}) = (a + c) + (b + d)\mathbf{i}$.

Let $\vec{u} = a + b\mathbf{i}$, $\vec{v} = c + d\mathbf{i}$, and $\vec{w} = e + f\mathbf{i}$. Then

$$\begin{aligned}\vec{u} \oplus (\vec{v} \oplus \vec{w}) &= (a + b\mathbf{i}) \oplus ((c + d\mathbf{i}) \oplus (e + f\mathbf{i})) \\ &= (a + b\mathbf{i}) \oplus ((c + e) + (d + f)\mathbf{i}) \\ &= (a + c + e) + (b + d + f)\mathbf{i}\end{aligned}$$

$$\begin{aligned}(\vec{u} \oplus \vec{v}) \oplus \vec{w} &= ((a + b\mathbf{i}) \oplus (c + d\mathbf{i})) \oplus (e + f\mathbf{i}) \\ &= ((a + c) + (b + d)\mathbf{i}) \oplus (e + f\mathbf{i}) \\ &= (a + c + e) + (b + d + f)\mathbf{i}\end{aligned}$$

This proves that complex addition is associative: $\vec{u} \oplus (\vec{v} \oplus \vec{w}) = (\vec{u} \oplus \vec{v}) \oplus \vec{w}$. The seven other vector space properties may also be verified, so $\mathbb{C}$ is an example of a vector space.

Vector Spaces (AT5)

Remark 3.5.9 The following sets are just a few examples of vector spaces, with the usual/natural operations for addition and scalar multiplication.

- $\mathbb{R}^n$: Euclidean vectors with n components.
- $\mathbb{C}$: Complex numbers.
- $M_{m,n}$: Matrices of real numbers with m rows and n columns.
- $\mathcal{P}_n$: Polynomials of degree n or less.
- $\mathcal{P}$: Polynomials of any degree.
- $C(\mathbb{R})$: Real-valued continuous functions.

Activity 3.5.10 Consider the set $V = \{(x, y) \,|\, y = 2^x\}$.
Which of the following vectors is not in V?

A. $(0, 0)$

B. $(1, 2)$

C. $(2, 4)$

D. $(3, 8)$

Vector Spaces (AT5)

Activity 3.5.11 Consider the set $V = \{(x, y) \,|\, y = 2^x\}$ with the operation $\oplus$ defined by

$$(x_1, y_1) \oplus (x_2, y_2) = (x_1 + x_2, y_1 y_2).$$

Let $\vec{u}, \vec{v}$ be in V with $\vec{u} = (1, 2)$ and $\vec{v} = (2, 4)$. Using the operations defined for V, which of the following is $\vec{u} \oplus \vec{v}$?

A. $(2, 6)$

B. $(2, 8)$

C. $(3, 6)$

D. $(3, 8)$

Vector Spaces (AT5)

Activity 3.5.12 Consider the set $V = \{(x, y) \,|\, y = 2^x\}$ with operations $\oplus, \odot$ defined by

$$(x_1, y_1) \oplus (x_2, y_2) = (x_1 + x_2, y_1 y_2) \qquad c \odot (x, y) = (cx, y^c).$$

Let $a = 2, b = -3$ be scalars and $\vec{u} = (1, 2) \in V$.

(a) Verify that

$$(a + b) \odot \vec{u} = \left(-1, \frac{1}{2}\right).$$

(b) Compute the value of

$$(a \odot \vec{u}) \oplus (b \odot \vec{u}).$$

Activity 3.5.13 Consider the set $V = \{(x, y) \,|\, y = 2^x\}$ with operations $\oplus, \odot$ defined by

$$(x_1, y_1) \oplus (x_2, y_2) = (x_1 + x_2, y_1 y_2) \qquad c \odot (x, y) = (cx, y^c).$$

Let a, b be unspecified scalars in $\mathbb{R}$ and $\vec{u} = (x, y)$ be an unspecified vector in V.

(a) Show that both sides of the equation

$$(a + b) \odot (x, y) = (a \odot (x, y)) \oplus (b \odot (x, y))$$

simplify to the expression $(ax + bx, y^a y^b)$.

(b) Show that V contains an additive identity element $\vec{z} = (\,?\,,\,?\,)$ satisfying

$$(x, y) \oplus (\,?\,,\,?\,) = (x, y)$$

for all $(x, y) \in V$.

That is, pick appropriate values for $\vec{z} = (\,?\,,\,?\,)$ and then simplify $(x, y) \oplus (\,?\,,\,?\,)$ into just (x, y).

(c) Is V a vector space?

A. Yes

B. No

C. More work is required

Remark 3.5.14 It turns out $V = \{(x, y) \,|\, y = 2^x\}$ with operations $\oplus, \odot$ defined by

$$(x_1, y_1) \oplus (x_2, y_2) = (x_1 + x_2, y_1 y_2) \qquad\qquad c \odot (x, y) = (cx, y^c)$$

satisfies all eight properties from Definition 3.5.7.

Thus, V is a vector space.

Vector Spaces (AT5)

Activity 3.5.15 Let $V = \{(x, y) \,|\, x, y \in \mathbb{R}\}$ have operations defined by

$$(x_1, y_1) \oplus (x_2, y_2) = (x_1 + y_1 + x_2 + y_2, x_1^2 + x_2^2)$$

$$c \odot (x, y) = (x^c, y + c - 1).$$

(a) Show that 1 is the scalar multiplication identity element by simplifying $1 \odot (x, y)$ to (x, y).

(b) Show that V does not have an additive identity element $\vec{z} = (z, w)$ by showing that $(0, -1) \oplus (z, w) \neq (0, -1)$ no matter what the values of z, w are.

(c) Is V a vector space?

A. Yes

B. No

C. More work is required

Vector Spaces (AT5)

Activity 3.5.16 Let $V = \{(x, y) \,|\, x, y \in \mathbb{R}\}$ have operations defined by

$$(x_1, y_1) \oplus (x_2, y_2) = (x_1 + x_2, y_1 + 3y_2) \qquad c \odot (x, y) = (cx, cy).$$

(a) Show that scalar multiplication distributes over vector addition, i.e.

$$c \odot ((x_1, y_1) \oplus (x_2, y_2)) = c \odot (x_1, y_1) \oplus c \odot (x_2, y_2)$$

for *all* $c \in \mathbb{R}$, $(x_1, y_1), (x_2, y_2) \in V$.

(b) Show that vector addition is not associative, i.e.

$$(x_1, y_1) \oplus ((x_2, y_2) \oplus (x_3, y_3)) \neq ((x_1, y_1) \oplus (x_2, y_2)) \oplus (x_3, y_3)$$

for *some* vectors $(x_1, y_1), (x_2, y_2), (x_3, y_3) \in V$.

(c) Is V a vector space?

A. Yes

B. No

C. More work is required

Vector Spaces (AT5)

Activity 3.5.17

(a) What are some objects that are important to you personally, academically, or otherwise that appear vector-like to you? What makes them feel vector-like? Which axiom for vector spaces does not hold for these objects, if any?

(b) Our vector space axioms have eight properties. While these eight properties are enough to capture vectors, the objects that we study in the real world often have additional structures not captured by these axioms. What are some structures that you have encountered in other classes, or in previous experiences, that are not captured by these eight axioms?

3.6 Polynomial and Matrix Spaces (AT6)

Learning Outcomes

- Answer questions about vector spaces of polynomials or matrices.

Polynomial and Matrix Spaces (AT6)

Activity 3.6.1 Consider the following vector equation and statements about it:

$$x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_n\vec{v}_n = \vec{w}$$

1. The above vector equation is consistent for every choice of $\vec{w}$.
2. When the right hand is equal to $\vec{0}$, the equation has a unique solution.
3. The given equation always has a unique solution, no matter what $\vec{w}$ is.

Which, if any, of these statements make sense if we no longer assume that the vectors $\vec{v}_1, \ldots, \vec{v}_n$ are Euclidean vectors, but rather elements of a vector space?

Polynomial and Matrix Spaces (AT6)

Observation 3.6.2 Nearly every term we've defined for Euclidean vector spaces $\mathbb{R}^n$ was actually defined for all kinds of vector spaces:

- Definition 2.1.3
- Definition 2.1.4
- Definition 2.3.7
- Definition 2.4.3
- Definition 2.5.5
- Definition 3.1.3
- Definition 3.1.4
- Definition 3.3.3
- Definition 3.3.8
- Definition 3.4.2
- Definition 3.4.5
- Definition 3.4.12

Polynomial and Matrix Spaces (AT6)

Activity 3.6.3 Let V be a vector space with the basis $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$. Which of these completes the following definition for a bijective linear map $T : V \to \mathbb{R}^3$?

$$T(\vec{v}) = T(a\vec{v}_1 + b\vec{v}_2 + c\vec{v}_3) = \,?\,\vec{e}_1 + \,?\,\vec{e}_2 + \,?\,\vec{e}_3 = \begin{bmatrix} ? \\ ? \\ ? \end{bmatrix}$$

A. $0\vec{e}_1 + 0\vec{e}_2 + 0\vec{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

B. $(a + b + c)\vec{e}_1 + 0\vec{e}_2 + 0\vec{e}_3 = \begin{bmatrix} a + b + c \\ 0 \\ 0 \end{bmatrix}$

C. $a\vec{e}_1 + b\vec{e}_2 + c\vec{e}_3 = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

Fact 3.6.4 *Every vector space with finite dimension, that is, every vector space V with a basis of the form $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ has a linear bijection T with Euclidean space $\mathbb{R}^n$ that simply swaps its basis with the standard basis $\{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\}$ for $\mathbb{R}^n$:*

$$T(c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n) = c_1\vec{e}_1 + c_2\vec{e}_2 + \cdots + c_n\vec{e}_n = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

*This transformation (in fact, any linear bijection between vector spaces) is called an **isomorphism**, and V is said to be **isomorphic** to $\mathbb{R}^n$.*

Note, in particular, that every *vector space of dimension n is isomorphic to $\mathbb{R}^n$.*

Polynomial and Matrix Spaces (AT6)

Activity 3.6.5 Consider the matrix space $M_{2,2} = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| a, b, c, d \in \mathbb{R} \right\}$ and the following set of matrices:

$$S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}.$$

(a) Does the set S span $M_{2,2}$?

A. No; the matrix $\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$ is not a linear combination of the matrices in S.

B. No; the matrix $\begin{bmatrix} 7 & 1 \\ 0 & -1 \end{bmatrix}$ is not a linear combination of the matrices in S.

C. No; the matrix $\begin{bmatrix} -1 & 5 \\ 2 & 9 \end{bmatrix}$ is not a linear combination of the matrices in S.

D. Yes, every matrix in $M_{2,2}$ is a linear combination of the matrices in S.

(b) Is the set S linearly independent?

A. No; the matrix $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \in S$ is a linear combination of the other matrices in S.

B. No; the matrix $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \in S$ is a linear combination of the other matrices in S.

C. No; the matrix $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \in S$ is a linear combination of the other matrices in S.

D. Yes; no matrix in S is a linear combination of the other matrices in S.

(c) What statement do you think best describes the set

$$S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}?$$

A. S is linearly independent

B. S spans $M_{2,2}$

C. S is a basis of $M_{2,2}$

D. S is a basis of $\mathbb{R}^4$

(d) What is the dimension of $M_{2,2}$?

A. 2

B. 3

C. 4

D. 5

(e) Which Euclidean space is $M_{2,2}$ isomorphic to?

A. $\mathbb{R}^2$

B. $\mathbb{R}^3$

C. $\mathbb{R}^4$

D. $\mathbb{R}^5$

Polynomial and Matrix Spaces (AT6)

(f) Describe an isomorphism $T : M_{2,2} \to \mathbb{R}^{?}$:

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} ? \\ \vdots \\ ? \end{bmatrix}$$

Polynomial and Matrix Spaces (AT6)

Activity 3.6.6 Consider polynomial space $\mathcal{P}^4 = \{a + by + cy^2 + dy^3 + ey^4 | a, b, c, d, e \in \mathbb{R}\}$ and the following set:

$$S = \left\{1, y, y^2, y^3, y^4\right\}.$$

(a) Does the set S span $\mathcal{P}^4$?

A. No; the polynomial $1+y^2+2y^3$ is not a linear combination of the polynomials in S.

B. No; the polynomial $6+y-y^3+y^4$ is not a linear combination of the polynomials in S.

C. No; the polynomial $y^2+2y^3-y^4$ is not a linear combination of the polynomials in S.

D. Yes; every polynomial in $\mathcal{P}^4$ is a linear combination of the polynomials in S.

(b) Is the set S linearly independent?

A. No; the polynomial y^2 is a linear combination of the other polynomials in S.

B. No; the polynomial y^3 is a linear combination of the other polynomials in S.

C. No; the polynomial 1 is a linear combination of the other polynomials in S.

D. Yes; no polynomial in S is a linear combination of the other polynomials in S.

(c) What statement do you think best describes the set

$$S = \left\{1, y, y^2, y^3, y^4\right\}?$$

A. S is linearly independent

B. S spans $\mathcal{P}^4$

C. S is a basis of $\mathcal{P}^4$

(d) What is the dimension of $\mathcal{P}^4$?

A. 2

B. 3

C. 4

D. 5

(e) Which Euclidean space is $\mathcal{P}^4$ isomorphic to?

A. $\mathbb{R}^2$

B. $\mathbb{R}^3$

C. $\mathbb{R}^4$

D. $\mathbb{R}^5$

(f) Describe an isomorphism $T : \mathcal{P}^4 \to \mathbb{R}^{?}$:

$$T\left(a + by + cy^2 + dy^3 + ey^4\right) = \begin{bmatrix} ? \\ \vdots \\ ? \end{bmatrix}$$

Polynomial and Matrix Spaces (AT6)

Remark 3.6.7 Since any finite-dimensional vector space is isomorphic to a Euclidean space $\mathbb{R}^n$, one approach to answering questions about such spaces is to answer the corresponding question about $\mathbb{R}^n$.

Activity 3.6.8 Consider how to construct the polynomial $x^3 + x^2 + 5x + 1$ as a linear combination of polynomials from the set

$$\left\{x^3 - 2\,x^2 + x + 2, 2\,x^2 - 1, -x^3 + 3\,x^2 + 3\,x - 2, x^3 - 6\,x^2 + 9\,x + 5\right\}.$$

(a) Describe the vector space involved in this problem, and an isomorphic Euclidean space, and relevant Euclidean vectors that can be used to solve this problem.

(b) Show how to construct an appropriate Euclidean vector from an appropriate set of Euclidean vectors.

(c) Use this result to answer the original question.

Polynomial and Matrix Spaces (AT6)

Observation 3.6.9 The space of polynomials $\mathcal{P}$ (of *any* degree) has the basis $\{1, x, x^2, x^3, \dots\}$, so it is a natural example of an infinite-dimensional vector space.

Since $\mathcal{P}$ and other infinite-dimensional vector spaces cannot be treated as an isomorphic finite-dimensional Euclidean space $\mathbb{R}^n$, vectors in such vector spaces cannot be studied by converting them into Euclidean vectors. Fortunately, most of the examples we will be interested in for this course will be finite-dimensional.

Activity 3.6.10 Let $A = \begin{bmatrix} -2 & -1 & 1 \\ 1 & 0 & 0 \\ 0 & -4 & -2 \\ 0 & 1 & 3 \end{bmatrix}$ and let $T\colon \mathbb{R}^3 \to \mathbb{R}^4$ denote the corresponding linear transformation. Note that

$$\mathrm{RREF}(A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

The following statements are all *invalid* for at least one reason. Determine what makes them invalid and, suggest alternative *valid* statements that the author may have meant instead.

(a) The matrix A is injective because $\mathrm{RREF}(A)$ has a pivot in each column.

(b) The matrix A does not span $\mathbb{R}^4$ because $\mathrm{RREF}(A)$ has a row of zeroes.

(c) The transformation T does not span $\mathbb{R}^4$.

(d) The transformation T is linearly independent.

Chapter 4

Matrices (MX)

Learning Outcomes

What algebraic structure do matrices have?
By the end of this chapter, you should be able to...

1. Multiply matrices.
2. Determine if a matrix is invertible, and if so, compute its inverse and use it to solve an appropriate system of equations.
3. Calculate the change-of-basis matrix for the standard basis to a non-standard basis of $\mathbb{R}^n$.
4. Express row operations through matrix multiplication.

4.1 Matrices and Multiplication (MX1)

Learning Outcomes

- Multiply matrices.

Matrices and Multiplication (MX1)

Activity 4.1.1 Suppose that $T \colon V \to W$ is a linear transformation.

(a) What is the definition of $\ker T$? How does it relate to the domain of T?

(b) What is definition of $\operatorname{Im} T$? How does it relate to the codomain of T?

Observation 4.1.2 If $T : \mathbb{R}^n \to \mathbb{R}^m$ and $S : \mathbb{R}^m \to \mathbb{R}^k$ are linear maps, then the composition map $S \circ T$ computed as $(S \circ T)(\vec{v}) = S(T(\vec{v}))$ is a linear map from $\mathbb{R}^n \to \mathbb{R}^k$.

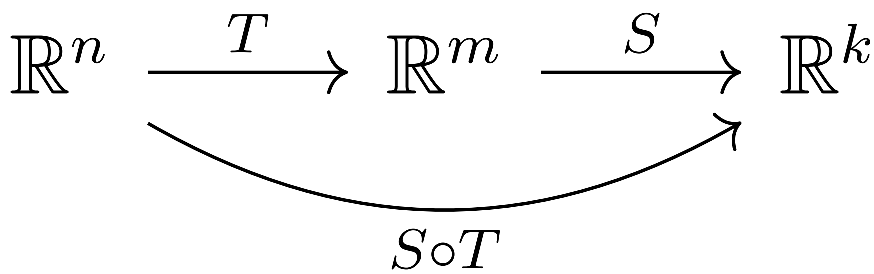


Figure 16 The composition of two linear maps.

Matrices and Multiplication (MX1)

Activity 4.1.3 Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be defined by the 2×3 standard matrix B and $S : \mathbb{R}^2 \to \mathbb{R}^4$ be defined by the 4×2 standard matrix A:

$$B = \begin{bmatrix} 2 & 1 & -3 \\ 5 & -3 & 4 \end{bmatrix} \qquad A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 5 \\ -1 & -2 \end{bmatrix}.$$

(a) What are the domain and codomain of the composition map $S \circ T$?

A. The domain is $\mathbb{R}^3$ and the codomain is $\mathbb{R}^2$

B. The domain is $\mathbb{R}^2$ and the codomain is $\mathbb{R}^4$

C. The domain is $\mathbb{R}^3$ and the codomain is $\mathbb{R}^4$

D. The domain is $\mathbb{R}^4$ and the codomain is $\mathbb{R}^3$

(b) What size will the standard matrix of $S \circ T$ be?

A. 4 (rows) × 3 (columns)

B. 3 (rows) × 4 (columns)

C. 3 (rows) × 2 (columns)

D. 2 (rows) × 4 (columns)

(c) Compute

$$(S \circ T)(\vec{e}_1) = S(T(\vec{e}_1)) = S\left(\begin{bmatrix} 2 \\ 5 \end{bmatrix}\right) = \begin{bmatrix} ? \\ ? \\ ? \\ ? \end{bmatrix}.$$

(d) Compute $(S \circ T)(\vec{e}_2)$.

(e) Compute $(S \circ T)(\vec{e}_3)$.

(f) Use $(S \circ T)(\vec{e}_1), (S \circ T)(\vec{e}_2), (S \circ T)(\vec{e}_3)$ to write the standard matrix for $S \circ T$.

Definition 4.1.4 We define the **product** AB of a $m \times n$ matrix A and a $n \times k$ matrix B to be the $m \times k$ standard matrix of the composition map of the two corresponding linear functions.

For the previous activity, T was a map $\mathbb{R}^3 \to \mathbb{R}^2$, and S was a map $\mathbb{R}^2 \to \mathbb{R}^4$, so $S \circ T$ gave a map $\mathbb{R}^3 \to \mathbb{R}^4$ with a 4×3 standard matrix:

$$AB = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 5 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} 2 & 1 & -3 \\ 5 & -3 & 4 \end{bmatrix}$$

$$= \left[(S \circ T)(\vec{e}_1) \quad (S \circ T)(\vec{e}_2) \quad (S \circ T)(\vec{e}_3)\right] = \begin{bmatrix} 12 & -5 & 5 \\ 5 & -3 & 4 \\ 31 & -12 & 11 \\ -12 & 5 & -5 \end{bmatrix}.$$

◊

Matrices and Multiplication (MX1)

Activity 4.1.5 Let $S : \mathbb{R}^3 \to \mathbb{R}^2$ be given by the matrix $A = \begin{bmatrix} -4 & -2 & 3 \\ 0 & 1 & 1 \end{bmatrix}$ and $T : \mathbb{R}^2 \to \mathbb{R}^3$ be given by the matrix $B = \begin{bmatrix} 2 & 3 \\ 1 & -1 \\ 0 & -1 \end{bmatrix}$.

(a) Write the dimensions (rows $\times$ columns) for A, B, AB, and BA.

(b) Find the standard matrix AB of $S \circ T$.

(c) Find the standard matrix BA of $T \circ S$.

Matrices and Multiplication (MX1)

Activity 4.1.6 Consider the following three matrices.

$$A = \begin{bmatrix} 1 & 0 & -3 \\ 3 & 2 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 2 & 1 & 0 & 1 \\ 1 & 1 & 1 & -1 & 0 \\ 0 & 0 & 3 & 2 & 1 \\ -1 & 5 & 7 & 2 & 1 \end{bmatrix} \qquad C = \begin{bmatrix} 2 & 2 \\ 0 & -1 \\ 3 & 1 \\ 4 & 0 \end{bmatrix}$$

(a) Find the domain and codomain of each of the three linear maps corresponding to A, B, and C.

(b) Only one of the matrix products AB, AC, BA, BC, CA, CB can actually be computed. Compute it.

Matrices and Multiplication (MX1)

Activity 4.1.7 Let $B = \begin{bmatrix} 3 & -4 & 0 \\ 2 & 0 & -1 \\ 0 & -3 & 3 \end{bmatrix}$, and let $A = \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix}$.

(a) Compute the product BA by hand.

(b) Check your work using technology. Using Octave:

```
B = [3 -4 0 ; 2 0 -1 ; 0 -3 3]
A = [2 7 -1 ; 0 3 2  ; 1 1 -1]
B*A
```

Matrices and Multiplication (MX1)

Activity 4.1.8 Of the following three matrices, only two may be multiplied.

$$A = \begin{bmatrix} -1 & 3 & -2 & -3 \\ 1 & -4 & 2 & 3 \end{bmatrix} \quad B = \begin{bmatrix} 1 & -6 & -1 \\ 0 & 1 & 0 \end{bmatrix} \quad C = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -2 \\ -2 & 4 & -1 \\ -2 & 3 & -1 \end{bmatrix}$$

Explain which two can be multiplied and why. Then show how to find their product.

Activity 4.1.9 Let $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x+2y \\ y \\ 3x+5y \\ -x-2y \end{bmatrix}$ In Fact 3.2.12 we adopted the notation

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x+2y \\ y \\ 3x+5y \\ -x-2y \end{bmatrix} = A\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 5 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}.$$

Verify that $\begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 5 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+2y \\ y \\ 3x+5y \\ -x-2y \end{bmatrix}$ in terms of matrix multiplication.

Matrices and Multiplication (MX1)

Activity 4.1.10 Given two $n \times n$ matrices A and B, explain why the sentence "Multiply the matrices A and B together." is ambiguous. How could you re-write the sentence in order to eliminate the ambiguity?

4.2 The Inverse of a Matrix (MX2)

Learning Outcomes

- Determine if a matrix is invertible, and if so, compute its inverse and use it to solve an appropriate system of equations.

The Inverse of a Matrix (MX2)

Activity 4.2.1 Consider the matrices:

$$A = \begin{bmatrix} 1 & 5 & -1 \\ 0 & 3 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 7 & 2 & -1 & 1 \\ 0 & 3 & 2 & -2 \\ 1 & 1 & -1 & -3 \end{bmatrix}.$$

Without using technology, what is the third column of the product AB?

The Inverse of a Matrix (MX2)

Activity 4.2.2 Let $A = \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix}$. Find a 3×3 matrix B such that $BA = A$, that is,

$$\begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix}$$

Check your guess using technology.

Definition 4.2.3 The identity matrix I_n (or just I when n is obvious from context) is the $n \times n$ matrix

$$I_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix}.$$

It has a 1 on each diagonal element and a 0 in every other position. $\lozenge$

The Inverse of a Matrix (MX2)

Fact 4.2.4 *For any square matrix A, $IA = AI = A$:*

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix}$$

The Inverse of a Matrix (MX2)

Activity 4.2.5 Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map with standard matrix A. Sort the following items into three groups of statements: a group that means T is *injective*, a group that means T is *surjective*, and a group that means T is *bijective*.

A. $T(\vec{x}) = \vec{b}$ has a solution for all $\vec{b} \in \mathbb{R}^m$

B. $T(\vec{x}) = \vec{b}$ has a unique solution for all $\vec{b} \in \mathbb{R}^m$

C. $T(\vec{x}) = \vec{0}$ has a unique solution.

D. The columns of A span $\mathbb{R}^m$

E. The columns of A are linearly independent

F. The columns of A are a basis of $\mathbb{R}^m$

G. Every column of $\text{RREF}(A)$ has a pivot

H. Every row of $\text{RREF}(A)$ has a pivot

I. $m = n$ and $\text{RREF}(A) = I$

The Inverse of a Matrix (MX2)

Definition 4.2.6 Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear bijection with standard matrix A.

By item (B) from Activity 4.2.5 we may define an **inverse map** $T^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ that defines $T^{-1}(\vec{b})$ as the unique solution $\vec{x}$ satisfying $T(\vec{x}) = \vec{b}$, that is, $T(T^{-1}(\vec{b})) = \vec{b}$.

Furthermore, let

$$A^{-1} = [T^{-1}(\vec{e}_1) \quad \cdots \quad T^{-1}(\vec{e}_n)]$$

be the standard matrix for T^{-1}. We call A^{-1} the **inverse matrix** of A, and we also say that A is an **invertible** matrix. $\Diamond$

The Inverse of a Matrix (MX2)

Activity 4.2.7 Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear bijection given by the standard matrix $A = \begin{bmatrix} 2 & -1 & -6 \\ 2 & 1 & 3 \\ 1 & 1 & 4 \end{bmatrix}$.

(a) To find $\vec{x} = T^{-1}(\vec{e}_1)$, we need to find the unique solution for $T(\vec{x}) = \vec{e}_1$. Which of these linear systems can be used to find this solution?

A. $\begin{alignedat}{4} 2x_1 & -1x_2 & -6x_3 & = x_1 \\ 2x_1 & +1x_2 & +3x_3 & = 0 \\ 1x_1 & +1x_2 & +4x_3 & = 0 \end{alignedat}$

B. $\begin{alignedat}{4} 2x_1 & -1x_2 & -6x_3 & = x_1 \\ 2x_1 & +1x_2 & +3x_3 & = x_2 \\ 1x_1 & +1x_2 & +4x_3 & = x_3 \end{alignedat}$

C. $\begin{alignedat}{4} 2x_1 & -1x_2 & -6x_3 & = 1 \\ 2x_1 & +1x_2 & +3x_3 & = 0 \\ 1x_1 & +1x_2 & +4x_3 & = 0 \end{alignedat}$

D. $\begin{alignedat}{4} 2x_1 & -1x_2 & -6x_3 & = 1 \\ 2x_1 & +1x_2 & +3x_3 & = 1 \\ 1x_1 & +1x_2 & +4x_3 & = 1 \end{alignedat}$

(b) Use that system to find the solution $\vec{x} = T^{-1}(\vec{e}_1)$ for $T(\vec{x}) = \vec{e}_1$.

(c) Similarly, solve $T(\vec{x}) = \vec{e}_2$ to find $T^{-1}(\vec{e}_2)$, and solve $T(\vec{x}) = \vec{e}_3$ to find $T^{-1}(\vec{e}_3)$.

(d) Use these to write

$$A^{-1} = [T^{-1}(\vec{e}_1) \quad T^{-1}(\vec{e}_2) \quad T^{-1}(\vec{e}_3)],$$

the standard matrix for T^{-1}.

The Inverse of a Matrix (MX2)

Activity 4.2.8 Let $T\colon \mathbb{R}^4 \to \mathbb{R}^4$ be the linear bijection given by the standard matrix:

$$A = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & -1 & -4 \\ 1 & 1 & 0 & -4 \\ 1 & -1 & -1 & 2 \end{bmatrix}.$$

(a) Calculate $T^{-1}(\vec{e}_2)$ by row reducing an appropriate augmented matrix.

(b) Calculate $T^{-1}(\vec{e}_4)$ by row reducing an appropriate augmented matrix.

(c) Suppose we completed the previous two tasks by hand. Which of the following statements best describes the row operations we would use?

A. We had to use the same row operations in (a) as we did in (b).

B. We could have used the same row operation in (a) as we did in (b).

C. The row operations used in (a) have nothing to do with with the row operations used in part (b).

(d) So far, we have only considered augmented matrices with a single augmented column. Write down an augmented matrix with more than one augmented column whose RREF would help us find A^{-1}.

Observation 4.2.9 Our exploration above yields a succinct way to calculate the inverse of a matrix. Indeed, if A is an invertible matrix, then we have

$$\mathrm{RREF}\,[A|I] = \left[I|A^{-1}\right].$$

The Inverse of a Matrix (MX2)

Activity 4.2.10 Is the matrix $\begin{bmatrix} 2 & 3 & 1 \\ -1 & -4 & 2 \\ 0 & -5 & 5 \end{bmatrix}$ invertible?

A. Yes, because its transformation is a bijection.

B. Yes, because its transformation is not a bijection.

C. No, because its transformation is a bijection.

D. No, because its transformation is not a bijection.

The Inverse of a Matrix (MX2)

Observation 4.2.11 An $n \times n$ matrix A is invertible if and only if $\mathrm{RREF}(A) = I_n$.

The Inverse of a Matrix (MX2)

Observation 4.2.12 Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear bijection with standard matrix A and suppose $\vec{b} \in \mathbb{R}^n$. By definition of the inverse map and inverse matrix, the vector $\vec{x} = A^{-1}\vec{b}$ is the unique solution to the equation $A\vec{x} = \vec{b}$.

In other words, when the matrix A is invertible, we have a new method for solving the equation $A\vec{x} = \vec{b}$: we can first calculate A^{-1} and then calculate the product $\vec{x} = A^{-1}\vec{b}$.

Activity 4.2.13 Consider the vector equation

$$x_1 \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ -1 \\ 0 \\ -1 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ -4 \\ -4 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

with a unique solution.

(a) Use technology to both verify that the coefficient matrix is invertible and calculate its inverse.

(b) Explain and demonstrate how to use the inverse to find the unique solution to the given vector equation.

The Inverse of a Matrix (MX2)

Activity 4.2.14 Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the bijective linear map defined by $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x - 3y \\ -3x + 5y \end{bmatrix}$, with the inverse map $T^{-1}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 5x + 3y \\ 3x + 2y \end{bmatrix}$.

(a) Compute $(T^{-1} \circ T)\left(\begin{bmatrix} -2 \\ 1 \end{bmatrix}\right)$.

(b) If A is the standard matrix for T and A^{-1} is the standard matrix for T^{-1}, find the 2×2 matrix

$$A^{-1}A = \begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix}.$$

Observation 4.2.15 $T^{-1} \circ T = T \circ T^{-1}$ is the identity map for any bijective linear transformation T. Therefore $A^{-1}A = AA^{-1}$ equals the identity matrix I for any invertible matrix A.

The Inverse of a Matrix (MX2)

Activity 4.2.16 Now that we have defined the inverse of a matrix, we have the ability to solve matrix equations. In the following equations, A, B all denote square matrices of the same size and I denotes the identity matrix. For each equation, solve for X.

(a) $A^{-1}XA = B$

(b) $AXA^{-1} = B$

(c) $ABX = I$

(d) $BAX = I$

The Inverse of a Matrix (MX2)

Activity 4.2.17 Solving linear systems using matrix multiplication is most useful when we are working with one common coefficient matrix, and varying the right-hand side. That is, when we have $A\vec{x} = \vec{b}$ for several different values of $\vec{b}$.

In the following, let $A = \begin{bmatrix} 2 & -1 & -6 \\ 2 & 1 & 3 \\ 1 & 1 & 4 \end{bmatrix}$ and consider the following questions about various equations of the form $A\vec{x} = \vec{b}$?

(a) Suppose that $\vec{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. If asked to solve the equation $A\vec{x} = \vec{b}$, which of the following approaches do you prefer?

A. Calculate RREF$[A|\vec{b}]$.

B. Calculate A^{-1} and then compute $\vec{x} = A^{-1}\vec{b}$

(b) Suppose that $\vec{b}_1, \vec{b}_2, \vec{b}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ 5 \end{bmatrix}$. If asked to solve each of the equations $A\vec{x} = \vec{b}_1, A\vec{x} = \vec{b}_2, A\vec{x} = \vec{b}_3$, which of the following approaches do you prefer?

A. Calculate RREF$[A|\vec{b}_1]$, RREF$[A|\vec{b}_2]$, and RREF$[A|\vec{b}_3]$

B. Calculate A^{-1} and then compute $\vec{x} = A^{-1}\vec{b}_1$, $\vec{x} = A^{-1}\vec{b}_2$, and $\vec{x} = A^{-1}\vec{b}_3$

(c) Suppose that $\vec{b}_1, \ldots, \vec{b}_{10}$ are 10 distinct vectors. If asked to solve each of the equations $A\vec{x} = \vec{b}_1, \ldots, A\vec{x} = \vec{b}_{10}$, which of the following approaches do you prefer?

A. Calculate RREF$[A|\vec{b}_1]$, ... RREF$[A|\vec{b}_{10}]$.

B. Calculate A^{-1} and then compute $\vec{x} = A^{-1}\vec{b}_1$, ... $\vec{x} = A^{-1}\vec{b}_{10}$.

4.3 Change of Basis (MX3)

Learning Outcomes

- Calculate the change-of-basis matrix for the standard basis to a non-standard basis of $\mathbb{R}^n$.

Change of Basis (MX3)

Activity 4.3.1 Let $T\colon \mathbb{R}^4 \to \mathbb{R}^4$ be the linear bijection given by the standard matrix:

$$A = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & -1 & -4 \\ 1 & 1 & 0 & -4 \\ 1 & -1 & -1 & 2 \end{bmatrix}.$$

(a) If $\vec{b} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 2 \end{bmatrix}$, what is the meaning of the vector $T^{-1}(\vec{b})$?

(b) Explain and demonstrate how to find the third column of A^{-1}.

Change of Basis (MX3)

Remark 4.3.2 So far, when working with the Euclidean vector space $\mathbb{R}^n$, we have primarily worked with the standard basis $\mathcal{E} = \{\vec{e}_1, \ldots, \vec{e}_n\}$. We can explore alternative perspectives more easily if we expand our toolkit to analyze different bases.

Change of Basis (MX3)

Activity 4.3.3 Let $\mathcal{B} = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$.

(a) Is $\mathcal{B}$ a basis of $\mathbb{R}^3$?

A. Yes.

B. No.

(b) Since $\mathcal{B}$ is a basis, we know that if $\vec{v} \in \mathbb{R}^3$, the following vector equation will have a unique solution:

$$x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3 = \vec{v}$$

Given this, we define a map $C_{\mathcal{B}}\colon \mathbb{R}^3 \to \mathbb{R}^3$ via the rule that $C_{\mathcal{B}}(\vec{v})$ is equal to the unique solution to the above vector equation. The map $C_{\mathcal{B}}$ is a linear map.

Compute $C_{\mathcal{B}}\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right)$, the unique solution to

$$x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

(c) Compute $C_{\mathcal{B}}(\vec{e}_1), C_{\mathcal{B}}(\vec{e}_2), C_{\mathcal{B}}(\vec{e}_3)$ and, in doing so, write down the standard matrix $M_{\mathcal{B}}$ of $C_{\mathcal{B}}$.

Change of Basis (MX3)

Definition 4.3.4 Given a basis $\mathcal{B} = \left\{\vec{\sqsubseteq}_{\infty}, \ldots, \vec{\sqsubseteq}_{\setminus}\right\}$ of $\mathbb{R}^n$, the **change of basis/coordinate** transformation *from* the standard basis *to* $\mathcal{B}$ is the transformation $C_{\mathcal{B}} \colon \mathbb{R}^n \to \mathbb{R}^n$ defined by the property that, for any vector $\vec{v} \in \mathbb{R}^n$, the vector $C_{\mathcal{B}}(\vec{v})$ is the unique solution to the vector equation:

$$x_1\vec{v}_1 + \cdots + x_n\vec{v}_n = \vec{v}.$$

Its standard matrix is called the change-of-basis matrix from the standard basis to $\mathcal{B}$ and is denoted by $M_{\mathcal{B}}$. It satisfies the following:

$$M_{\mathcal{B}} = [\vec{v}_1 \ \cdots \ \vec{v}_n]^{-1}.$$

$\Diamond$

Change of Basis (MX3)

Remark 4.3.5 The vector $C_{\mathcal{B}}(\vec{v})$ is the $\mathcal{B}$-coordinates of $\vec{v}$. If you work with standard coordinates, and I work with $\mathcal{B}$-coordinates, then to build the vector that you call $\vec{v} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = a_1\vec{e}_1 + \cdots + a_n\vec{e}_n$, I would first compute $C_{\mathcal{B}}(\vec{v}) = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ and then build $\vec{v} = x_1\vec{v}_1 + \cdots + x_n\vec{v}_n$.

In particular, notation as above, we would have:

$$a_1\vec{e}_1 + \cdots a_n\vec{e}_n = \vec{v} = x_1\vec{v}_1 + \cdots + x_n\vec{v}_n.$$

Change of Basis (MX3)

Activity 4.3.6 Let $\vec{v}_1 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$, and $\mathcal{B} = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$

(a) Calculate $M_{\mathcal{B}}$ using technology.

(b) Use your result to calculate $C_{\mathcal{B}}\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right)$ and express the vector $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ as a linear combination of $\vec{v}_1, \vec{v}_2, \vec{v}_3$.

Change of Basis (MX3)

Observation 4.3.7 Let $T\colon \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation and let A denote its standard matrix. If $\mathcal{B} = \left\{\vec{\sqsubseteq}_{\infty}, \ldots, \vec{\sqsubseteq}_{\backslash}\right\}$ is some other basis, then we have:

$$\begin{aligned} M_{\mathcal{B}} A M_{\mathcal{B}}^{-1} &= M_{\mathcal{B}} A [\vec{v}_1 \cdots \vec{v}_n] \\ &= M_{\mathcal{B}} [T(\vec{v}_1) \cdots T(\vec{v}_n)] \\ &= [C_{\mathcal{B}}(T(\vec{v}_1)) \cdots C_{\mathcal{B}}(T(\vec{v}_n))] \end{aligned}$$

In other words, the matrix $M_{\mathcal{B}} A M_{\mathcal{B}}^{-1}$ is the matrix whose columns consist of $\mathcal{B}$-*coordinate* vectors of the image vectors $T(\vec{v}_i)$. The matrix $M_{\mathcal{B}} A M_{\mathcal{B}}^{-1}$ is called the ***matrix of*** T ***with respect to*** $\mathcal{B}$***-coordinates***.

Change of Basis (MX3)

Activity 4.3.8 Let $\mathcal{B} = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\} = \left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}$ be basis from the previous Activity. Let T denote the linear transformation whose standard matrix is given by:

$$A = \begin{bmatrix} 9 & 4 & 4 \\ 6 & 9 & 2 \\ -18 & -16 & -9 \end{bmatrix}.$$

(a) Calculate the matrix $M_{\mathcal{B}} A M_{\mathcal{B}}^{-1}$.

(b) The matrix A describes how T transforms the standard basis of $\mathbb{R}^3$. The matrix $M_{\mathcal{B}} A M_{\mathcal{B}}^{-1}$ describes how T transforms the basis $\mathcal{B}$ (in $\mathcal{B}$-coordinates).

Which of these two descriptions of T is most helpful to you in describing/understanding/visualizing the transformation T and why?

Change of Basis (MX3)

Definition 4.3.9 Suppose that A and B are two $n \times n$ matrix. We say that A is **similar** to B if there exists an invertible matrix P that satisfies:

$$PAP^{-1} = B.$$

The results of this section demonstrate that similar matrices can be viewed as describing the same linear transformation with respect to different bases. Specifically, if A describes a transformation with respect to the standard basis of $\mathbb{R}^n$, then the matrix B describes the same linear transformation with respect to the basis consisting of the columns of P^{-1}. $\diamondsuit$

Change of Basis (MX3)

Activity 4.3.10 Suppose that $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ is a linear transformation and you knew that $\mathcal{B} = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ was a basis of $\mathbb{R}^3$ that satisfied:

$$T(\vec{v}_1) = 3\vec{v}_1,\ T(\vec{v}_2) = -5\vec{v}_2,\ T(\vec{v}_3) = 7\vec{v}_3.$$

If A is the standard matrix of T, do you have enough information to determine the matrix $M_{\mathcal{B}} A M_{\mathcal{B}}^{-1}$? If yes, write it down; if not, describe what additional information is needed.

Change of Basis (MX3)

Activity 4.3.11 Suppose that A is similar to B. Prove that B is also similar to A. Thus, we may simply that A and B are similar matrices.

4.4 Row Operations as Matrix Multiplication (MX4)

Learning Outcomes

- Express row operations through matrix multiplication.

Row Operations as Matrix Multiplication (MX4)

Activity 4.4.1 Given a linear transformation T, how did we define its standard matrix A? How do we compute the standard matrix A from T?

Row Operations as Matrix Multiplication (MX4)

Activity 4.4.2 Tweaking the identity matrix slightly allows us to write row operations in terms of matrix multiplication.

(a) Which of these tweaks of the identity matrix yields a matrix that doubles the third row of A when left-multiplying? $(2R_3 \to R_3)$

$$\begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix}\begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 2 & 2 & -2 \end{bmatrix}$$

A. $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

B. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

C. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

D. $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

(b) Which of these tweaks of the identity matrix yields a matrix that swaps the second and third rows of A when left-multiplying? $(R_2 \leftrightarrow R_3)$

$$\begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix}\begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 7 & -1 \\ 1 & 1 & -1 \\ 0 & 3 & 2 \end{bmatrix}$$

A. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

B. $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$

C. $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

D. $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(c) Which of these tweaks of the identity matrix yields a matrix that adds 5 times the third row of A to the first row when left-multiplying? $(R_1 + 5R_3 \to R_1)$

$$\begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix}\begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 2+5(1) & 7+5(1) & -1+5(-1) \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix}$$

A. $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

B. $\begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

C. $\begin{bmatrix} 5 & 5 & 5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

D. $\begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

Row Operations as Matrix Multiplication (MX4)

Fact 4.4.3 *If R is the result of applying a row operation to I, then RA is the result of applying the same row operation to A.*

- *Scaling a row:* $R = \begin{bmatrix} c & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
- *Swapping rows:* $R = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
- *Adding a row multiple to another row:* $R = \begin{bmatrix} 1 & 0 & c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Such matrices can be chained together to emulate multiple row operations. In particular,

$$\text{RREF}(A) = R_k \dots R_2 R_1 A$$

for some sequence of matrices $R_1, R_2, \dots, R_k$.

Row Operations as Matrix Multiplication (MX4)

Activity 4.4.4 What would happen if you *right*-multiplied by the tweaked identity matrix rather than left-multiplied?

A. The manipulated rows would be reversed.

B. Columns would be manipulated instead of rows.

C. The entries of the resulting matrix would be rotated 180 degrees.

Row Operations as Matrix Multiplication (MX4)

Activity 4.4.5 Consider the two row operations $R_2 \leftrightarrow R_3$ and $R_1 + R_2 \to R_1$ applied as follows to show $A \sim B$:

$$A = \begin{bmatrix} -1 & 4 & 5 \\ 0 & 3 & -1 \\ 1 & 2 & 3 \end{bmatrix} \sim \begin{bmatrix} -1 & 4 & 5 \\ 1 & 2 & 3 \\ 0 & 3 & -1 \end{bmatrix}$$

$$\sim \begin{bmatrix} -1+1 & 4+2 & 5+3 \\ 1 & 2 & 3 \\ 0 & 3 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 6 & 8 \\ 1 & 2 & 3 \\ 0 & 3 & -1 \end{bmatrix} = B$$

Express these row operations as matrix multiplication by expressing B as the product of two matrices and A:

$$B = \begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} A$$

Check your work using technology.

Row Operations as Matrix Multiplication (MX4)

Activity 4.4.6

(a) Give a 3×3 matrix B that may be used to perform the row operation $R_1 \leftrightarrow R_3$.

(b) Give a 3×3 matrix C that may be used to perform the row operation $R_3 + 5R_2 \to R_3$.

(c) Give a 3×3 matrix P that may be used to perform the row operation $-4R_1 \to R_1$.

(d) Give a single 3×3 matrix that may be used to first apply $R_1 \leftrightarrow R_3$, then $-4R_1 \to R_1$, and finally $R_3 + 5R_2 \to R_3$ (note the order).

(e) Show how to manually apply those row operations to $A = \begin{bmatrix} 0 & 1 & 2 \\ 2 & -5 & -8 \\ 1 & -4 & -7 \end{bmatrix}$, then use technology to verify that your matrix in the previous task gives the same result.

Row Operations as Matrix Multiplication (MX4)

Activity 4.4.7 Consider the matrix $A = \begin{bmatrix} 2 & 6 & -1 & 6 \\ 1 & 3 & -1 & 2 \\ -1 & -3 & 2 & 0 \end{bmatrix}$. Illustrate Fact 4.4.3 by finding row operation matrices $R_1, \ldots, R_k$ for which

$$\mathrm{RREF}(A) = R_k \cdots R_2 R_1 A.$$

If you and a teammate were to do this independently, would you necessarily come up with the same sequence of matrices $R_1, \ldots, R_k$?

Chapter 5

Geometric Properties of Linear Maps (GT)

Learning Outcomes

How do we understand linear maps geometrically?
By the end of this chapter, you should be able to...

1. Describe how a row operation affects the determinant of a matrix.
2. Compute the determinant of a 4×4 matrix.
3. Find the eigenvalues of a 2×2 matrix.
4. Find a basis for the eigenspace of a 4×4 matrix associated with a given eigenvalue.

5.1 Row Operations and Determinants (GT1)

Learning Outcomes

- Describe how a row operation affects the determinant of a matrix.

Row Operations and Determinants (GT1)

Activity 5.1.1 Consider the linear transformation $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ corresponding to the standard matrix $A = \begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix}$.

(a) Draw a figure that depicts how T transforms the unit square.

(b) What geometric features of the unit square were preserved by the transformation? Which geometric features changed?

Observation 5.1.2 The tool in Figure 46 can be used to visualize the effect of a linear transformation (defined by its standard matrix) on the geometry of the unit square defined by the standard basic vectors $\vec{e}_1, \vec{e}_2$.

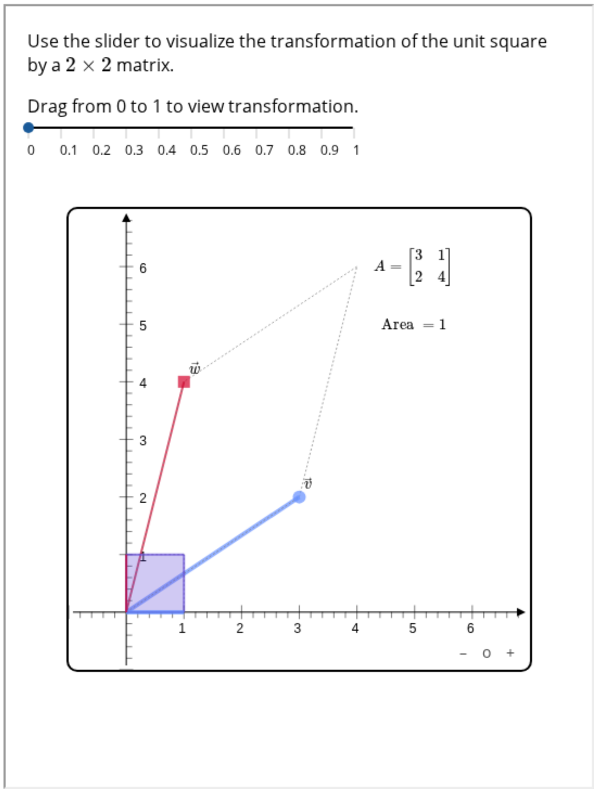


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Figure 17 Tool to visualize a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$

Activity 5.1.3 The image in Figure 47 illustrates how the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by the standard matrix $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ transforms the unit square.

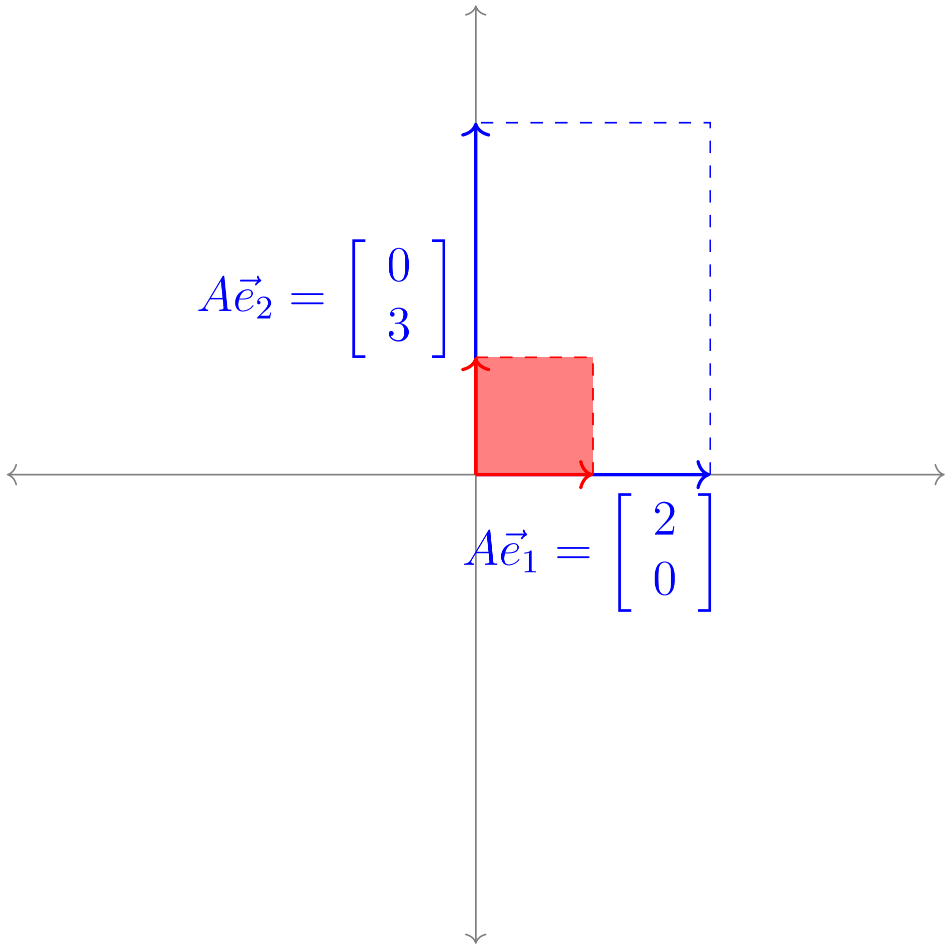


Figure 18 Transformation of the unit square by the matrix A.

(a) What are the lengths of $A\vec{e}_1$ and $A\vec{e}_2$?

(b) What is the area of the transformed unit square?

Activity 5.1.4 The image below illustrates how the linear transformation $S:\mathbb{R}^2\to\mathbb{R}^2$ given by the standard matrix $B=\left[\begin{array}{cc}2&3\\0&4\end{array}\right]$ transforms the unit square.

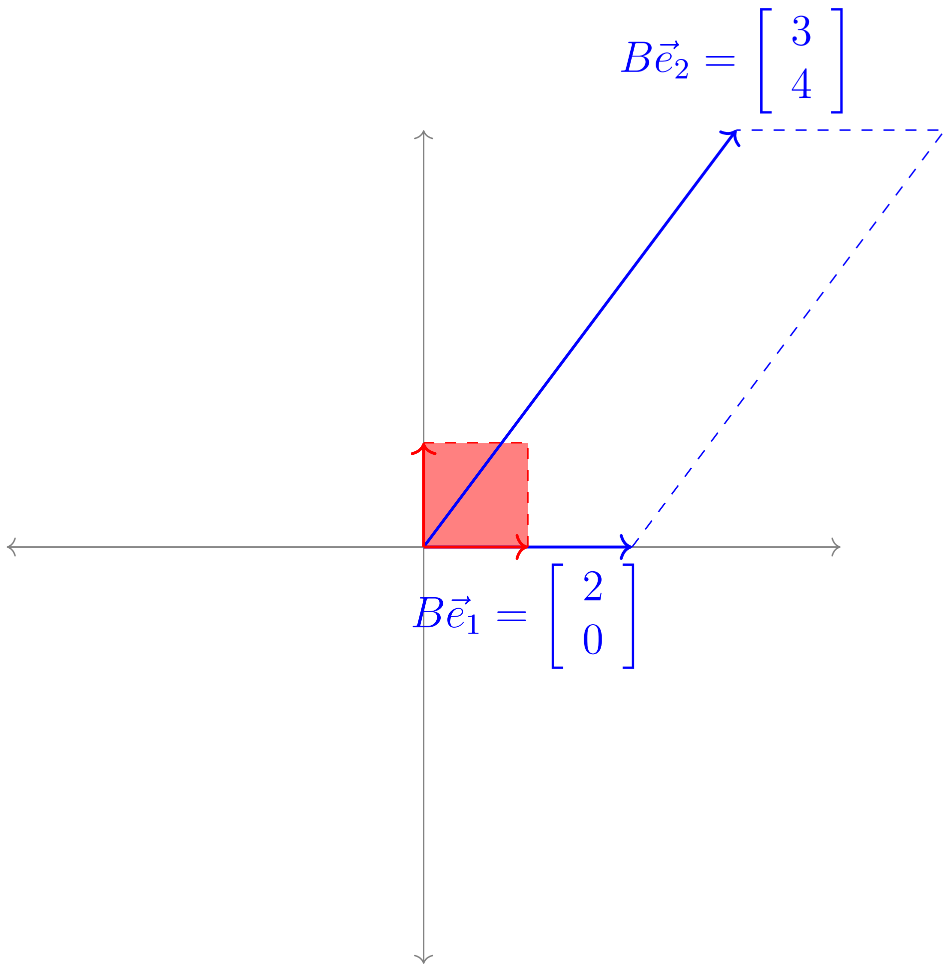


Figure 19 Transformation of the unit square by the matrix B

(a) What are the lengths of $B\vec{e}_1$ and $B\vec{e}_2$?

(b) What is the area of the transformed unit square?

Observation 5.1.5 It is possible to find two nonparallel vectors that are scaled but not rotated by the linear map given by B.

$$B\vec{e}_1 = \begin{bmatrix} 2 & 3 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} = 2\vec{e}_1$$

$$B \begin{bmatrix} \frac{3}{4} \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} \frac{3}{4} \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} = 4 \begin{bmatrix} \frac{3}{4} \\ \frac{1}{2} \end{bmatrix}$$

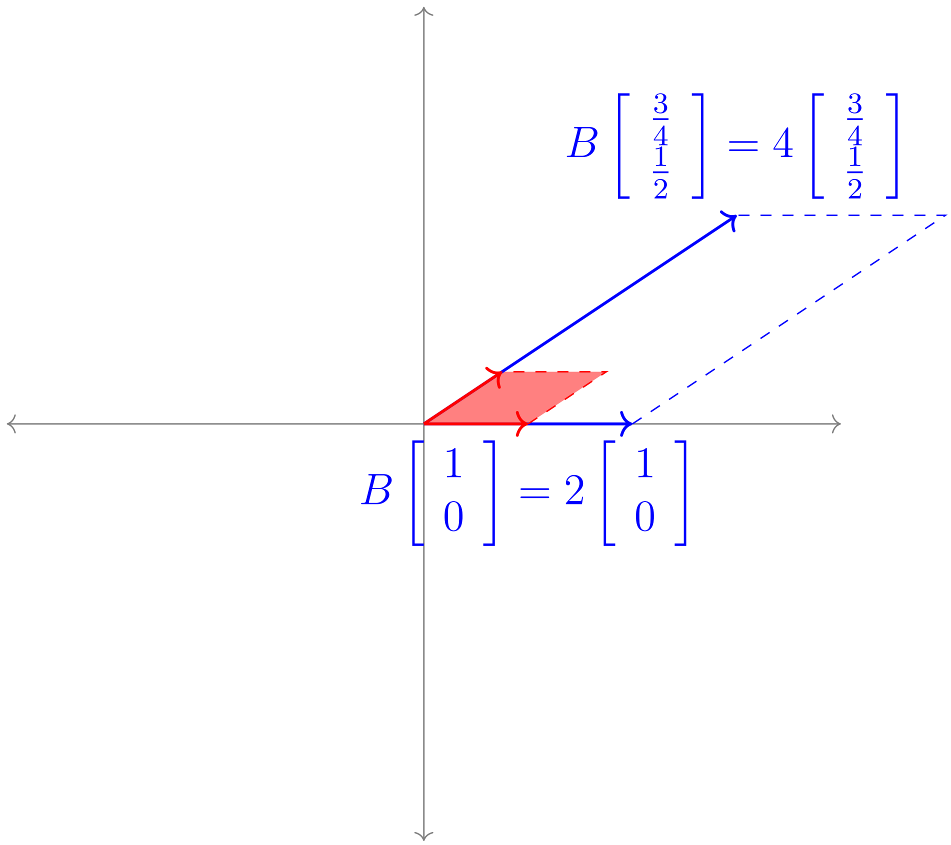


Figure 20 Certain vectors are stretched out without being rotated.

The process for finding such vectors will be covered later in this chapter.

Observation 5.1.6 Notice that while a linear map can transform vectors in various ways, linear maps always transform parallelograms into parallelograms, and these areas are always transformed by the same factor: in the case of $B = \begin{bmatrix} 2 & 3 \\ 0 & 4 \end{bmatrix}$, this factor is 8.

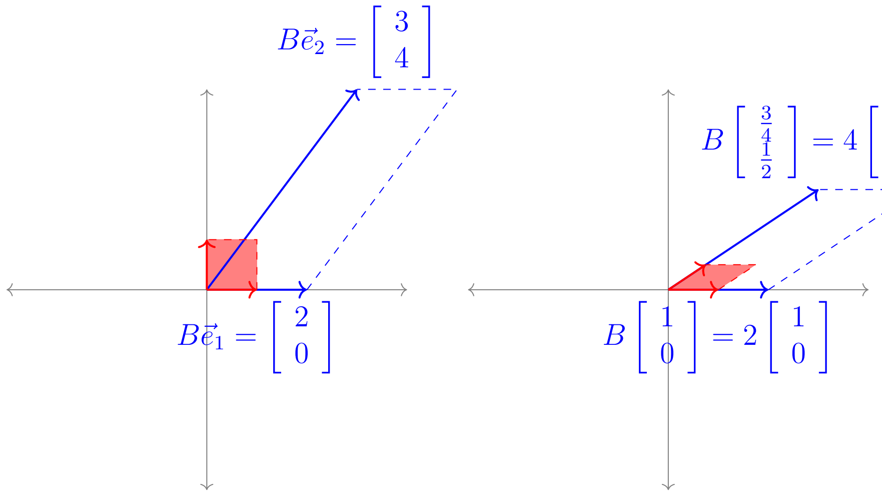


Figure 21 A linear map transforming parallelograms into parallelograms.

Since this change in area is always the same for a given linear map, it will be equal to the value of the transformed unit square (which begins with area 1).

Remark 5.1.7 We will define the **determinant** of a square matrix B, or $\det(B)$ for short, to be the factor by which B scales areas. In order to figure out how to compute it, we first figure out the properties it must satisfy.

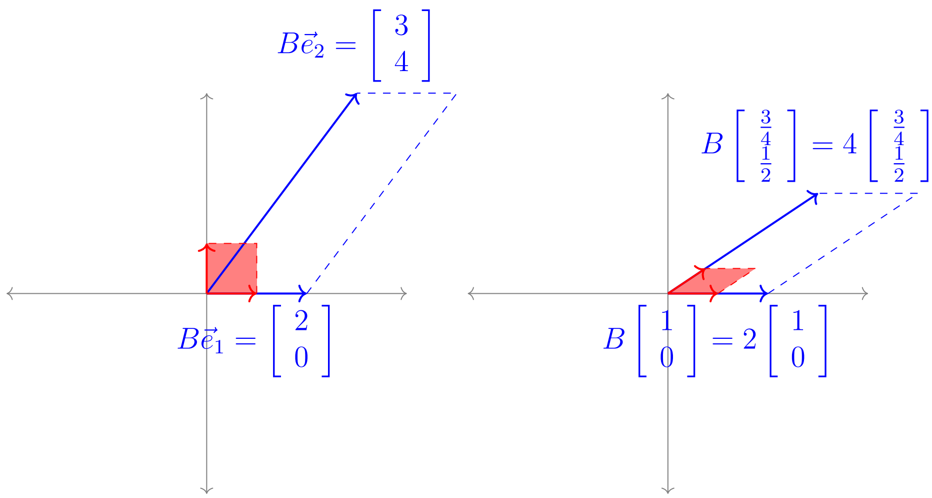


Figure 22 The linear transformation B scaling areas by a constant factor, which we call the **determinant**

Activity 5.1.8 The transformation of the unit square by the standard matrix $[\vec{e}_1 \ \ \vec{e}_2] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$ is illustrated below. If $\det([\vec{e}_1 \ \ \vec{e}_2]) = \det(I)$ is the area of resulting parallelogram, what is the value of $\det([\vec{e}_1 \ \ \vec{e}_2]) = \det(I)$?

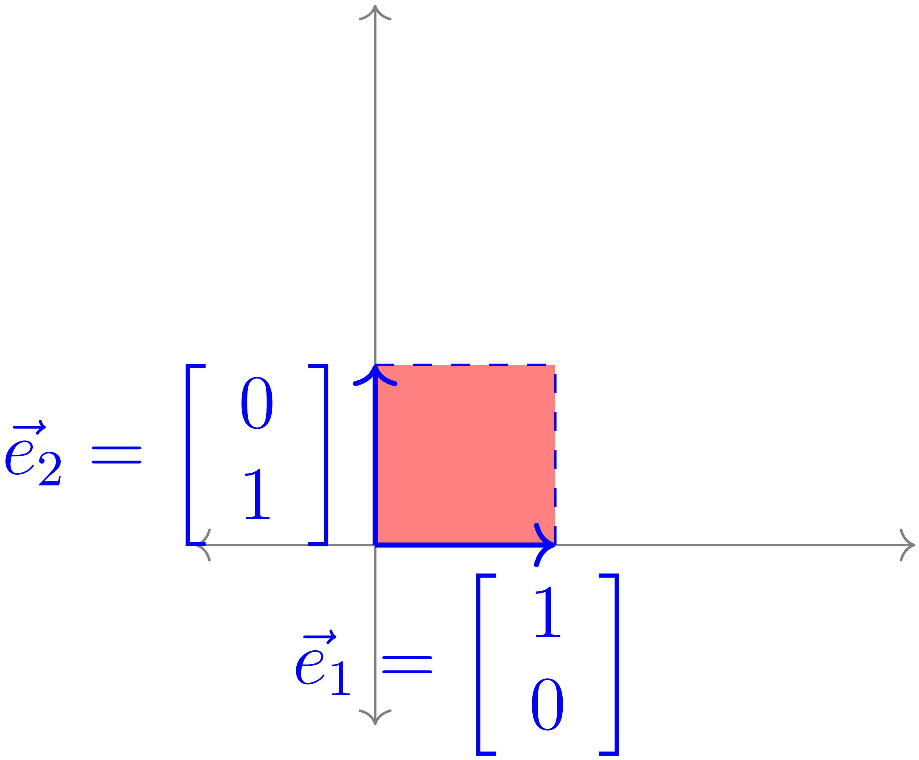


Figure 23 The transformation of the unit square by the identity matrix.

The value for $\det([\vec{e}_1 \ \ \vec{e}_2]) = \det(I)$ is:

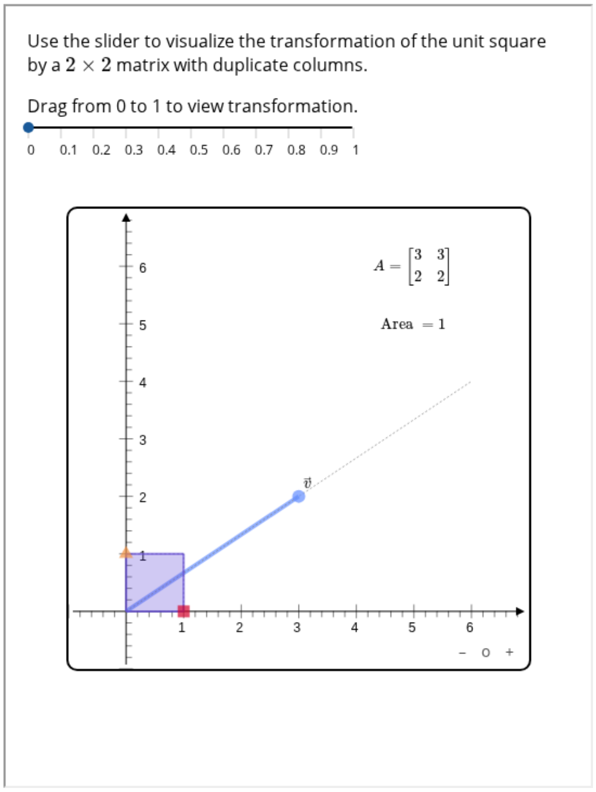
A. 0

B. 1

C. 2

D. 4

Activity 5.1.9



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Which of the following is true?

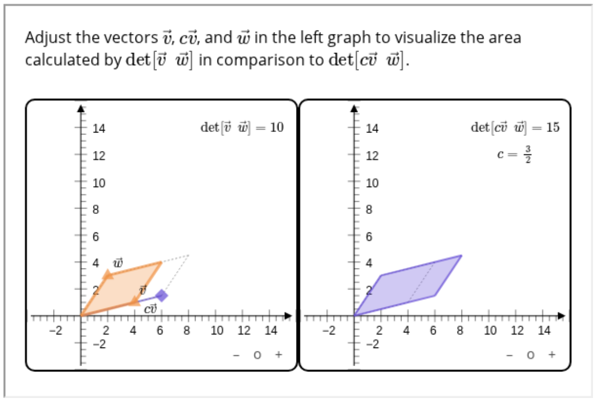
A. $\det([\vec{v}\ \ \vec{v}]) = 0$

B. $\det([\vec{v}\ \ \vec{v}]) = 1$

C. $\det([\vec{v}\ \ \vec{v}]) = 2$

D. $\det([\vec{v}\ \ \vec{v}]) = 4$

Activity 5.1.10



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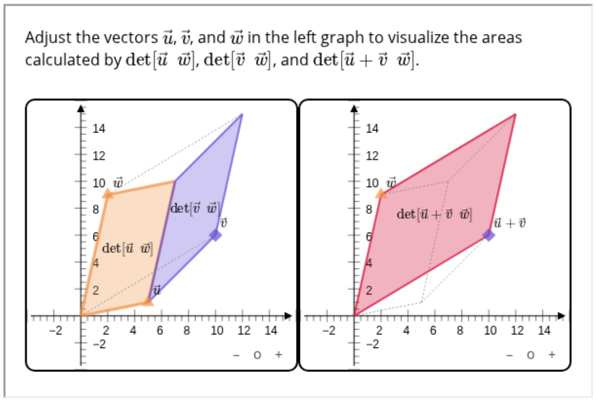
Which of the following is true?

A. $\det([c\vec{v}\ \ \vec{w}]) = \det([\vec{v}\ \ \vec{w}])$

B. $\det([c\vec{v}\ \ \vec{w}]) = c\det([\vec{v}\ \ \vec{w}])$

C. $\det([c\vec{v}\ \ \vec{w}]) = c^2\det([\vec{v}\ \ \vec{w}])$

Activity 5.1.11



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Which of the following is true?

A. $\det([\vec{u}+\vec{v}\ \ \vec{w}]) = \det([\vec{u}\ \ \vec{w}]) = \det([\vec{v}\ \ \vec{w}])$

B. $\det([\vec{u}+\vec{v}\ \ \vec{w}]) = \det([\vec{u}\ \ \vec{w}]) + \det([\vec{v}\ \ \vec{w}])$

C. $\det([\vec{u}+\vec{v}\ \ \vec{w}]) = \det([\vec{u}\ \ \vec{w}])\det([\vec{v}\ \ \vec{w}])$

Row Operations and Determinants (GT1)

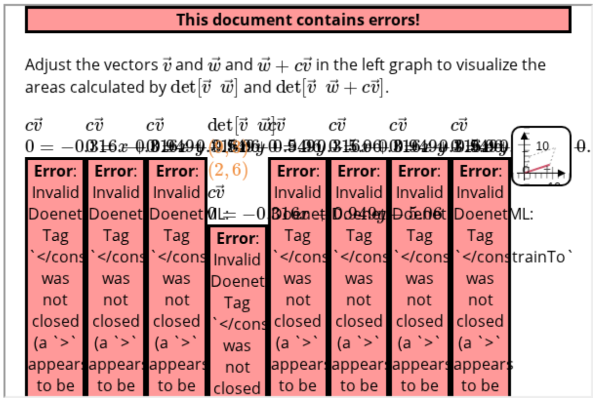
Definition 5.1.12 The **determinant** is the unique function $\det : M_{n,n} \to \mathbb{R}$ satisfying these properties:

1. $\det(I) = 1$
2. $\det(A) = 0$ whenever two columns of the matrix are identical.
3. $\det[\cdots \ c\vec{v} \ \cdots] = c\det[\cdots \ \vec{v} \ \cdots]$, assuming no other columns change.
4. $\det[\cdots \ \vec{v}+\vec{w} \ \cdots] = \det[\cdots \ \vec{v} \ \cdots] + \det[\cdots \ \vec{w} \ \cdots]$, assuming no other columns change.

Note that these last two properties together can be phrased as "The determinant is linear in each column."

Essentially, the determinant measures the change in "size" caused by a transformation, where "size" means area for 2×2 matrices and volume for 3×3 matrices. ◊

Observation 5.1.13 The determinant must also satisfy other properties. Consider $\det([\vec{v} \quad \vec{w}+c\vec{v}])$ and $\det([\vec{v} \quad \vec{w}])$.



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The base of both parallelograms is $\vec{v}$, while the height has not changed, so the determinant does not change either. This can also be proven using the other properties of the determinant:

$$\begin{aligned}
\det([\vec{v} \quad \vec{w}+c\vec{v}]) &= \det([\vec{v} \quad \vec{w}]) + \det([\vec{v} \quad c\vec{v}]) \\
&= \det([\vec{v} \quad \vec{w}]) + c\det([\vec{v} \quad \vec{v}]) \\
&= \det([\vec{v} \quad \vec{w}]) + c \cdot 0 \\
&= \det([\vec{v} \quad \vec{w}])
\end{aligned}$$

Remark 5.1.14 Swapping columns may be thought of as a reflection, which is represented by a negative determinant. For example, the following matrices transform the unit square into the same parallelogram, but the second matrix reflects its orientation.

$$A = \begin{bmatrix} 2 & 3 \\ 0 & 4 \end{bmatrix} \quad \det A = 8 \qquad B = \begin{bmatrix} 3 & 2 \\ 4 & 0 \end{bmatrix} \quad \det B = -8$$

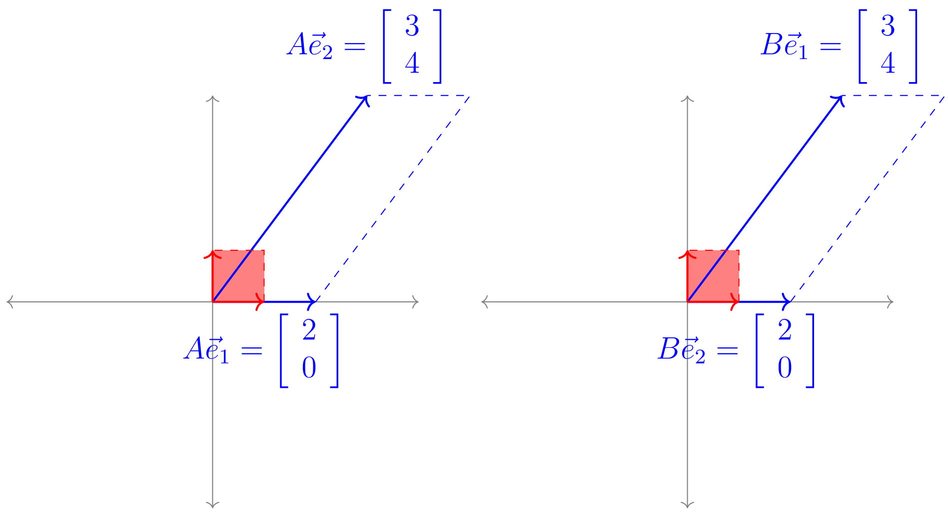


Figure 24 Reflection of a parallelogram as a result of swapping columns.

Observation 5.1.15 The fact that swapping columns multiplies determinants by a negative may be verified by adding and subtracting columns.

$$\begin{aligned}\det([\vec{v} \quad \vec{w}]) &= \det([\vec{v}+\vec{w} \quad \vec{w}]) \\ &= \det([\vec{v}+\vec{w} \quad \vec{w}-(\vec{v}+\vec{w})]) \\ &= \det([\vec{v}+\vec{w} \quad -\vec{v}]) \\ &= \det([\vec{v}+\vec{w}-\vec{v} \quad -\vec{v}]) \\ &= \det([\vec{w} \quad -\vec{v}]) \\ &= -\det([\vec{w} \quad \vec{v}])\end{aligned}$$

Row Operations and Determinants (GT1)

Fact 5.1.16 *To summarize, we've shown that the column versions of the three row-reducing operations a matrix may be used to simplify a determinant in the following way:*

1. *Multiplying a column by a scalar multiplies the determinant by that scalar:*

$$c\det([\cdots \quad \vec{v} \quad \cdots]) = \det([\cdots \quad c\vec{v} \quad \cdots])$$

2. *Swapping two columns changes the sign of the determinant:*

$$\det([\cdots \quad \vec{v} \quad \cdots \quad \vec{w} \quad \cdots]) = -\det([\cdots \quad \vec{w} \quad \cdots \quad \vec{v} \quad \cdots])$$

3. *Adding a multiple of a column to another column does not change the determinant:*

$$\det([\cdots \quad \vec{v} \quad \cdots \quad \vec{w} \quad \cdots]) = \det([\cdots \quad \vec{v}+c\vec{w} \quad \cdots \quad \vec{w} \quad \cdots])$$

Activity 5.1.17 The transformation given by the standard matrix A scales areas by 4, and the transformation given by the standard matrix B scales areas by 3. By what factor does the transformation given by the standard matrix AB scale areas?

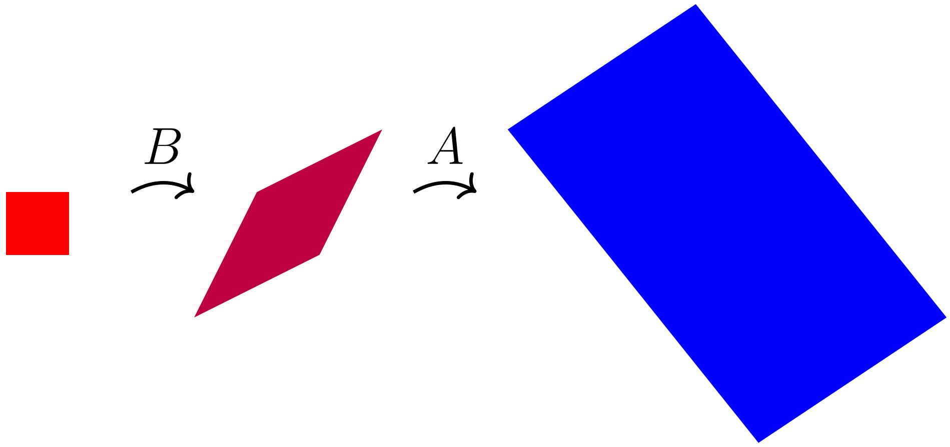


Figure 25 Area changing under the composition of two linear maps

A. 1

B. 7

C. 12

D. Cannot be determined

Row Operations and Determinants (GT1)

Fact 5.1.18 *Since the transformation given by the standard matrix AB is obtained by applying the transformations given by A and B, it follows that*

$$\det(AB) = \det(A)\det(B) = \det(B)\det(A) = \det(BA).$$

Row Operations and Determinants (GT1)

Remark 5.1.19 Recall that row operations may be produced by matrix multiplication.

- Multiply the first row of A by c: $\begin{bmatrix} c & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} A$
- Swap the first and second row of A: $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} A$
- Add c times the third row to the first row of A: $\begin{bmatrix} 1 & 0 & c & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} A$

Row Operations and Determinants (GT1)

Fact 5.1.20 *The determinants of row operation matrices may be computed by manipulating columns to reduce each matrix to the identity:*

- *Scaling a row:* $\det \begin{bmatrix} c & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = c \det \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = c$

- *Swapping rows:* $\det \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = -1 \det \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = -1$

- *Adding a row multiple to another row:* $\det \begin{bmatrix} 1 & 0 & c & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} =$

  $\det \begin{bmatrix} 1 & 0 & c-1c & 0 \\ 0 & 1 & 0-0c & 0 \\ 0 & 0 & 1-0c & 0 \\ 0 & 0 & 0-0c & 1 \end{bmatrix} = \det(I) = 1$

Row Operations and Determinants (GT1)

Activity 5.1.21 Consider the row operation $R_1 + 4R_3 \to R_1$ applied as follows to show $A \sim B$:

$$A = \begin{bmatrix} 1 & 2 & 0 & -3 \\ 4 & 1 & 3 & 0 \\ 0 & 0 & -3 & -5 \\ 1 & 1 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1+4(0) & 2+4(0) & 0+4(-3) & -3+4(-5) \\ 4 & 1 & 3 & 0 \\ 0 & 0 & -3 & -5 \\ 1 & 1 & 1 & 3 \end{bmatrix} = B$$

(a) Find a matrix R such that $B = RA$, by applying the same row operation to $I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

(b) The determinant of A is 70. Complete the following computation to calculate the determinant of B:

$$\begin{aligned} \det(B) &= \det(RA) \\ &= \det(R)\det(A) \\ &= (\ ?\ )(\ ?\ ) \\ &= \ ? \end{aligned}$$

Row Operations and Determinants (GT1)

Activity 5.1.22 Consider the row operation $R_1 \leftrightarrow R_4$ applied as follows to show $A \sim B$:

$$A = \begin{bmatrix} 1 & 2 & 0 & -3 \\ 4 & 1 & 3 & 0 \\ 0 & 0 & -3 & -5 \\ 1 & 1 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 3 \\ 4 & 1 & 3 & 0 \\ 0 & 0 & -3 & -5 \\ 1 & 2 & 0 & -3 \end{bmatrix} = B$$

(a) Find a matrix R such that $B = RA$, by applying the same row operation to I.

(b) The determinant of A is 70. Show how to compute the determinant of B.

Row Operations and Determinants (GT1)

Activity 5.1.23 Consider the row operation $3R_2 \to R_2$ applied as follows to show $A \sim B$:

$$A = \begin{bmatrix} 1 & 2 & 0 & -3 \\ 4 & 1 & 3 & 0 \\ 0 & 0 & -3 & -5 \\ 1 & 1 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & -3 \\ 3(4) & 3(1) & 3(3) & 3(0) \\ 0 & 0 & -3 & -5 \\ 1 & 1 & 1 & 3 \end{bmatrix} = B$$

(a) Find a matrix R such that $B = RA$.

(b) The determinant of A is 70. Show how to compute the determinant of B.

Row Operations and Determinants (GT1)

Activity 5.1.24 Let A be *any* 4×4 matrix with determinant 2.

(a) Let B be the matrix obtained from A by applying the row operation $R_1 - 5R_3 \to R_1$. What is $\det B$?

A 4 B -2 C 2 D 10

(b) Let M be the matrix obtained from A by applying the row operation $R_3 \leftrightarrow R_1$. What is $\det M$?

A 4 B -2 C 2 D 10

(c) Let P be the matrix obtained from A by applying the row operation $2R_4 \to R_4$. What is $\det P$?

A 4 B -2 C 2 D 10

Row Operations and Determinants (GT1)

Remark 5.1.25 Recall that the column versions of the three row-reducing operations a matrix may be used to simplify a determinant:

1. Multiplying columns by scalars:

$$\det([\cdots \quad c\vec{v} \quad \cdots]) = c\det([\cdots \quad \vec{v} \quad \cdots])$$

2. Swapping two columns:

$$\det([\cdots \quad \vec{v} \quad \cdots \quad \vec{w} \quad \cdots]) = -\det([\cdots \quad \vec{w} \quad \cdots \quad \vec{v} \quad \cdots])$$

3. Adding a multiple of a column to another column:

$$\det([\cdots \quad \vec{v} \quad \cdots \quad \vec{w} \quad \cdots]) = \det([\cdots \quad \vec{v}+c\vec{w} \quad \cdots \quad \vec{w} \quad \cdots])$$

Row Operations and Determinants (GT1)

Remark 5.1.26 The determinants of row operation matrices may be computed by manipulating columns to reduce each matrix to the identity:

- Scaling a row: $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$
- Swapping rows: $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$
- Adding a row multiple to another row: $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & c & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Row Operations and Determinants (GT1)

Fact 5.1.27 *Thus we can also use both row operations to simplify determinants:*

- *Multiplying rows by scalars:*

$$\det \begin{bmatrix} \vdots \\ cR \\ \vdots \end{bmatrix} = c \det \begin{bmatrix} \vdots \\ R \\ \vdots \end{bmatrix}$$

- *Swapping two rows:*

$$\det \begin{bmatrix} \vdots \\ R \\ \vdots \\ S \\ \vdots \end{bmatrix} = -\det \begin{bmatrix} \vdots \\ S \\ \vdots \\ R \\ \vdots \end{bmatrix}$$

- *Adding multiples of rows/columns to other rows:*

$$\det \begin{bmatrix} \vdots \\ R \\ \vdots \\ S \\ \vdots \end{bmatrix} = \det \begin{bmatrix} \vdots \\ R + cS \\ \vdots \\ S \\ \vdots \end{bmatrix}$$

Row Operations and Determinants (GT1)

Activity 5.1.28 Complete the following derivation for a formula calculating 2×2 determinants:

$$\begin{aligned}
\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} &= ? \det \begin{bmatrix} 1 & b/a \\ c & d \end{bmatrix} \\
&= ? \det \begin{bmatrix} 1 & b/a \\ c-c & d-bc/a \end{bmatrix} \\
&= ? \det \begin{bmatrix} 1 & b/a \\ 0 & d-bc/a \end{bmatrix} \\
&= ? \det \begin{bmatrix} 1 & b/a \\ 0 & 1 \end{bmatrix} \\
&= ? \det \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\
&= ? \det I \\
&= ?
\end{aligned}$$

Row Operations and Determinants (GT1)

Observation 5.1.29 So we may compute the determinant of $\begin{bmatrix} 2 & 4 \\ 2 & 3 \end{bmatrix}$ by using determinant properties to manipulate its rows/columns to reduce the matrix to I:

$$\begin{aligned} \det \begin{bmatrix} 2 & 4 \\ 2 & 3 \end{bmatrix} &= 2 \det \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \\ &= 2 \det \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} \\ &= -2 \det \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \\ &= -2 \det \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= -2 \end{aligned}$$

Or we may use a formula:

$$\det \begin{bmatrix} 2 & 4 \\ 2 & 3 \end{bmatrix} = (2)(3) - (4)(2) = -2$$

Activity 5.1.30 Suppose we have a linear transformation $T\colon \mathbb{R}^2 \to \mathbb{R}^2$. Given some shape S in the plane $\mathbb{R}^2$, we can use to T to transform it into some new shape $T(S)$. Consider the following questions about properties that may or may not be preserved by T.

(a) If S is a straight line segment, explain why $T(S)$ is also a straight line segment.

(b) If S is a straight line segment, does $T(S)$ necessarily have to have the same length as that of S?

(c) If S is a triangle, explain why $T(S)$ is also a triangle.

(d) Continuing as above, do the angles of $T(S)$ necessarily have to be the same as those of S?

5.2 Computing Determinants (GT2)

Learning Outcomes

- Compute the determinant of a 4×4 matrix.

Computing Determinants (GT2)

Activity 5.2.1 Consider the matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.

(a) Use a combination of row and column operations to transform A into the identity matrix. Use this to calculate the determinant of A.

(b) Check your work using the formula for the determinant of a 2×2 matrix.

Computing Determinants (GT2)

Remark 5.2.2 We've seen that row reducing all the way into RREF gives us a method of computing determinants.

However, we learned in Chapter 1 that this can be tedious for large matrices. Thus, we will try to figure out how to turn the determinant of a larger matrix into the determinant of a smaller matrix.

Activity 5.2.3 The following image illustrates the transformation of the unit cube by the matrix $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 0 & 1 \end{bmatrix}$.

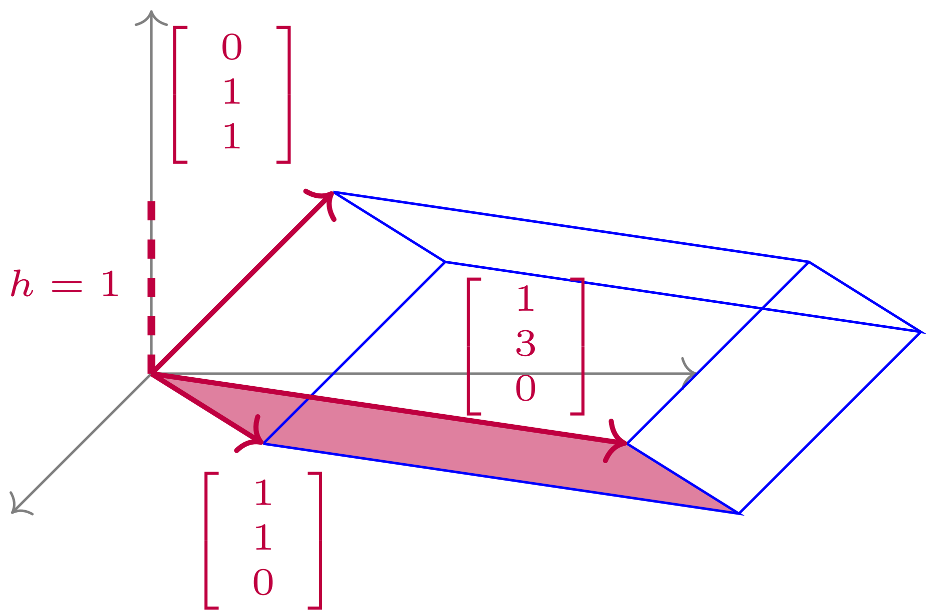


Figure 26 Transformation of the unit cube by the linear transformation.

Recall that for this solid $V = Bh$, where h is the height of the solid and B is the area of its parallelogram base. So what must its volume be?

A. $\det \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix}$

B. $\det \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$

C. $\det \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

D. $\det \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix}$

Computing Determinants (GT2)

Fact 5.2.4 *If row i contains all zeros except for a 1 on the main (upper-left to lower-right) diagonal, then both column and row i may be removed without changing the value of the determinant.*

$$\det\begin{bmatrix} 3 & 2 & -1 & 3 \\ 0 & 1 & 0 & 0 \\ -1 & 4 & 1 & 0 \\ 5 & 0 & 11 & 1 \end{bmatrix} = \det\begin{bmatrix} 3 & -1 & 3 \\ -1 & 1 & 0 \\ 5 & 11 & 1 \end{bmatrix}$$

Since row and column operations affect the determinants in the same way, the same technique works for a column of all zeros except for a 1 on the main diagonal.

$$\det\begin{bmatrix} 3 & 0 & -1 & 5 \\ 2 & 1 & 4 & 0 \\ -1 & 0 & 1 & 11 \\ 3 & 0 & 0 & 1 \end{bmatrix} = \det\begin{bmatrix} 3 & -1 & 5 \\ -1 & 1 & 11 \\ 3 & 0 & 1 \end{bmatrix}$$

Put another way, if you have either a column or row from the identity matrix, you can cancel both the column and row containing the 1.

Computing Determinants (GT2)

Activity 5.2.5 Remove an appropriate row and column of $\det \begin{bmatrix} 1 & 0 & 0 \\ 1 & 5 & 12 \\ 3 & 2 & -1 \end{bmatrix}$ to simplify the determinant to a 2×2 determinant.

Computing Determinants (GT2)

Activity 5.2.6 Simplify $\det \begin{bmatrix} 0 & 3 & -2 \\ 2 & 5 & 12 \\ 0 & 2 & -1 \end{bmatrix}$ to a multiple of a 2×2 determinant by first doing the following:

(a) Factor out a 2 from a column.

(b) Swap rows or columns to put a 1 on the main diagonal.

Computing Determinants (GT2)

Activity 5.2.7 Simplify $\det \begin{bmatrix} 4 & -2 & 2 \\ 3 & 1 & 4 \\ 1 & -1 & 3 \end{bmatrix}$ to a multiple of a 2×2 determinant by first doing the following:

(a) Use row/column operations to create two zeroes in the same row or column.

(b) Factor/swap as needed to get a row/column of all zeroes except a 1 on the main diagonal.

Observation 5.2.8 Using row/column operations, you can introduce zeros and reduce dimension to whittle down the determinant of a large matrix to a determinant of a smaller matrix.

$$\det\begin{bmatrix} 4 & 3 & 0 & 1 \\ 2 & -2 & 4 & 0 \\ -1 & 4 & 1 & 5 \\ 2 & 8 & 0 & 3 \end{bmatrix} = \det\begin{bmatrix} 4 & 3 & 0 & 1 \\ 6 & -18 & 0 & -20 \\ -1 & 4 & 1 & 5 \\ 2 & 8 & 0 & 3 \end{bmatrix} = \det\begin{bmatrix} 4 & 3 & 1 \\ 6 & -18 & -20 \\ 2 & 8 & 3 \end{bmatrix}$$

$$= \cdots = -2\det\begin{bmatrix} 1 & 3 & 4 \\ 0 & 21 & 43 \\ 0 & -1 & -10 \end{bmatrix} = -2\det\begin{bmatrix} 21 & 43 \\ -1 & -10 \end{bmatrix}$$

$$= \cdots = -2\det\begin{bmatrix} -167 & 21 \\ 0 & 1 \end{bmatrix} = -2\det[-167]$$

$$= -2(-167)\det(I) = 334$$

Computing Determinants (GT2)

Activity 5.2.9 Rewrite

$$\det \begin{bmatrix} 2 & 1 & -2 & 1 \\ 3 & 0 & 1 & 4 \\ -2 & 2 & 3 & 0 \\ -2 & 0 & -3 & -3 \end{bmatrix}$$

as a multiple of a determinant of a 3×3 matrix.

Computing Determinants (GT2)

Activity 5.2.10 Compute $\det \begin{bmatrix} 2 & 3 & 5 & 0 \\ 0 & 3 & 2 & 0 \\ 1 & 2 & 0 & 3 \\ -1 & -1 & 2 & 2 \end{bmatrix}$ by using any combination of row/column operations.

Observation 5.2.11 Another option is to take advantage of the fact that the determinant is linear in each row or column. This approach is called **Laplace expansion** or **cofactor expansion**.

For example, since $\begin{bmatrix} 1 & 2 & 4 \end{bmatrix} = 1\begin{bmatrix} 1 & 0 & 0 \end{bmatrix} + 2\begin{bmatrix} 0 & 1 & 0 \end{bmatrix} + 4\begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$,

$$\begin{aligned}
\det\begin{bmatrix} 2 & 3 & 5 \\ -1 & 3 & 5 \\ 1 & 2 & 4 \end{bmatrix} &= 1\det\begin{bmatrix} 2 & 3 & 5 \\ -1 & 3 & 5 \\ 1 & 0 & 0 \end{bmatrix} + 2\det\begin{bmatrix} 2 & 3 & 5 \\ -1 & 3 & 5 \\ 0 & 1 & 0 \end{bmatrix} + 4\det\begin{bmatrix} 2 & 3 & 5 \\ -1 & 3 & 5 \\ 0 & 0 & 1 \end{bmatrix} \\
&= -1\det\begin{bmatrix} 5 & 3 & 2 \\ 5 & 3 & -1 \\ 0 & 0 & 1 \end{bmatrix} - 2\det\begin{bmatrix} 2 & 5 & 3 \\ -1 & 5 & 3 \\ 0 & 0 & 1 \end{bmatrix} + 4\det\begin{bmatrix} 2 & 3 & 5 \\ -1 & 3 & 5 \\ 0 & 0 & 1 \end{bmatrix} \\
&= -\det\begin{bmatrix} 5 & 3 \\ 5 & 3 \end{bmatrix} - 2\det\begin{bmatrix} 2 & 5 \\ -1 & 5 \end{bmatrix} + 4\det\begin{bmatrix} 2 & 3 \\ -1 & 3 \end{bmatrix}
\end{aligned}$$

Computing Determinants (GT2)

Observation 5.2.12 Recall the formula for a 2×2 determinant found in Observation 5.1.29:

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc.$$

There are formulas and algorithms for the determinants of larger matrices, but they can be pretty tedious to use. For example, writing out a formula for a 4×4 determinant would require 24 different terms!

$$\det \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} = a_{11}(a_{22}(a_{33}a_{44} - a_{43}a_{34}) - a_{23}(a_{32}a_{44} - a_{42}a_{34}) + \dots) + \dots$$

Computing Determinants (GT2)

Activity 5.2.13 Based on the previous activities, which technique is easier for computing determinants?

A. Memorizing formulas.

B. Using row/column operations.

C. Laplace expansion.

D. Some other technique.

Computing Determinants (GT2)

Activity 5.2.14 Use your preferred technique to compute $\det \begin{bmatrix} 4 & -3 & 0 & 0 \\ 1 & -3 & 2 & -1 \\ 3 & 2 & 0 & 3 \\ 0 & -3 & 2 & -2 \end{bmatrix}$.

Computing Determinants (GT2)

Activity 5.2.15 A *diagonal* matrix is a matrix that has zeroes in every position except (possibly) the main upper-left to lower-right diagonal. A matrix is *upper (resp. lower) triangular* if every entry below (resp. above) the main diagonal is zero.

(a) Explain why the determinant of a diagonal matrix is always equal to the product of the entries on the main diagonal.

(b) Explain why the determinant of an upper (or lower) triangular matrix is always equal to the product of the entries on the main diagonal.

5.3 Eigenvalues and Characteristic Polynomials (GT3)

Learning Outcomes

- Find the eigenvalues of a 2×2 matrix.

Eigenvalues and Characteristic Polynomials (GT3)

Activity 5.3.1 Let $R\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the transformation given by rotating vectors about the origin through and angle of $45°$, and let $S\colon \mathbb{R}^2 \to \mathbb{R}^2$ denote the transformation that reflects vectors about the line $x_1 = x_2$.

(a) If L is a line, let $R(L)$ denote the line obtained by applying R to it. Are there any lines L for which $R(L)$ is parallel to L?

(b) Now consider the transformation S. Are there any lines L for which $S(L)$ is parallel to L?

Eigenvalues and Characteristic Polynomials (GT3)

Activity 5.3.2 An invertible matrix M and its inverse M^{-1} are given below:

$$M = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad M^{-1} = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}$$

Which of the following is equal to $\det(M)\det(M^{-1})$?

A. -1

B. 0

C. 1

D. 4

Fact 5.3.3 *For every invertible matrix* M*,*

$$\det(M)\det(M^{-1}) = \det(I) = 1$$

so $\det(M^{-1}) = \frac{1}{\det(M)}$.

Furthermore, a square matrix M *is invertible if and only if* $\det(M) \neq 0$*.*

Observation 5.3.4 Consider the linear transformation $A : \mathbb{R}^2 \to \mathbb{R}^2$ given by the matrix $A = \begin{bmatrix} 2 & 2 \\ 0 & 3 \end{bmatrix}$.

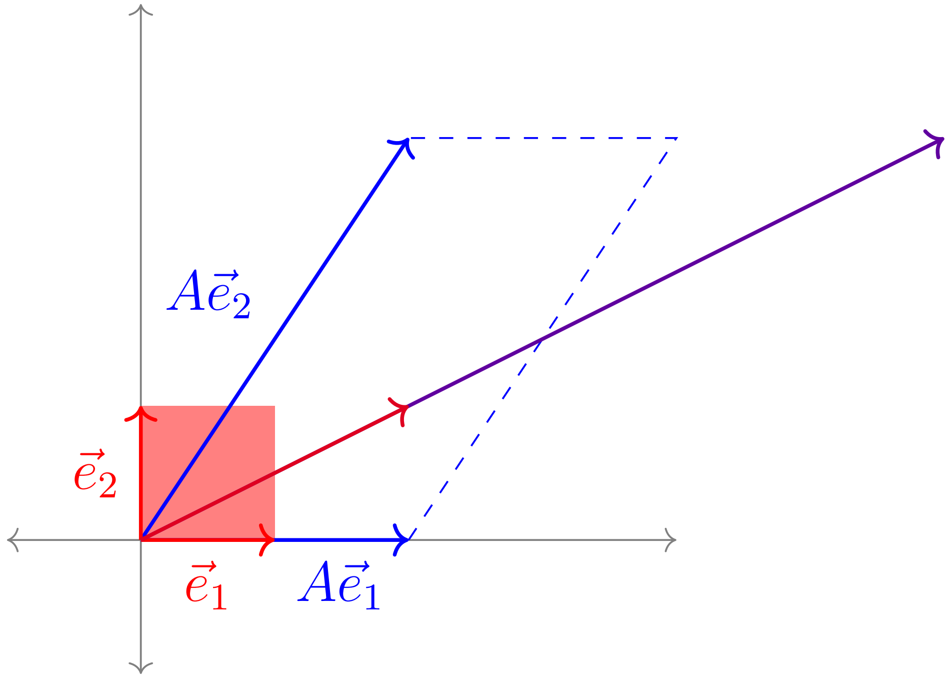


Figure 27 Transformation of the unit square by the linear transformation A

It is easy to see geometrically that

$$A\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

It is less obvious (but easily checked once you find it) that

$$A\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix} = 3\begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

Definition 5.3.5 Let $A \in M_{n,n}$. An **eigenvector** for A is a vector $\vec{x} \in \mathbb{R}^n$ such that $A\vec{x}$ is parallel to $\vec{x}$.

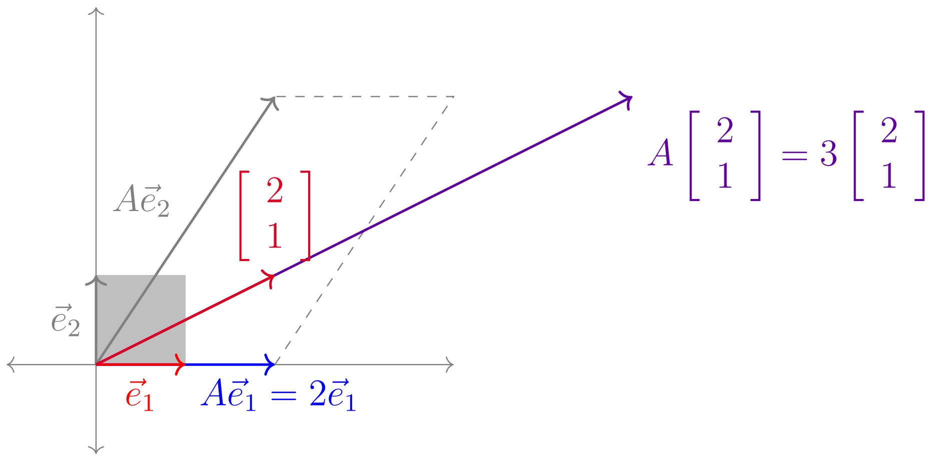
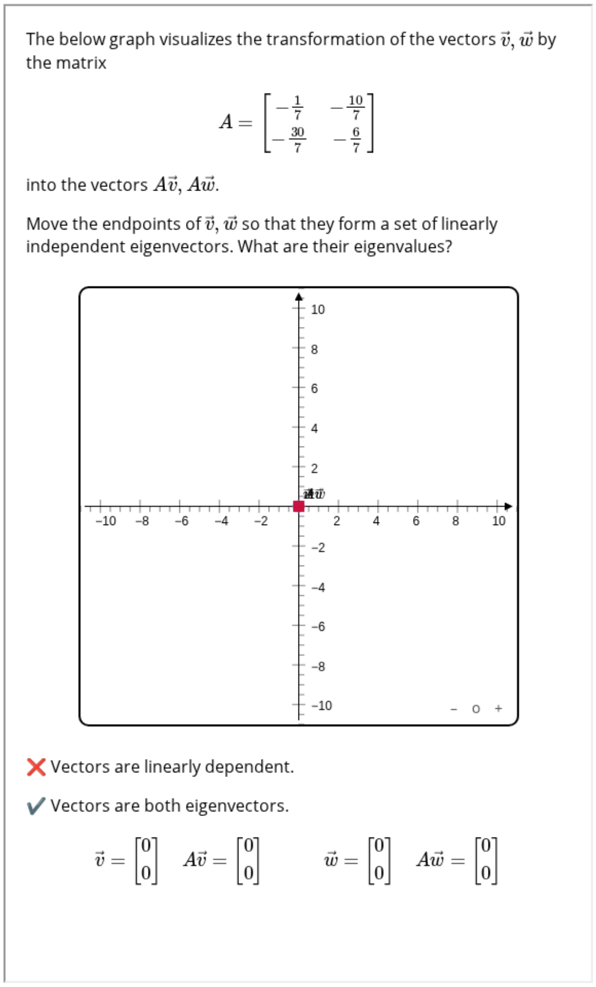


Figure 28 The map A stretches out the eigenvector $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ by a factor of 3 (the corresponding eigenvalue).

In other words, $A\vec{x} = \lambda\vec{x}$ for some scalar λ. If $\vec{x} \neq \vec{0}$, then we say $\vec{x}$ is a **nontrivial eigenvector** and we call this λ an **eigenvalue** of A. ◊

Activity 5.3.6



Standalone
Embed

What are the eigenvalues for this matrix?

A. $1, -2$

B. $-1, 3$

C. $2, -3$

D. $-1, -2$

Eigenvalues and Characteristic Polynomials (GT3)

Activity 5.3.7 Finding the eigenvalues λ that satisfy

$$A\vec{x} = \lambda\vec{x} = \lambda(I\vec{x}) = (\lambda I)\vec{x}$$

for some nontrivial eigenvector $\vec{x}$ is equivalent to finding nonzero solutions for the matrix equation

$$(A - \lambda I)\vec{x} = \vec{0}.$$

(a) If λ is an eigenvalue, and T is the transformation with standard matrix $A - \lambda I$, which of these must contain a non-zero vector?

A. The kernel of T

B. The image of T

C. The domain of T

D. The codomain of T

(b) Therefore, what can we conclude?

A. A is invertible

B. A is not invertible

C. $A - \lambda I$ is invertible

D. $A - \lambda I$ is not invertible

(c) And what else?

A. $\det A = 0$

B. $\det A = 1$

C. $\det(A - \lambda I) = 0$

D. $\det(A - \lambda I) = 1$

Eigenvalues and Characteristic Polynomials (GT3)

Fact 5.3.8 *The eigenvalues* λ *for a matrix* A *are exactly the values that make* $A - \lambda I$ *non-invertible.*

Thus the eigenvalues λ *for a matrix* A *are the solutions to the equation*

$$\det(A - \lambda I) = 0.$$

Eigenvalues and Characteristic Polynomials (GT3)

Definition 5.3.9 The expression $\det(A - \lambda I)$ is called the **characteristic polynomial** of A.

For example, when $A = \begin{bmatrix} 1 & 2 \\ 5 & 4 \end{bmatrix}$, we have

$$A - \lambda I = \begin{bmatrix} 1 & 2 \\ 5 & 4 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 1-\lambda & 2 \\ 5 & 4-\lambda \end{bmatrix}.$$

Thus the characteristic polynomial of A is

$$\det \begin{bmatrix} 1-\lambda & 2 \\ 5 & 4-\lambda \end{bmatrix} = (1-\lambda)(4-\lambda) - (2)(5) = \lambda^2 - 5\lambda - 6$$

and its eigenvalues are the solutions $-1, 6$ to $\lambda^2 - 5\lambda - 6 = 0$. $\diamondsuit$

Eigenvalues and Characteristic Polynomials (GT3)

Activity 5.3.10 Let $A = \begin{bmatrix} 5 & 2 \\ -3 & -2 \end{bmatrix}$.

(a) Compute $\det(A - \lambda I)$ to determine the characteristic polynomial of A.

(b) Set this characteristic polynomial equal to zero and factor to determine the eigenvalues of A.

Eigenvalues and Characteristic Polynomials (GT3)

Activity 5.3.11 Find all the eigenvalues for the matrix $A = \begin{bmatrix} 3 & -3 \\ 2 & -4 \end{bmatrix}$.

Eigenvalues and Characteristic Polynomials (GT3)

Activity 5.3.12 Find all the eigenvalues for the matrix $A = \begin{bmatrix} 1 & -4 \\ 0 & 5 \end{bmatrix}$.

Eigenvalues and Characteristic Polynomials (GT3)

Activity 5.3.13 Find all the eigenvalues for the matrix $A = \begin{bmatrix} 3 & -3 & 1 \\ 0 & -4 & 2 \\ 0 & 0 & 7 \end{bmatrix}$.

Eigenvalues and Characteristic Polynomials (GT3)

Activity 5.3.14 Let $A \in M_{n,n}$ and $\lambda \in \mathbb{R}$. The eigenvalues of A that correspond to λ are the vectors that get stretched by a factor of λ. Consider the following special cases for which we can make more geometric meaning.

(a) What are some other ways we can think of the eigenvectors corresponding to eigenvalue $\lambda = 0$?

(b) What are some other ways we can think of the eigenvectors corresponding to eigenvalue $\lambda = 1$?

(c) What are some other ways we can think of the eigenvectors corresponding to eigenvalue $\lambda = -1$?

(d) How might we interpret a matrix that has no (real) eigenvectors/values?

5.4 Eigenvectors and Eigenspaces (GT4)

Learning Outcomes

- Find a basis for the eigenspace of a 4×4 matrix associated with a given eigenvalue.

Eigenvectors and Eigenspaces (GT4)

Activity 5.4.1 Which of the following vectors is an eigenvector for $A = \begin{bmatrix} 2 & 4 & -1 & -5 \\ 0 & 0 & -3 & -9 \\ 1 & 1 & 0 & 2 \\ -2 & -2 & 3 & 5 \end{bmatrix}$?

A. $\begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix}$

B. $\begin{bmatrix} -3 \\ 3 \\ -2 \\ 1 \end{bmatrix}$

Eigenvectors and Eigenspaces (GT4)

Activity 5.4.2 It's possible to show that -2 is an eigenvalue for $A = \begin{bmatrix} -1 & 4 & -2 \\ 2 & -7 & 9 \\ 3 & 0 & 4 \end{bmatrix}$.

Compute the kernel of the transformation with standard matrix

$$A - (-2)I = \begin{bmatrix} ? & 4 & -2 \\ 2 & ? & 9 \\ 3 & 0 & ? \end{bmatrix}$$

to find all the eigenvectors $\vec{x}$ such that $A\vec{x} = -2\vec{x}$.

Eigenvectors and Eigenspaces (GT4)

Definition 5.4.3 Since the kernel of a linear map is a subspace of $\mathbb{R}^n$, and the kernel obtained from $A - \lambda I$ contains all the eigenvectors associated with λ, we call this kernel the **eigenspace** of A associated with λ. $\Diamond$

Eigenvectors and Eigenspaces (GT4)

Activity 5.4.4 Find a basis for the eigenspace for the matrix $\begin{bmatrix} 0 & 0 & 3 \\ 1 & 0 & -1 \\ 0 & 1 & 3 \end{bmatrix}$ associated with the eigenvalue 3.

Eigenvectors and Eigenspaces (GT4)

Activity 5.4.5 Find a basis for the eigenspace for the matrix $\begin{bmatrix} 5 & -2 & 0 & 4 \\ 6 & -2 & 1 & 5 \\ -2 & 1 & 2 & -3 \\ 4 & 5 & -3 & 6 \end{bmatrix}$ associated with the eigenvalue 1.

Eigenvectors and Eigenspaces (GT4)

Activity 5.4.6 Find a basis for the eigenspace for the matrix $\begin{bmatrix} 4 & 3 & 0 & 0 \\ 3 & 3 & 0 & 0 \\ 0 & 0 & 2 & 5 \\ 0 & 0 & 0 & 2 \end{bmatrix}$ associated with the eigenvalue 2.

Activity 5.4.7 Suppose that $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation with standard matrix A. Further, suppose that we know that $\vec{u} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$ are eigenvectors corresponding to eigenvalues 2 and -3 respectively.

(a) Express the vector $\vec{w} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ as a linear combination of $\vec{u}, \vec{v}$.

(b) Determine $T(\vec{w})$.

Appendix A

Applications

A.1 Civil Engineering: Trusses and Struts

Definition A.1.1 In engineering, a **truss** is a structure designed from several beams of material called **struts**, assembled to behave as a single object.

Figure 29 A simple truss

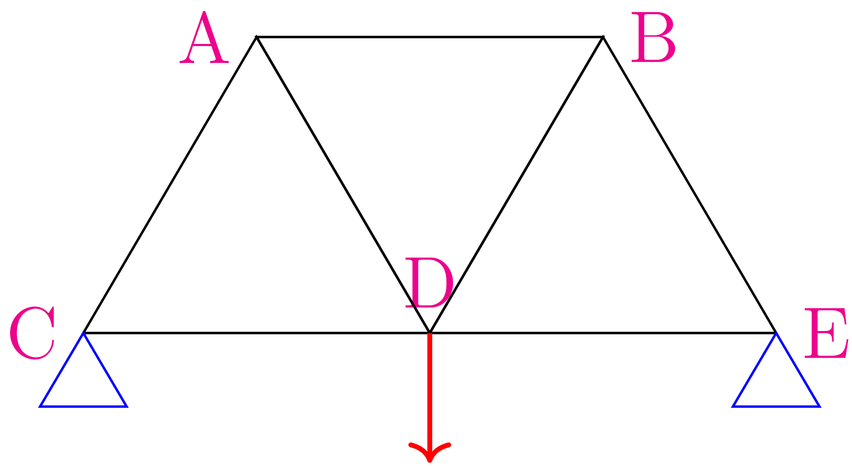


Figure 30 A simple truss

◊

Activity A.1.2 Consider the representation of a simple truss pictured below. All of the seven struts are of equal length, affixed to two anchor points applying a normal force to nodes C and E, and with a $10000N$ load applied to the node given by D.

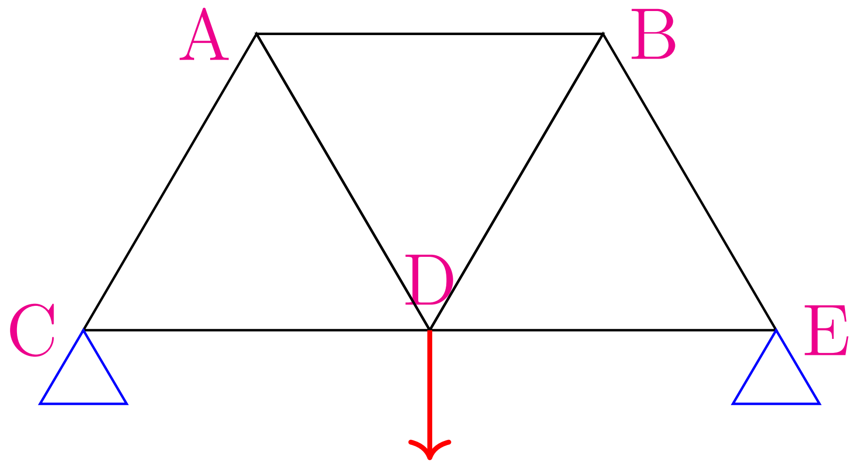


Figure 31 A simple truss

Which of the following must hold for the truss to be stable?

1. All of the struts will experience compression.
2. All of the struts will experience tension.
3. Some of the struts will be compressed, but others will be tensioned.

Observation A.1.3 Since the forces must balance at each node for the truss to be stable, some of the struts will be compressed, while others will be tensioned.

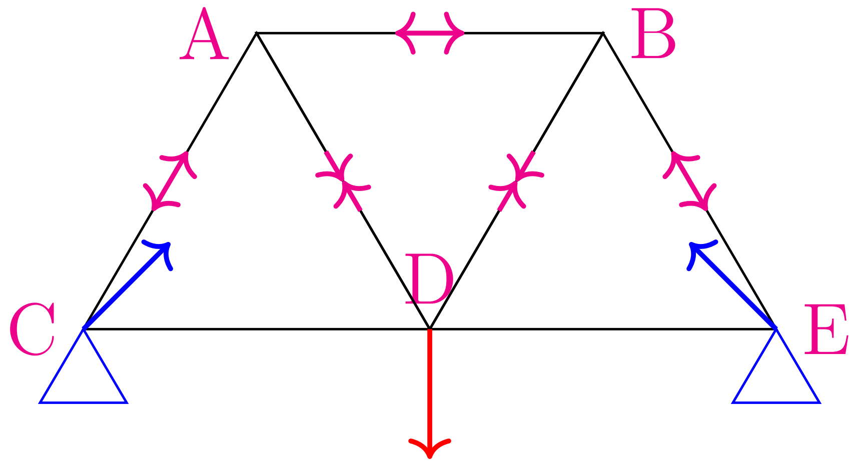


Figure 32 Completed truss

By finding vector equations that must hold at each node, we may determine many of the forces at play.

Remark A.1.4 For example, at the bottom left node, 3 forces are acting.

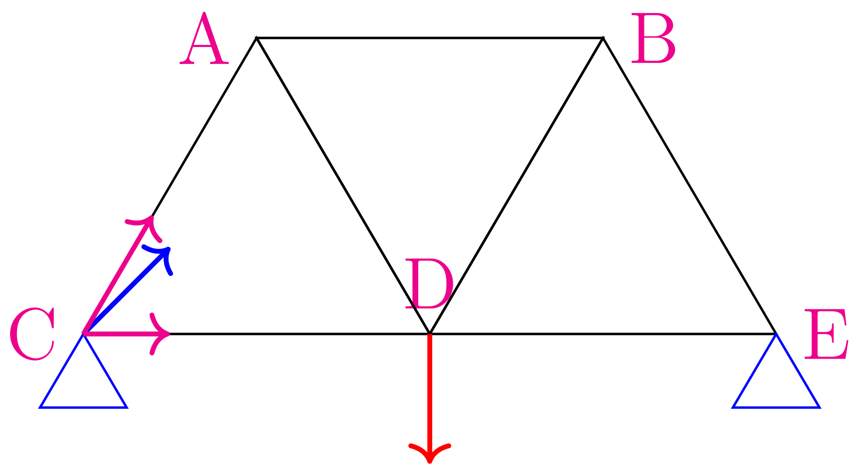


Figure 33 Truss with forces

Let $\vec{F}_{CA}$ be the force on C given by the compression/tension of the strut CA, let $\vec{F}_{CD}$ be defined similarly, and let $\vec{N}_C$ be the normal force of the anchor point on C.

For the truss to be stable, we must have:

$$\vec{F}_{CA} + \vec{F}_{CD} + \vec{N}_C = \vec{0}$$

Activity A.1.5 Using the conventions of the previous remark, and where $\vec{L}$ represents the load vector on node D, find four more vector equations that must be satisfied for each of the other four nodes of the truss.

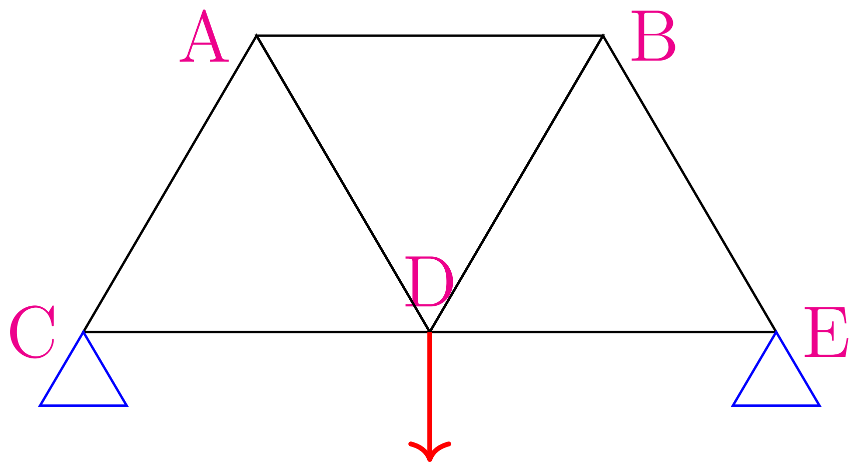


Figure 34 A simple truss

$$A : ?$$

$$B : ?$$

$$C : \vec{F}_{CA} + \vec{F}_{CD} + \vec{N}_C = \vec{0}$$

$$D : ?$$

$$E : ?$$

Remark A.1.6 The five vector equations may be written as follows.

$$A : \vec{F}_{AC} + \vec{F}_{AD} + \vec{F}_{AB} = \vec{0}$$

$$B : \vec{F}_{BA} + \vec{F}_{BD} + \vec{F}_{BE} = \vec{0}$$

$$C : \vec{F}_{CA} + \vec{F}_{CD} + \vec{N}_{C} = \vec{0}$$

$$D : \vec{F}_{DC} + \vec{F}_{DA} + \vec{F}_{DB} + \vec{F}_{DE} + \vec{L} = \vec{0}$$

$$E : \vec{F}_{EB} + \vec{F}_{ED} + \vec{N}_{E} = \vec{0}$$

Observation A.1.7 Each vector has a vertical and horizontal component, so it may be treated as a vector in $\mathbb{R}^2$. Note that $\vec{F}_{CA}$ must have the same magnitude (but opposite direction) as $\vec{F}_{AC}$.

$$\vec{F}_{CA} = x \begin{bmatrix} \cos(60°) \\ \sin(60°) \end{bmatrix} = x \begin{bmatrix} 1/2 \\ \sqrt{3}/2 \end{bmatrix}$$

$$\vec{F}_{AC} = x \begin{bmatrix} \cos(-120°) \\ \sin(-120°) \end{bmatrix} = x \begin{bmatrix} -1/2 \\ -\sqrt{3}/2 \end{bmatrix}$$

Activity A.1.8 To write a linear system that models the truss under consideration with constant load 10000 newtons, how many scalar variables will be required?

- 7: 5 from the nodes, 2 from the anchors
- 9: 7 from the struts, 2 from the anchors
- 11: 7 from the struts, 4 from the anchors
- 12: 7 from the struts, 4 from the anchors, 1 from the load
- 13: 5 from the nodes, 7 from the struts, 1 from the load

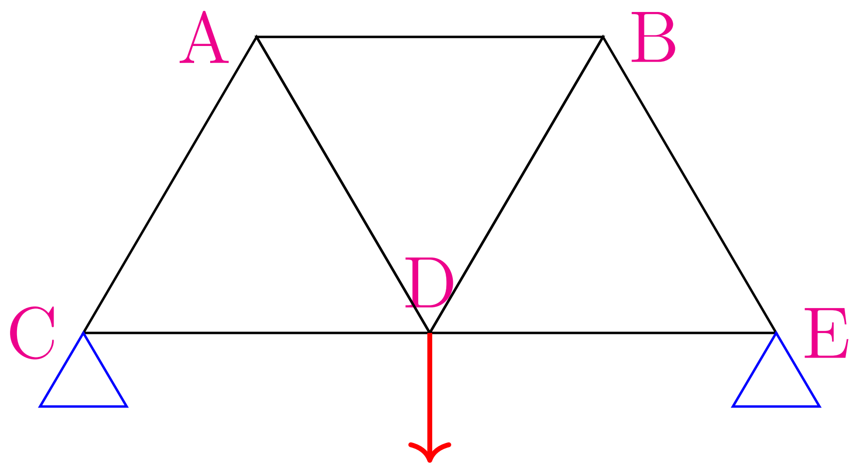


Figure 35 A simple truss

Observation A.1.9 Since the angles for each strut are known, one variable may be used to represent each.

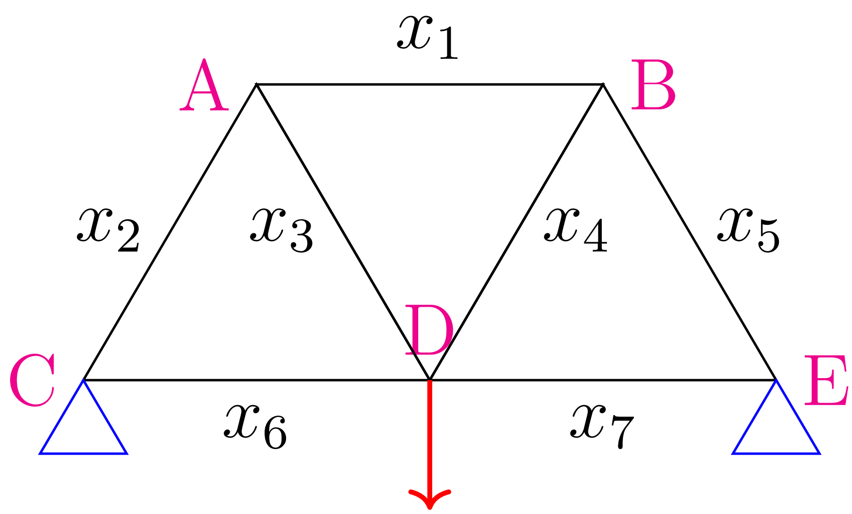


Figure 36 Variables for the truss

For example:

$$\vec{F}_{AB} = -\vec{F}_{BA} = x_1 \begin{bmatrix} \cos(0) \\ \sin(0) \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\vec{F}_{BE} = -\vec{F}_{EB} = x_5 \begin{bmatrix} \cos(-60^\circ) \\ \sin(-60^\circ) \end{bmatrix} = x_5 \begin{bmatrix} 1/2 \\ -\sqrt{3}/2 \end{bmatrix}$$

Observation A.1.10 Since the angle of the normal forces for each anchor point is unknown, two variables may be used to represent each.

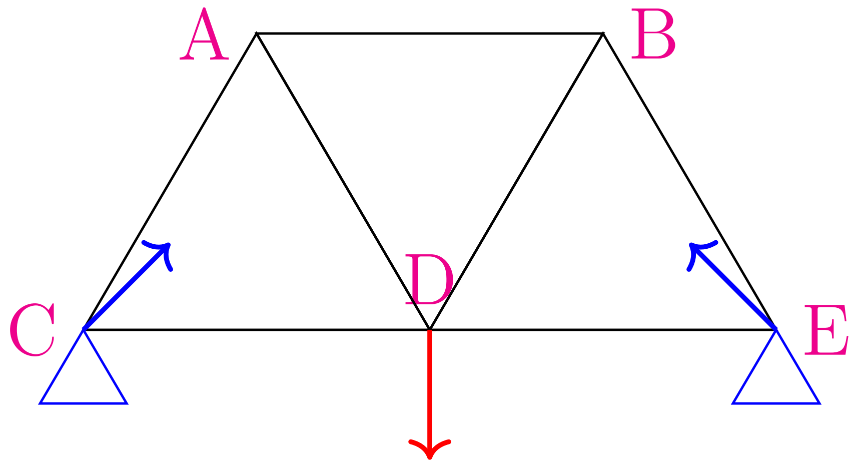


Figure 37 Truss with normal forces

$$\vec{N}_C = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \qquad \vec{N}_D = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

The load vector is constant.

$$\vec{L} = \begin{bmatrix} 0 \\ -10000 \end{bmatrix}$$

Remark A.1.11 Each of the five vector equations found previously represent two linear equations: one for the horizontal component and one for the vertical.

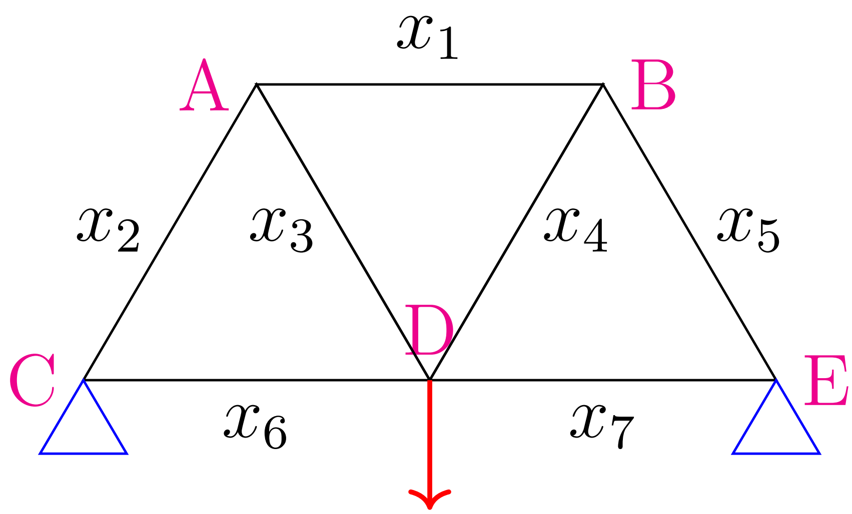


Figure 38 Variables for the truss

$$C : \vec{F}_{CA} + \vec{F}_{CD} + \vec{N}_C = \vec{0}$$

$$\Leftrightarrow x_2 \begin{bmatrix} \cos(60°) \\ \sin(60°) \end{bmatrix} + x_6 \begin{bmatrix} \cos(0°) \\ \sin(0°) \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\sqrt{3}/2 \approx 0.866$

$$\Leftrightarrow x_2 \begin{bmatrix} 0.5 \\ 0.866 \end{bmatrix} + x_6 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + y_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + y_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Activity A.1.12 Expand the vector equation given below using sine and cosine of appropriate angles, then compute each component (approximating $\sqrt{3}/2 \approx 0.866$).

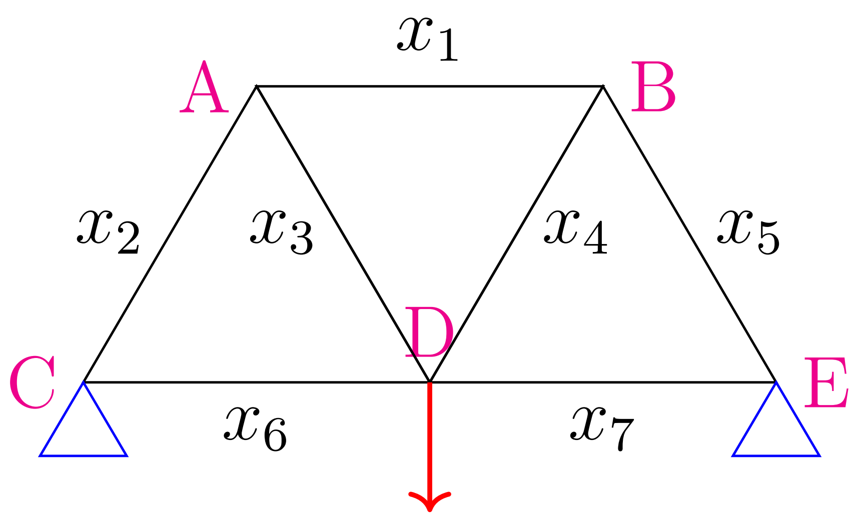


Figure 39 Variables for the truss

$$D : \vec{F}_{DA} + \vec{F}_{DB} + \vec{F}_{DC} + \vec{F}_{DE} = -\vec{L}$$

$$\Leftrightarrow x_3 \begin{bmatrix} \cos(\,?\,) \\ \sin(\,?\,) \end{bmatrix} + x_4 \begin{bmatrix} \cos(\,?\,) \\ \sin(\,?\,) \end{bmatrix} + x_6 \begin{bmatrix} \cos(\,?\,) \\ \sin(\,?\,) \end{bmatrix} + x_7 \begin{bmatrix} \cos(\,?\,) \\ \sin(\,?\,) \end{bmatrix} = \begin{bmatrix} ? \\ ? \end{bmatrix}$$

$$\Leftrightarrow x_3 \begin{bmatrix} ? \\ ? \end{bmatrix} + x_4 \begin{bmatrix} ? \\ ? \end{bmatrix} + x_6 \begin{bmatrix} ? \\ ? \end{bmatrix} + x_7 \begin{bmatrix} ? \\ ? \end{bmatrix} = \begin{bmatrix} ? \\ ? \end{bmatrix}$$

Observation A.1.13 The full augmented matrix given by the ten equations in this linear system is shown below, where the eleven columns correspond to $x_1, \ldots, x_7, y_1, y_2, z_1, z_2$, and the ten rows correspond to the horizontal and vertical components of the forces acting at $A, \ldots, E$.

$$\left[\begin{array}{ccccccccccc|c}
1 & -0.5 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -0.866 & -0.866 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & -0.5 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -0.866 & -0.866 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0.5 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0.866 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & -0.5 & 0.5 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0.866 & 0.866 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 10000 \\
0 & 0 & 0 & 0 & -0.5 & 0 & -1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0.866 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{array}\right]$$

Observation A.1.14 This matrix row-reduces to the following.

$$\sim \left[\begin{array}{ccccccccccc|c} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -5773.7 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -5773.7 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5773.7 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5773.7 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -5773.7 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 2886.8 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 2886.8 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 5000 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 5000 \end{array}\right]$$

Observation A.1.15 Thus we know the truss must satisfy the following conditions.

$$\begin{aligned} x_1 = x_2 = x_5 &= -5882.4 \\ x_3 = x_4 &= 5882.4 \\ x_6 = x_7 &= 2886.8 + z_1 \\ y_1 &= -z_1 \\ y_2 = z_2 &= 5000 \end{aligned}$$

In particular, the negative x_1, x_2, x_5 represent tension (forces pointing into the nodes), and the positive x_3, x_4 represent compression (forces pointing out of the nodes). The vertical normal forces $y_2 + z_2$ counteract the 10000 load.

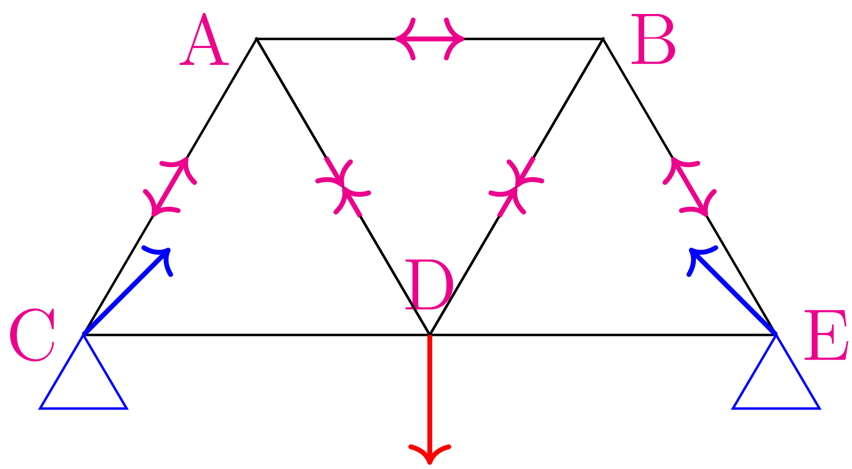


Figure 40 Completed truss

A.2 Computer Science: PageRank

Activity A.2.1 The $2,110,000,000,000 Problem.
In the picture below, each circle represents a webpage, and each arrow represents a link from one page to another.

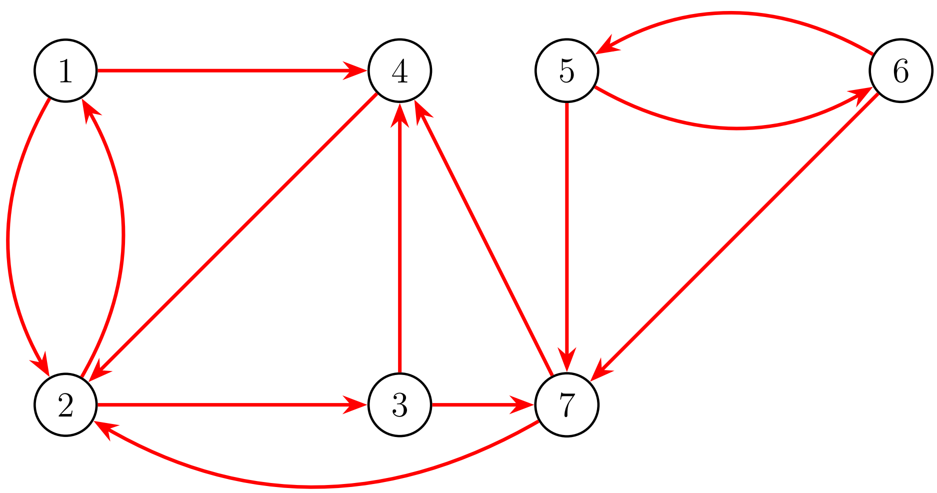


Figure 41 A seven-webpage network

Based on how these pages link to each other, write a list of the 7 webpages in order from most important to least important.

Computer Science: PageRank

Observation A.2.2 The $2,110,000,000,000 Idea. Links are endorsements. That is:

1. A webpage is important if it is linked to (endorsed) by important pages.
2. A webpage distributes its importance equally among all the pages it links to (endorses).

Example A.2.3 Consider this small network with only three pages. Let x_1, x_2, x_3 be the importance of the three pages respectively.

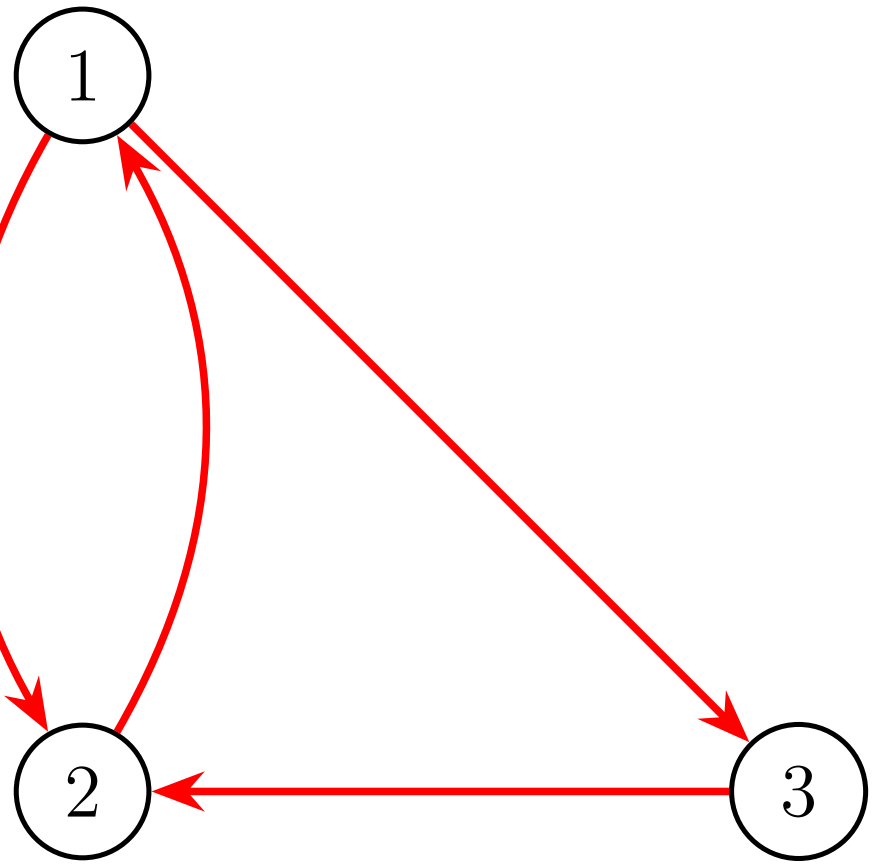


Figure 42 A three-webpage network

1. x_1 splits its endorsement in half between x_2 and x_3
2. x_2 sends all of its endorsement to x_1
3. x_3 sends all of its endorsement to x_2.

This corresponds to the **page rank system**:

$$\begin{aligned} x_2 \quad &= x_1 \\ \frac{1}{2}x_1 \quad + x_3 &= x_2 \\ \frac{1}{2}x_1 \quad &= x_3 \end{aligned}$$

□

Observation A.2.4

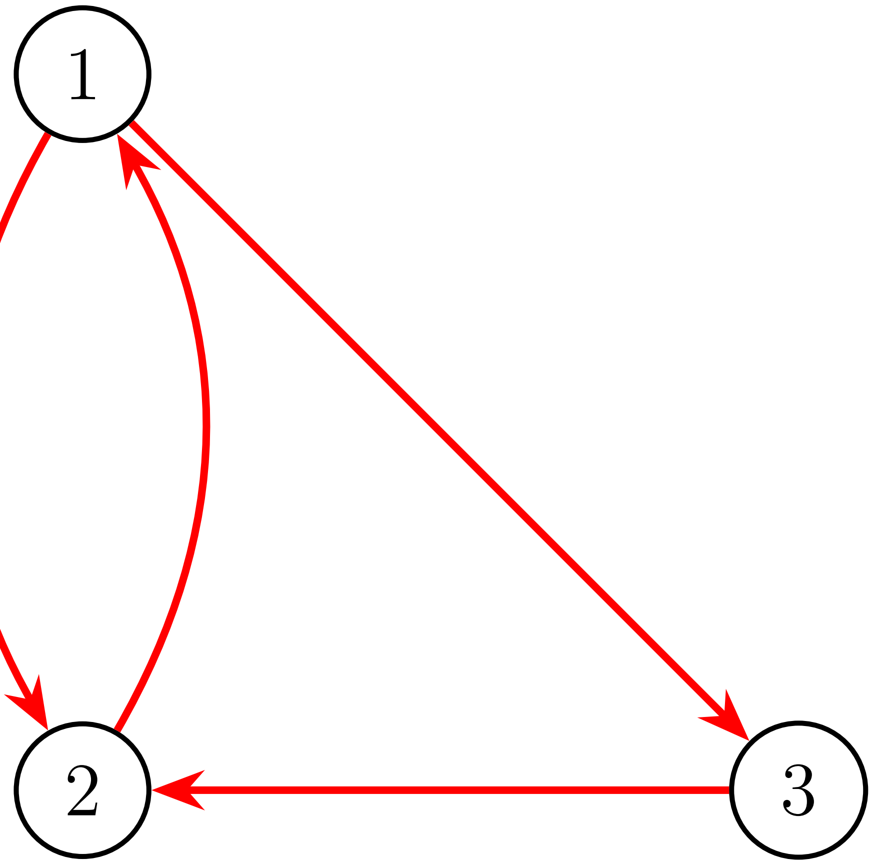


Figure 43 A three-webpage network

$$\begin{aligned} x_2 &= x_1 \\ \frac{1}{2}x_1 \quad + x_3 &= x_2 \\ \frac{1}{2}x_1 \quad &= x_3 \end{aligned} \qquad \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & 1 \\ \frac{1}{2} & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

By writing this linear system in terms of matrix multiplication, we obtain the **page rank matrix** $A = \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & 1 \\ \frac{1}{2} & 0 & 0 \end{bmatrix}$ and page rank vector $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$.

Thus, computing the importance of pages on a network is equivalent to solving the matrix equation $A\vec{x} = 1\vec{x}$.

Activity A.2.5 Thus, our \$2,110,000,000,000 problem is what kind of problem?

$$\begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 1 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

A. An antiderivative problem

B. A bijection problem

C. A cofactoring problem

D. A determinant problem

E. An eigenvector problem

Activity A.2.6 Find a page rank vector $\vec{x}$ satisfying $A\vec{x} = 1\vec{x}$ for the following network's page rank matrix A.

That is, find the eigenspace associated with $\lambda = 1$ for the matrix A, and choose a vector from that eigenspace.

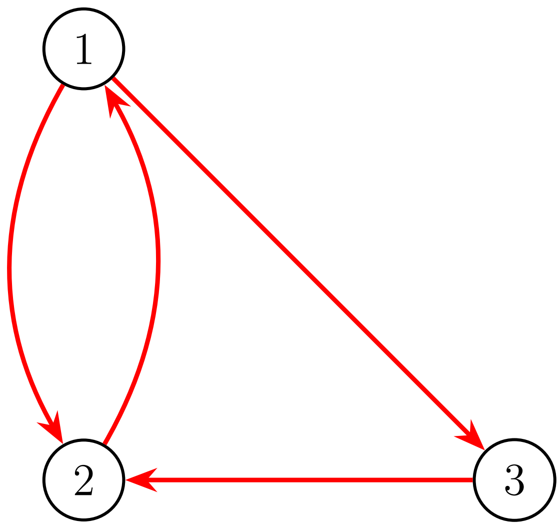


Figure 44 A three-webpage network

$$A = \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & 1 \\ \frac{1}{2} & 0 & 0 \end{bmatrix}$$

Observation A.2.7 Row-reducing $A - I = \begin{bmatrix} -1 & 1 & 0 \\ \frac{1}{2} & -1 & 1 \\ \frac{1}{2} & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$ yields the basic eigenvector $\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$.

Therefore, we may conclude that pages 1 and 2 are equally important, and both pages are twice as important as page 3.

Activity A.2.8 Compute the 7×7 page rank matrix for the following network.

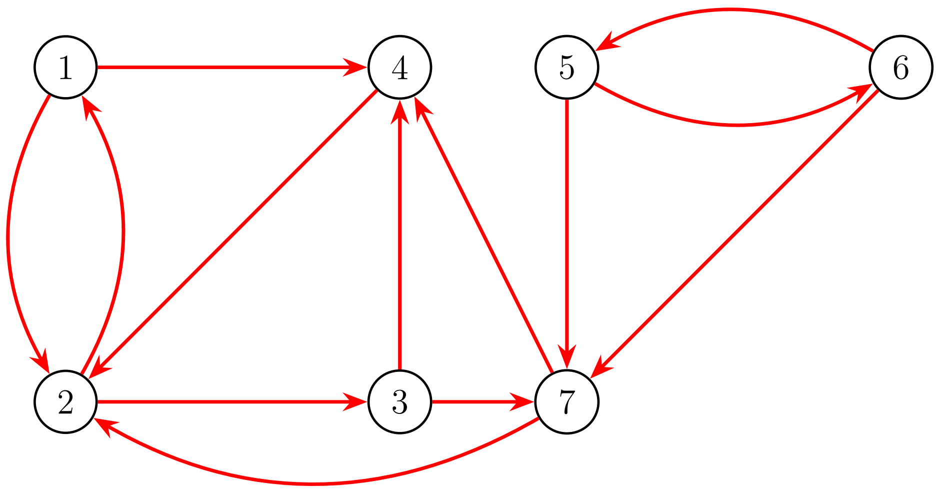


Figure 45 A seven-webpage network

For example, since website 1 distributes its endorsement equally between 2 and 4, the first column is $\begin{bmatrix} 0 \\ \frac{1}{2} \\ 0 \\ \frac{1}{2} \\ 0 \\ 0 \\ 0 \end{bmatrix}$.

Activity A.2.9 Find a page rank vector for the given page rank matrix.

$$A = \begin{bmatrix} 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 1 & 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$$

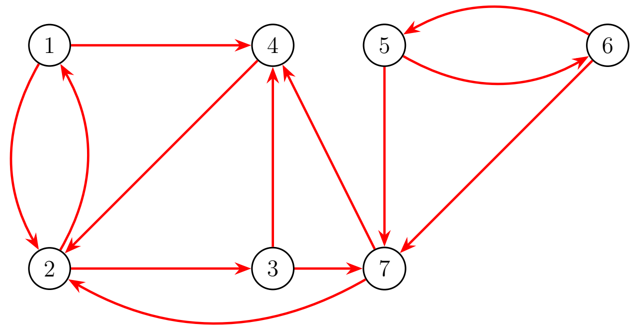


Figure 46 A seven-webpage network

Which webpage is most important?

Observation A.2.10 Since a page rank vector for the network is given by $\vec{x}$, it's reasonable to consider page 2 as the most important page.

$$\vec{x} = \begin{bmatrix} 2 \\ 4 \\ 2 \\ 2.5 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Based on this page rank vector, here is a complete ranking of all seven pages from most important to least important:

$$2, 4, 1, 3, 7, 5, 6$$

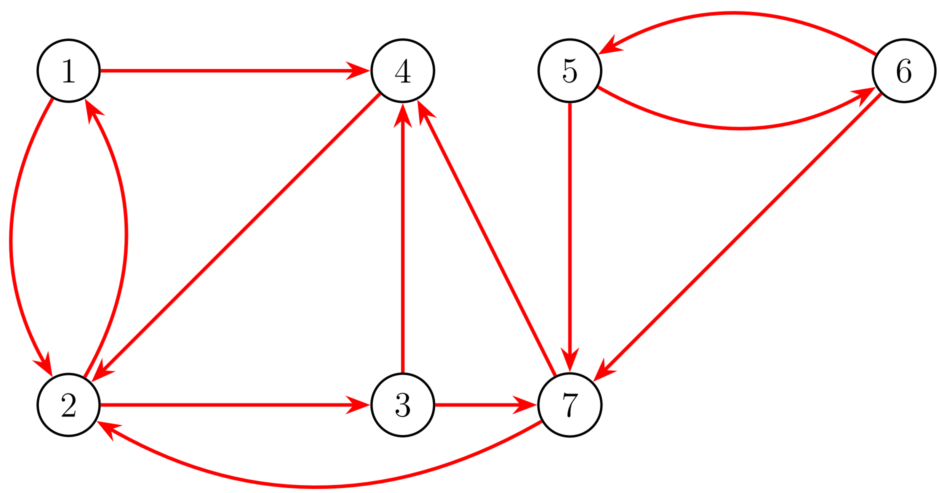


Figure 47 A seven-webpage network

Activity A.2.11 Given the following diagram, use a page rank vector to rank the pages 1 through 7 in order from most important to least important.

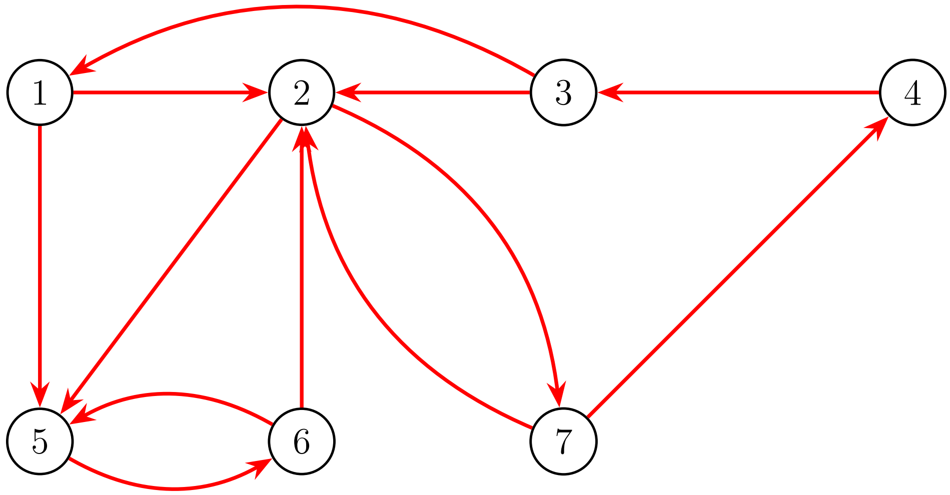


Figure 48 Another seven-webpage network

A.3 Geology: Phases and Components

Definition A.3.1 In geology, a **phase** is any physically separable material in the system, such as various minerals or liquids.

A **component** is a chemical compound necessary to make up the phases; these are usually oxides such as Calcium Oxide (CaO) or Silicon Dioxide (SiO_2).

In a typical application, a geologist knows how to build each phase from the components, and is interested in determining reactions among the different phases. ◊

Geology: Phases and Components

Observation A.3.2 Consider the 3 components

$$\vec{c}_1 = \mathrm{CaO} \quad \vec{c}_2 = \mathrm{MgO} \quad \text{and } \vec{c}_3 = \mathrm{SiO_2}$$

and the 5 phases:

$$\vec{p}_1 = \mathrm{Ca_3MgSi_2O_8} \qquad \vec{p}_2 = \mathrm{CaMgSiO_4} \qquad \vec{p}_3 = \mathrm{CaSiO_3}$$
$$\vec{p}_4 = \mathrm{CaMgSi_2O_6} \qquad \vec{p}_5 = \mathrm{Ca_2MgSi_2O_7}$$

Geologists already know (or can easily deduce) that

$$\vec{p}_1 = 3\vec{c}_1 + \vec{c}_2 + 2\vec{c}_3 \qquad \vec{p}_2 = \vec{c}_1 + \vec{c}_2 + \vec{c}_3 \qquad \vec{p}_3 = \vec{c}_1 + 0\vec{c}_2 + \vec{c}_3$$
$$\vec{p}_4 = \vec{c}_1 + \vec{c}_2 + 2\vec{c}_3 \qquad \vec{p}_5 = 2\vec{c}_1 + \vec{c}_2 + 2\vec{c}_3$$

since, for example:

$$\vec{c}_1 + \vec{c}_3 = \mathrm{CaO} + \mathrm{SiO_2} = \mathrm{CaSiO_3} = \vec{p}_3$$

Geology: Phases and Components

Activity A.3.3 To study this vector space, each of the three components $\vec{c}_1, \vec{c}_2, \vec{c}_3$ may be considered as the three components of a Euclidean vector.

$$\vec{p}_1 = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}, \vec{p}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \vec{p}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \vec{p}_4 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \vec{p}_5 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}.$$

Determine if the set of phases is linearly dependent or linearly independent.

Geology: Phases and Components

Activity A.3.4 Geologists are interested in knowing all the possible chemical reactions among the 5 phases:

$$\vec{p}_1 = Ca_3MgSi_2O_8 = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} \quad \vec{p}_2 = CaMgSiO_4 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \vec{p}_3 = CaSiO_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$\vec{p}_4 = CaMgSi_2O_6 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \quad \vec{p}_5 = Ca_2MgSi_2O_7 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}.$$

That is, they want to find numbers x_1, x_2, x_3, x_4, x_5 such that

$$x_1\vec{p}_1 + x_2\vec{p}_2 + x_3\vec{p}_3 + x_4\vec{p}_4 + x_5\vec{p}_5 = 0.$$

(a) Set up a system of equations equivalent to this vector equation.

(b) Find a basis for its solution space.

(c) Interpret each basis vector as a vector equation and a chemical equation.

Activity A.3.5 We found two basis vectors $\begin{bmatrix} 1 \\ -2 \\ -2 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ -1 \\ -1 \\ 0 \\ 1 \end{bmatrix}$, corresponding to the vector and chemical equations

$$\begin{aligned} 2\vec{p}_2 + 2\vec{p}_3 &= \vec{p}_1 + \vec{p}_4 & \qquad 2\mathrm{CaMgSiO_4} + 2\mathrm{CaSiO_3} &= \mathrm{Ca_3MgSi_2O_8} + \mathrm{CaMgSi_2O_6} \\ \vec{p}_2 + \vec{p}_3 &= \vec{p}_5 & \qquad \mathrm{CaMgSiO_4} + \mathrm{CaSiO_3} &= \mathrm{Ca_2MgSi_2O_7} \end{aligned}$$

Combine the basis vectors to produce a chemical equation among the five phases that does not involve $\vec{p}_2 = \mathrm{CaMgSiO_4}$.

Appendix B

Appendix

B.1 Sample Exercises with Solutions

Here we model one exercise and solution for each learning objective. Your solutions should not look identical to those shown below, but these solutions can give you an idea of the level of detail required for a complete solution.

Example B.1.1 LE1. Consider the vector equation

$$x_1 \begin{bmatrix} 4 \\ -3 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 4 \\ -3 \\ 3 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix} + x_4 \begin{bmatrix} 18 \\ -7 \\ 15 \end{bmatrix} = \begin{bmatrix} -11 \\ 5 \\ -9 \end{bmatrix}$$

(a) Write a corresponding system of equations.

Solution.

$$\begin{matrix} 4\,x_1 & + & 4\,x_2 & + & 3\,x_3 & + & 18\,x_4 & = & -11 \\ -3\,x_1 & - & 3\,x_2 & + & x_3 & - & 7\,x_4 & = & 5 \\ 3\,x_1 & + & 3\,x_2 & + & 3\,x_3 & + & 15\,x_4 & = & -9 \end{matrix}$$

(b) Write a corresponding augmented matrix.

Solution.

$$\left[\begin{array}{cccc|c} 4 & 4 & 3 & 18 & -11 \\ -3 & -3 & 1 & -7 & 5 \\ 3 & 3 & 3 & 15 & -9 \end{array}\right]$$

□

Sample Exercises with Solutions

Example B.1.2 LE2.

(a) For each of the following matrices, explain why it is not in reduced row echelon form.

(i)

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 & -2 \\ 1 & 5 & 0 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Solution. $A = \begin{bmatrix} 0 & 0 & \boxed{1} & 0 & -2 \\ \boxed{1} & 5 & 0 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ is not in reduced row echelon form because the pivots are not descending to the right.

(ii)

$$B = \begin{bmatrix} 1 & -6 & 3 & 0 & -1 \\ 0 & 0 & 0 & 7 & 14 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Solution. $B = \begin{bmatrix} \boxed{1} & -6 & 3 & 0 & -1 \\ 0 & 0 & 0 & \boxed{7} & 14 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ is not in reduced row echelon form because a leading term has a value besides 1.

(iii)

$$C = \begin{bmatrix} 1 & 7 & -4 & 1 & 12 \\ 0 & 1 & -1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Solution. $C = \begin{bmatrix} \boxed{1} & 7 & -4 & 1 & 12 \\ 0 & \boxed{1} & -1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ is not in reduced row echelon form because there is a non-zero entry above or below a pivot.

(b) Use technology to find

$$\text{RREF} \begin{bmatrix} 4 & 4 & 3 & 18 & -11 \\ -3 & -3 & 1 & -7 & 5 \\ 3 & 3 & 3 & 15 & -9 \end{bmatrix}$$

Solution.

$$\begin{bmatrix} 4 & 4 & 3 & 18 & -11 \\ -3 & -3 & 1 & -7 & 5 \\ 3 & 3 & 3 & 15 & -9 \end{bmatrix} \sim \begin{bmatrix} \boxed{1} & 1 & 0 & 3 & -2 \\ 0 & 0 & \boxed{1} & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(c) Show step by step how to find

$$\text{RREF} \begin{bmatrix} 4 & 4 & 3 & 18 & -11 \\ -3 & -3 & 1 & -7 & 5 \\ 3 & 3 & 3 & 15 & -9 \end{bmatrix}$$

Solution.

$$\begin{bmatrix} 4 & 4 & 3 & 18 & -11 \\ -3 & -3 & 1 & -7 & 5 \\ 3 & 3 & 3 & 15 & -9 \end{bmatrix} \overset{R_1+R_2\to R_1}{\sim} \begin{bmatrix} \boxed{1} & 1 & 4 & 11 & -6 \\ -3 & -3 & 1 & -7 & 5 \\ 3 & 3 & 3 & 15 & -9 \end{bmatrix}$$

$$\overset{\substack{R_2+3R_1\to R_2 \\ R_3-3R_1\to R_3}}{\sim} \begin{bmatrix} \boxed{1} & 1 & 4 & 11 & -6 \\ 0 & 0 & 13 & 26 & -13 \\ 0 & 0 & -9 & -18 & 9 \end{bmatrix}$$

$$\overset{\substack{\frac{1}{13}R_2\to R_2 \\ \frac{1}{9}R_3\to R_3}}{\sim} \begin{bmatrix} \boxed{1} & 1 & 4 & 11 & -6 \\ 0 & 0 & \boxed{1} & 2 & -1 \\ 0 & 0 & -1 & -2 & 1 \end{bmatrix}$$

$$\overset{\substack{R_1-4R_2\to R_1 \\ R_3+R_1\to R_3}}{\sim} \begin{bmatrix} \boxed{1} & 1 & 0 & 3 & -2 \\ 0 & 0 & \boxed{1} & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

□

Sample Exercises with Solutions

Example B.1.3 LE3. Consider each of the following systems of linear equations or vector equations.

(a)

$$\begin{array}{rcrcrcr} x_1 & - & x_2 & + & x_3 & = & 4 \\ & & x_2 & - & 2\,x_3 & = & -1 \\ & & x_2 & - & 2\,x_3 & = & -3 \\ x_1 & + & 2\,x_2 & - & 5\,x_3 & = & 0 \end{array}$$

(i) Explain and demonstrate how to find a simpler linear system that has the same solution set.

Solution. The given linear system is represented by this augmented matrix, which row reduces as follows:

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 4 \\ 0 & 1 & -2 & -1 \\ 0 & 1 & -2 & -3 \\ 1 & 2 & -5 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The RREF matrix then yields the following simplified linear system with the same solution set:

$$\begin{array}{rcrcr} x_1 & & - & x_3 & = & 0 \\ & x_2 & - & 2\,x_3 & = & 0 \\ & & & 0 & = & 1 \\ & & & 0 & = & 0 \end{array}$$

(ii) Explain whether this solution set has no solutions, one solution, or infinitely-many solutions. If the set is finite, describe it using set notation.

Solution. Because $0 = 1$ is false, the solution set has no solutions. This means the solution set is $\emptyset$.

(b)

$$\begin{array}{rcrcrcr} -x_1 & + & x_2 & + & x_3 & = & 2 \\ -3\,x_1 & + & x_2 & - & 4\,x_3 & = & -9 \\ 2\,x_1 & - & x_2 & + & 2\,x_3 & = & 5 \\ -6\,x_1 & + & 3\,x_2 & - & 4\,x_3 & = & -9 \end{array}$$

(i) Explain and demonstrate how to find a simpler linear system that has the same solution set.

Solution. The given linear system is represented by this augmented matrix, which row reduces as follows:

$$\left[\begin{array}{ccc|c} -1 & 1 & 1 & 2 \\ -3 & 1 & -4 & -9 \\ 2 & -1 & 2 & 5 \\ -6 & 3 & -4 & -9 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The RREF matrix then yields the following simplified linear system with the same solution set:

$$\begin{array}{rcrcrcr} x_1 & & & & & = & -2 \\ & & x_2 & & & = & -3 \\ & & & & x_3 & = & 3 \\ & & & & 0 & = & 0 \end{array}$$

(ii) Explain whether this solution set has no solutions, one solution, or infinitely-many solutions. If the set is finite, describe it using set notation.

Solution. Since each variable is equal to a fixed value, there exists only one solution. The solution set is $\left\{ \begin{bmatrix} -2 \\ -3 \\ 3 \end{bmatrix} \right\}$.

(c)

$$\begin{array}{rcrcrcr} x_1 & + & 4\,x_2 & - & 14\,x_3 & = & 11 \\ -x_1 & - & 3\,x_2 & + & 11\,x_3 & = & -8 \\ -x_1 & - & 3\,x_2 & + & 11\,x_3 & = & -8 \\ & & 3\,x_2 & - & 9\,x_3 & = & 9 \end{array}$$

(i) Explain and demonstrate how to find a simpler linear system that has the same solution set.

Solution. The given linear system is represented by this augmented matrix, which row reduces as follows:

$$\left[\begin{array}{ccc|c} 1 & 4 & -14 & 11 \\ -1 & -3 & 11 & -8 \\ -1 & -3 & 11 & -8 \\ 0 & 3 & -9 & 9 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -2 & -1 \\ 0 & 1 & -3 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The RREF matrix then yields the following simplified linear system with the same solution set:

$$\begin{array}{rcrcrcr} x_1 & & & - & 2\,x_3 & = & -1 \\ & & x_2 & - & 3\,x_3 & = & 3 \\ & & & & 0 & = & 0 \\ & & & & 0 & = & 0 \end{array}$$

(ii) Explain whether this solution set has no solutions, one solution, or infinitely-many solutions. If the set is finite, describe it using set notation.

Solution. Since the simplified system obtained from the RREF calculation has no contradictions, but has equations with multiple variables, the solution set has infinitely-many solutions.

□

Sample Exercises with Solutions

Example B.1.4 LE4. Consider the following vector equation.

$$x_1 \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 0 \\ -2 \\ -2 \end{bmatrix} + x_3 \begin{bmatrix} -5 \\ 1 \\ -5 \\ -2 \end{bmatrix} + x_4 \begin{bmatrix} 13 \\ -2 \\ 13 \\ 7 \end{bmatrix} + x_5 \begin{bmatrix} -14 \\ 3 \\ -14 \\ -5 \end{bmatrix} = \begin{bmatrix} 18 \\ -3 \\ 18 \\ 9 \end{bmatrix}$$

(a) Explain how to find a simpler linear system that has the same solution set.

Solution. The given linear system is represented by this augmented matrix, which row reduces as follows:

$$\left[\begin{array}{ccccc|c} 1 & -2 & -5 & 13 & -14 & 18 \\ 0 & 0 & 1 & -2 & 3 & -3 \\ 1 & -2 & -5 & 13 & -14 & 18 \\ 1 & -2 & -2 & 7 & -5 & 9 \end{array}\right] \sim \left[\begin{array}{ccccc|c} 1 & -2 & 0 & 3 & 1 & 3 \\ 0 & 0 & 1 & -2 & 3 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The RREF matrix then yields the following simplified linear system with the same solution set:

$$\begin{array}{rcrcrcrcrcr} x_1 & - & 2\,x_2 & & & + & 3\,x_4 & + & x_5 & = & 3 \\ & & & & x_3 & - & 2\,x_4 & + & 3\,x_5 & = & -3 \\ & & & & & & & & 0 & = & 0 \\ & & & & & & & & 0 & = & 0 \end{array}$$

(b) Explain how to describe this solution set using set notation.

Solution. We can assign free variables for each of the non-pivot columns: $x_2 = a$, $x_4 = b$, and $x_5 = c$:

$$\begin{array}{rcrcrcrcrcr} x_1 & - & 2\,a & & & + & 3\,b & + & c & = & 3 \\ & & & & x_3 & - & 2\,b & + & 3\,c & = & -3 \end{array}$$

Then we may solve for the bound variables x_1 and x_3:

$$x_1 = 2a - 3b - c + 3$$

$$x_3 = 2b - 3c - 3$$

Therefore, the solution set is $\left\{ \begin{bmatrix} 2\,a - 3\,b - c + 3 \\ a \\ 2\,b - 3\,c - 3 \\ b \\ c \end{bmatrix} \middle| \, a, b, c \in \mathbb{R} \right\}$.

□

Sample Exercises with Solutions

Example B.1.5 EV1. Consider each of these claims about a vector equation.

(a) $\begin{bmatrix} -13 \\ 3 \\ -15 \end{bmatrix}$ is a linear combination of the vectors $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}$, $\begin{bmatrix} 3 \\ 0 \\ 3 \end{bmatrix}$, and $\begin{bmatrix} -5 \\ 1 \\ -5 \end{bmatrix}$.

(i) Write a statement involving the solutions of a vector equation that's equivalent to this claim.

Solution. The vector equation

$$x_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ 0 \\ 3 \end{bmatrix} + x_4 \begin{bmatrix} -5 \\ 1 \\ -5 \end{bmatrix} = \begin{bmatrix} -13 \\ 3 \\ -15 \end{bmatrix}$$

has at least one solution.

(ii) Determine if the statement you wrote is true or false.

Solution. $\text{RREF} \left[\begin{array}{cccc|c} 1 & 2 & 3 & -5 & -13 \\ 0 & 0 & 0 & 1 & 3 \\ 1 & 2 & 3 & -5 & -15 \end{array}\right] = \left[\begin{array}{cccc|c} 1 & 2 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$

The bottom row requires $0 = 1$. Therefore the vector equation has no solutions, so $\begin{bmatrix} -13 \\ 3 \\ -15 \end{bmatrix}$ is not a linear combination.

(iii) If your statement was true, explain and demonstrate how to construct a specific linear combination of $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}$, $\begin{bmatrix} 3 \\ 0 \\ 3 \end{bmatrix}$, and $\begin{bmatrix} -5 \\ 1 \\ -5 \end{bmatrix}$ that equals $\begin{bmatrix} -13 \\ 3 \\ -15 \end{bmatrix}$.

Solution. N/A

(b) $\begin{bmatrix} -13 \\ 3 \\ -13 \end{bmatrix}$ is a linear combination of the vectors $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}$, $\begin{bmatrix} 3 \\ 0 \\ 3 \end{bmatrix}$, and $\begin{bmatrix} -5 \\ 1 \\ -5 \end{bmatrix}$.

(i) Write a statement involving the solutions of a vector equation that's equivalent to this claim.

Solution. The vector equation

$$x_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ 0 \\ 3 \end{bmatrix} + x_4 \begin{bmatrix} -5 \\ 1 \\ -5 \end{bmatrix} = \begin{bmatrix} -13 \\ 3 \\ -13 \end{bmatrix}$$

has at least one solution.

(ii) Determine if the statement you wrote is true or false.

Solution. $\text{RREF} \left[\begin{array}{cccc|c} 1 & 2 & 3 & -5 & -13 \\ 0 & 0 & 0 & 1 & 3 \\ 1 & 2 & 3 & -5 & -13 \end{array}\right] = \left[\begin{array}{cccc|c} 1 & 2 & 3 & 0 & 2 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

No row requires $0 = 1$. Therefore vector equation has at least one solution, so $\begin{bmatrix} -13 \\ 3 \\ -13 \end{bmatrix}$ is a linear combination.

(iii) If your statement was true, explain and demonstrate how to construct a specific linear combination of $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}$, $\begin{bmatrix} 3 \\ 0 \\ 3 \end{bmatrix}$, and $\begin{bmatrix} -5 \\ 1 \\ -5 \end{bmatrix}$ that equals $\begin{bmatrix} -13 \\ 3 \\ -13 \end{bmatrix}$.

Solution 1. By setting the free variables $x_2 = 0$ and $x_3 = 0$, we obtain the equations $x_1 = 2$ and $x_4 = 3$. Therefore we may construct

$$2 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + 3 \begin{bmatrix} -5 \\ 1 \\ -5 \end{bmatrix} = \begin{bmatrix} -13 \\ 3 \\ -13 \end{bmatrix}.$$

Solution 2. By trial and error, we may find that

$$1 \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix} + 3 \begin{bmatrix} -5 \\ 1 \\ -5 \end{bmatrix} = \begin{bmatrix} 2 - 15 \\ 0 + 3 \\ 2 - 15 \end{bmatrix} = \begin{bmatrix} -13 \\ 3 \\ -13 \end{bmatrix}.$$

□

Example B.1.6 EV2.

(a) Consider the set of vectors

$$\left\{\begin{bmatrix}-1\\1\\0\\2\end{bmatrix},\begin{bmatrix}0\\-1\\1\\-3\end{bmatrix},\begin{bmatrix}2\\-2\\1\\-2\end{bmatrix},\begin{bmatrix}-3\\-1\\4\\-5\end{bmatrix},\begin{bmatrix}7\\8\\-15\\28\end{bmatrix}\right\}$$

(i) Write a statement involving the solutions of a vector equation that's equivalent to this claim: "The set of vectors spans $\mathbb{R}^4$."

Solution. The vector equation

$$x_1\begin{bmatrix}-1\\1\\0\\2\end{bmatrix}+x_2\begin{bmatrix}0\\-1\\1\\-3\end{bmatrix}+x_3\begin{bmatrix}2\\-2\\1\\-2\end{bmatrix}+x_4\begin{bmatrix}-3\\-1\\4\\-5\end{bmatrix}+x_5\begin{bmatrix}7\\8\\-15\\28\end{bmatrix}=\vec{w}$$

has at least one solution for every $\vec{w}\in\mathbb{R}^4$.

(ii) Explain and demonstrate how to determine whether or not this statement is true.

Solution. Note that $\operatorname{RREF}\begin{bmatrix}-1&0&2&-3&7\\1&-1&-2&-1&8\\0&1&1&4&-15\\2&-3&-2&-5&28\end{bmatrix}=\begin{bmatrix}1&0&0&0&2\\0&1&0&0&-3\\0&0&1&0&0\\0&0&0&1&-3\end{bmatrix}$ has no row of zeros that would allow a $0=1$ contradiction.

Therefore the vector equation has solutions for every $\vec{w}$, and thus the set of vectors *does* span $\mathbb{R}^4$.

(b) Consider the set of vectors

$$\left\{\begin{bmatrix}1\\1\\-2\\0\end{bmatrix},\begin{bmatrix}0\\1\\-2\\3\end{bmatrix},\begin{bmatrix}-4\\-2\\5\\5\end{bmatrix},\begin{bmatrix}12\\3\\-9\\-24\end{bmatrix}\right\}$$

(i) Write a statement involving the solutions of a vector equation that's equivalent to this claim: "The set of vectors spans $\mathbb{R}^4$."

Solution. The vector equation

$$x_1\begin{bmatrix}1\\1\\-2\\0\end{bmatrix}+x_2\begin{bmatrix}0\\1\\-2\\3\end{bmatrix}+x_3\begin{bmatrix}-4\\-2\\5\\5\end{bmatrix}+x_4\begin{bmatrix}12\\3\\-9\\-24\end{bmatrix}=\vec{w}$$

has at least one solution for every $\vec{w}\in\mathbb{R}^4$.

(ii) Explain and demonstrate how to determine whether or not this statement is true.

Solution. Note that $\operatorname{RREF}\begin{bmatrix} 1 & 0 & -4 & 12 \\ 1 & 1 & -2 & 3 \\ -2 & -2 & 5 & -9 \\ 0 & 3 & 5 & -24 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ has a row of zeros that makes a $0 = 1$ contradiction possible.

Therefore the vector equation will not have solutions for every $\vec{w}$, and thus the set of vectors does *not* span $\mathbb{R}^4$.

(c) Consider the set of vectors

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -5 \\ 1 \\ -5 \\ -2 \end{bmatrix}, \begin{bmatrix} 12 \\ -2 \\ 12 \\ 6 \end{bmatrix} \right\}$$

(i) Write a statement involving the solutions of a vector equation that's equivalent to this claim: "The set of vectors spans $\mathbb{R}^4$."

Solution. The vector equation

$$x_1 \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} -5 \\ 1 \\ -5 \\ -2 \end{bmatrix} + x_3 \begin{bmatrix} 12 \\ -2 \\ 12 \\ 6 \end{bmatrix} = \vec{w}$$

has at least one solution for every $\vec{w} \in \mathbb{R}^4$.

(ii) Explain and demonstrate how to determine whether or not this statement is true.

Solution 1. Note that $\operatorname{RREF}\begin{bmatrix} 1 & -5 & 12 \\ 0 & 1 & -2 \\ 1 & -5 & 12 \\ 1 & -2 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ has a row of zeros that makes a $0 = 1$ contradiction possible.

Therefore the vector equation will not have solutions for every $\vec{w}$, and thus the set of vectors does *not* span $\mathbb{R}^4$.

Solution 2. It takes at least 4 vectors to span $\mathbb{R}^4$, so the equation cannot always have solutions and the set cannot span.

□

Sample Exercises with Solutions

Example B.1.7 EV3. Answer the following questions about Euclidean subspaces.

(a) Consider the following subsets of Euclidean space $\mathbb{R}^4$ defined by

$$U = \left\{ \left[\begin{array}{c} x \\ y \\ z \\ w \end{array} \right] \middle| y^2 - 7\,z^2 = x \right\} \quad \text{and} \quad W = \left\{ \left[\begin{array}{c} x \\ y \\ z \\ w \end{array} \right] \middle| -5\,w - 7\,x - y = -7\,z \right\}$$

Without writing a proof, explain why only one of these subsets is likely to be a subspace.

Solution. W appears to be a subspace as its equation is a linear combination of variables and constant scalars, and U is likely not due to its equation having squared terms.

(b) Consider the following subset of Euclidean space $\mathbb{R}^3$

$$Q = \left\{ \left[\begin{array}{c} x \\ y \\ z \end{array} \right] \middle| 0 = 5\,y^2 - 5\,x + 3\,z \right\}$$

Prove that Q *is not* a subspace.

Solution. Note that $\left[\begin{array}{c} 0 \\ 3 \\ -15 \end{array} \right]$ belongs to Q, since

$$5(3)^2 - 5(0) + 3(-15) = 45 - 45 = 0,$$

but $2 \left[\begin{array}{c} 0 \\ 3 \\ -15 \end{array} \right] = \left[\begin{array}{c} 0 \\ 6 \\ -30 \end{array} \right]$ does not, since

$$5(6)^2 - 5(0) + 3(-30) = 180 - 90 = 90 \neq 0.$$

(c) Consider the following subset of Euclidean space $\mathbb{R}^3$

$$R = \left\{ \left[\begin{array}{c} x \\ y \\ z \end{array} \right] \middle| 5\,x - 5\,y = -4\,z \right\}$$

Prove that R *is* a subspace.

Solution.

- First, note that $\left[\begin{array}{c} 0 \\ 0 \\ 0 \end{array} \right] \in R$ since $5(0) - 5(0) = 0$ and $-4(0) = 0$ as well.

Sample Exercises with Solutions

- Let $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \in R$ so that $5\,x - 5\,y = -4\,z$, and let $\begin{bmatrix} a \\ b \\ c \end{bmatrix} \in R$ so that $5\,a - 5\,b = -4\,c$. We may then compute

$$\begin{aligned} 5(x+a) - 5(y+b) &= 5x + 5a - 5y - 5b \\ &= (5x - 5y) + (5a - 5b) \\ &= (-4z) + (-4c) \\ &= -4(z+c) \end{aligned}$$

  So $5(x+a) - 5(y+b) = -4(z+c)$ and therefore $\begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} x+a \\ y+b \\ z+c \end{bmatrix} \in R$, showing R is closed under addition.

- Let $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \in R$ so that $5\,x - 5\,y = -4\,z$, and let $k \in \mathbb{R}$ be a scalar. We may then compute

$$\begin{aligned} 5x - 5y &= -4z \\ \Rightarrow k[5x - 5y] &= k[-4z] \\ \Rightarrow 5kx - 5ky &= -4kz \\ \Rightarrow 5(kx) - 5(ky) &= -4(kz) \end{aligned}$$

  and therefore $k \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} kx \\ ky \\ kz \end{bmatrix} \in R$, showing R is closed under scalar multiplication.

$\square$

Sample Exercises with Solutions

Example B.1.8 EV4.

(a) Consider the set of vectors

$$\left\{\begin{bmatrix}-3\\3\\3\\-4\end{bmatrix},\begin{bmatrix}9\\-9\\-9\\12\end{bmatrix},\begin{bmatrix}1\\-2\\-3\\2\end{bmatrix},\begin{bmatrix}-11\\13\\15\\-16\end{bmatrix}\right\}.$$

(i) Write a statement involving the solutions of a vector equation that's equivalent to this claim: "The set of vectors is linearly independent."

Solution. The vector equation

$$x_1\begin{bmatrix}-3\\3\\3\\-4\end{bmatrix}+x_2\begin{bmatrix}9\\-9\\-9\\12\end{bmatrix}+x_3\begin{bmatrix}1\\-2\\-3\\2\end{bmatrix}+x_4\begin{bmatrix}-11\\13\\15\\-16\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

has exactly one solution: $\begin{bmatrix}0\\0\\0\\0\end{bmatrix}$.

(ii) Explain and demonstrate how to determine whether or not this statement is true.

Solution. $\operatorname{RREF}\begin{bmatrix}-3&9&1&-11\\3&-9&-2&13\\3&-9&-3&15\\-4&12&2&-16\end{bmatrix}=\begin{bmatrix}1&-3&0&3\\0&0&1&-2\\0&0&0&0\\0&0&0&0\end{bmatrix}$

Since the RREF matrix has two non-pivot columns (the second and fourth), the solution set has free variables and thus there are more than one solution. This means the set is linearly *dependent*.

(b) Consider the set of vectors

$$\left\{\begin{bmatrix}1\\-1\\0\\1\end{bmatrix},\begin{bmatrix}-3\\4\\3\\1\end{bmatrix},\begin{bmatrix}-5\\5\\1\\-2\end{bmatrix},\begin{bmatrix}-15\\17\\8\\-1\end{bmatrix},\begin{bmatrix}2\\-5\\-7\\-4\end{bmatrix}\right\}.$$

(i) Write a statement involving the solutions of a vector equation that's equivalent to this claim: "The set of vectors is linearly independent."

Solution. The vector equation

$$x_1\begin{bmatrix}1\\-1\\0\\1\end{bmatrix}+x_2\begin{bmatrix}-3\\4\\3\\1\end{bmatrix}+x_3\begin{bmatrix}-5\\5\\1\\-2\end{bmatrix}+x_4\begin{bmatrix}-15\\17\\8\\-1\end{bmatrix}+x_5\begin{bmatrix}2\\-5\\-7\\-4\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

has exactly one solution: $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$.

(ii) Explain and demonstrate how to determine whether or not this statement is true.

Solution 1. RREF $\begin{bmatrix} 1 & -3 & -5 & -15 & 2 \\ -1 & 4 & 5 & 17 & -5 \\ 0 & 3 & 1 & 8 & -7 \\ 1 & 1 & -2 & -1 & -4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 1 & 3 \\ 0 & 1 & 0 & 2 & -3 \\ 0 & 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

Since the RREF matrix has two non-pivot columns (the fourth and fifth), the solution set has free variables and thus there are more than one solution. This means the set is linearly *dependent*.

Solution 2. Since these vectors are from $\mathbb{R}^4$ and there are more than 4 vectors, the equation must have infinitely-many solutions and the set must be linearly *dependent*.

(c) Consider the set of vectors

$$\left\{ \begin{bmatrix} -3 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ -5 \\ -4 \\ -2 \end{bmatrix} \right\}$$

(i) Write a statement involving the solutions of a vector equation that's equivalent to this claim: "The set of vectors is linearly independent."

Solution. The vector equation

$$x_1 \begin{bmatrix} -3 \\ 2 \\ 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 5 \\ -5 \\ -4 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

has exactly one solution: $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

(ii) Explain and demonstrate how to determine whether or not this statement is true.

Solution. RREF $\begin{bmatrix} -3 & -2 & 5 \\ 2 & 1 & -5 \\ 1 & 0 & -4 \\ 0 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

Since the RREF matrix has all pivot columns, the solution set lacks free variables and thus there is exactly one solution. This means the set is linearly *independent*.

□

Sample Exercises with Solutions

Example B.1.9 EV5.

(a) Consider the set of vectors

$$\left\{\begin{bmatrix}1\\-2\\0\\1\end{bmatrix},\begin{bmatrix}2\\-4\\0\\2\end{bmatrix},\begin{bmatrix}3\\-6\\0\\3\end{bmatrix},\begin{bmatrix}5\\-5\\-2\\4\end{bmatrix}\right\}$$

(i) Write a statement involving the solutions of a vector equation that's equivalent to this claim: "The set of vectors is a basis for"

Solution. The vector equation

$$x_1\begin{bmatrix}1\\-2\\0\\1\end{bmatrix}+x_2\begin{bmatrix}2\\-4\\0\\2\end{bmatrix}+x_3\begin{bmatrix}3\\-6\\0\\3\end{bmatrix}+x_4\begin{bmatrix}5\\-5\\-2\\4\end{bmatrix}=\vec{w}$$

has exactly one solution for every $\vec{w}\in\mathbb{R}^4$.

(ii) Explain and demonstrate how to determine whether or not this statement is true.

Solution 1. Since $\text{RREF}\begin{bmatrix}1&2&3&5\\-2&-4&-6&-5\\0&0&0&-2\\1&2&3&4\end{bmatrix}=\begin{bmatrix}1&2&3&0\\0&0&0&1\\0&0&0&0\\0&0&0&0\end{bmatrix}$, we see from the zero row that there are some vectors $\vec{w}$ for which the equation is not true, so the set fails to span and therefore fails to be a basis.

Solution 2. Since $\text{RREF}\begin{bmatrix}1&2&3&5\\-2&-4&-6&-5\\0&0&0&-2\\1&2&3&4\end{bmatrix}=\begin{bmatrix}1&2&3&0\\0&0&0&1\\0&0&0&0\\0&0&0&0\end{bmatrix}$, we see from the non-pivot column that there are some vectors $\vec{w}$ for which the equation has infinitely-many solutions, so the set is linearly dependent and therefore fails to be a basis.

(b) Consider the set of vectors

$$\left\{\begin{bmatrix}1\\3\\4\\-4\end{bmatrix},\begin{bmatrix}-1\\-3\\-4\\4\end{bmatrix},\begin{bmatrix}0\\1\\3\\-3\end{bmatrix}\right\}$$

(i) Write a statement involving the solutions of a vector equation that's equivalent to this claim: "The set of vectors is a basis for"

Solution. The vector equation

$$x_1\begin{bmatrix}1\\3\\4\\-4\end{bmatrix}+x_2\begin{bmatrix}-1\\-3\\-4\\4\end{bmatrix}+x_3\begin{bmatrix}0\\1\\3\\-3\end{bmatrix}=\vec{w}$$

has exactly one solution for every $\vec{w} \in \mathbb{R}^4$.

(ii) Explain and demonstrate how to determine whether or not this statement is true.

Solution 1. Since RREF $\begin{bmatrix} 1 & -1 & 0 \\ 3 & -3 & 1 \\ 4 & -4 & 3 \\ -4 & 4 & -3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ we see from the zero row that there are some vectors $\vec{w}$ for which the equation is not true, so the set fails to span and therefore fails to be a basis.

Solution 2. The set has only three vectors, so the set cannot span and there must be vectors for which the equation has no solutions. Therefore the set is not a basis.

(c) Consider the set of vectors

$$\left\{ \begin{bmatrix} 3 \\ 2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ -1 \\ 0 \\ -2 \end{bmatrix} \right\}$$

(i) Write a statement involving the solutions of a vector equation that's equivalent to this claim: "The set of vectors is a basis for"

Solution. The vector equation

$$x_1 \begin{bmatrix} 3 \\ 2 \\ -1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ -1 \\ 0 \\ -1 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -4 \\ -1 \\ 0 \\ -2 \end{bmatrix} = \vec{w}$$

has exactly one solution for every $\vec{w} \in \mathbb{R}^4$.

(ii) Explain and demonstrate how to determine whether or not this statement is true.

Solution. Since RREF $\begin{bmatrix} 3 & -2 & -2 & -4 \\ 2 & -1 & -1 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ we see the equation always has exactly one solution (each row and column has a pivot). Therefore the set is spanning and linearly independent, and therefore the set is a basis.

$\square$

Example B.1.10 EV6. Consider the subspace

$$W = \operatorname{span}\left\{\begin{bmatrix}1\\-3\\-1\\2\end{bmatrix}, \begin{bmatrix}1\\0\\1\\-2\end{bmatrix}, \begin{bmatrix}3\\-6\\-1\\2\end{bmatrix}, \begin{bmatrix}1\\6\\1\\-1\end{bmatrix}, \begin{bmatrix}2\\3\\0\\1\end{bmatrix}\right\}.$$

(a) Explain how to find a basis of W.

Solution. Observe that

$$\operatorname{RREF}\begin{bmatrix}1 & 1 & 3 & 1 & 2\\-3 & 0 & -6 & 6 & 3\\-1 & 1 & -1 & 1 & 0\\2 & -2 & 2 & -1 & 1\end{bmatrix} = \begin{bmatrix}1 & 0 & 2 & 0 & 1\\0 & 1 & 1 & 0 & 0\\0 & 0 & 0 & 1 & 1\\0 & 0 & 0 & 0 & 0\end{bmatrix}$$

If we remove the vectors yielding non-pivot columns, the resulting set will span the same vectors while being linearly independent. Therefore

$$\left\{\begin{bmatrix}1\\-3\\-1\\2\end{bmatrix}, \begin{bmatrix}1\\0\\1\\-2\end{bmatrix}, \begin{bmatrix}1\\6\\1\\-1\end{bmatrix}\right\}$$

is a basis of W.

(b) Explain how to find the dimension of W.

Solution. Since this (and thus every other) basis has three vectors in it, the dimension of W is 3.

□

Example B.1.11 EV7. Consider the homogeneous system of equations

$$\begin{aligned} x_1 + x_2 + 3x_3 + x_4 + 2x_5 &= 0 \\ -3x_1 - 6x_3 + 6x_4 + 3x_5 &= 0 \\ -x_1 + x_2 - x_3 + x_4 &= 0 \\ 2x_1 - 2x_2 + 2x_3 - x_4 + x_5 &= 0 \end{aligned}$$

(a) Find the solution space of the system.

Solution. Observe that

$$\text{RREF}\left[\begin{array}{ccccc|c} 1 & 1 & 3 & 1 & 2 & 0 \\ -3 & 0 & -6 & 6 & 3 & 0 \\ -1 & 1 & -1 & 1 & 0 & 0 \\ 2 & -2 & 2 & -1 & 1 & 0 \end{array}\right] = \left[\begin{array}{ccccc|c} 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Letting $x_3 = a$ and $x_5 = b$ (since those correspond to the non-pivot columns), this is equivalent to the system

$$\begin{aligned} x_1 + 2x_3 + x_5 &= 0 \\ x_2 + x_3 &= 0 \\ x_3 &= a \\ x_4 + x_5 &= 0 \\ x_5 &= b \end{aligned}$$

Thus, the solution set is

$$\left\{ \begin{bmatrix} -2a - b \\ -a \\ a \\ -b \\ b \end{bmatrix} \middle| \, a, b \in \mathbb{R} \right\}.$$

(b) Find a basis of the solution space.

Solution. Since we can write

$$\begin{bmatrix} -2a - b \\ -a \\ a \\ -b \\ b \end{bmatrix} = a \begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} -1 \\ 0 \\ 0 \\ -1 \\ 1 \end{bmatrix},$$

a basis for the solution space is

$$\left\{ \begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\}.$$

□

Sample Exercises with Solutions

Example B.1.12 AT1. Answer the following questions about transformations.

1. Consider the following maps of Euclidean vectors $P : \mathbb{R}^3 \to \mathbb{R}^3$ and $Q : \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$P\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 3\,x - y + z \\ 2\,x - 2\,y + 4\,z \\ -2\,x - 2\,y - 3\,z \end{bmatrix} \quad \text{and} \quad Q\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} y - 2\,z \\ -3\,x - 4\,y + 12\,z \\ 5\,xy + 3\,z \end{bmatrix}.$$

   Without writing a proof, explain why only one of these maps is likely to be a linear transformation.

2. Consider the following map of Euclidean vectors $S : \mathbb{R}^2 \to \mathbb{R}^2$

$$S\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x + 2\,y \\ -3\,xy \end{bmatrix}.$$

   Prove that S *is not* a linear transformation.

3. Consider the following map of Euclidean vectors $T : \mathbb{R}^2 \to \mathbb{R}^2$

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -4\,x - 5\,y \\ 2\,x - 4\,y \end{bmatrix}.$$

   Prove that T *is* a linear transformation.

Solution.

1. A linear map between Euclidean spaces must consist of linear polynomials in each component. All three components of P are linear so P is likely to be linear; however, the third component of Q contains the nonlinear term xy, so Q is unlikely to be linear.

2. We need to show *either* that S fails to preserve either vector addition *or* that S fails to preserve scalar multiplication.

   We can test if S preserves scalar multiplication for $c = -1$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix} \in \mathbb{R}^2$. We compute

$$S\left(-1\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = S\left(\begin{bmatrix} -1 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} -1 - 2 \\ -3 \end{bmatrix} = \begin{bmatrix} -3 \\ -3 \end{bmatrix}$$

   whereas

$$-1S\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = -1\begin{bmatrix} 1 + 2 \\ -3 \end{bmatrix} = \begin{bmatrix} -3 \\ 3 \end{bmatrix}.$$

   Since $\begin{bmatrix} -3 \\ -3 \end{bmatrix} \neq \begin{bmatrix} -3 \\ 3 \end{bmatrix}$, S fails to preserve scalar multiplication and thus cannot be a linear transformation.

   Alternatively, we could test preservation of vector addition for $\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \end{bmatrix} \in \mathbb{R}^2$.

$$S\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 2 \end{bmatrix}\right) = S\left(\begin{bmatrix} 3 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 3 + 2(3) \\ -3(3)(3) \end{bmatrix} = \begin{bmatrix} 9 \\ -27 \end{bmatrix}$$

whereas

$$S\left(\begin{bmatrix}1\\1\end{bmatrix}\right)+S\left(\begin{bmatrix}2\\2\end{bmatrix}\right)=\begin{bmatrix}1+2(1)\\-3(1)(1)\end{bmatrix}+\begin{bmatrix}2+2(2)\\-3(2)(2)\end{bmatrix}=\begin{bmatrix}3\\-3\end{bmatrix}+\begin{bmatrix}6\\-12\end{bmatrix}=\begin{bmatrix}9\\-15\end{bmatrix}.$$

Since $\begin{bmatrix}9\\-27\end{bmatrix}\neq\begin{bmatrix}9\\-15\end{bmatrix}$, S fails to preserve addition and thus cannot be a linear transformation.

3. We need to show that T preserves *both* vector addition *and* that T preserves scalar multiplication.

First, let us take two vectors $\begin{bmatrix}x_1\\y_1\end{bmatrix},\begin{bmatrix}x_2\\y_2\end{bmatrix}\in\mathbb{R}^2$ and compute

$$T\left(\begin{bmatrix}x_1\\y_1\end{bmatrix}+\begin{bmatrix}x_2\\y_2\end{bmatrix}\right)=T\left(\begin{bmatrix}x_1+x_2\\y_1+y_2\end{bmatrix}\right)=\begin{bmatrix}-4(x_1+x_2)-5(y_1+y_2)\\2(x_1+x_2)-4(y_1+y_2)\end{bmatrix}$$

and

$$T\left(\begin{bmatrix}x_1\\y_1\end{bmatrix}\right)+T\left(\begin{bmatrix}x_2\\y_2\end{bmatrix}\right)=\begin{bmatrix}-4x_1-5y_1\\2x_1-4y_1\end{bmatrix}+\begin{bmatrix}-4x_2-5y_2\\2x_2-4y_2\end{bmatrix}=\begin{bmatrix}-4x_1-5y_1-4x_2-5y_2\\2x_1-4y_1+2x_2-4y_2\end{bmatrix}$$

So we see that $T\left(\begin{bmatrix}x_1\\y_1\end{bmatrix}+\begin{bmatrix}x_2\\y_2\end{bmatrix}\right)=T\left(\begin{bmatrix}x_1\\y_1\end{bmatrix}\right)+T\left(\begin{bmatrix}x_2\\y_2\end{bmatrix}\right)$, so T preserves addition.

Now, take a scalar $c\in\mathbb{R}$ and a vector $\begin{bmatrix}x\\y\end{bmatrix}\in\mathbb{R}^2$, and compute

$$T\left(c\begin{bmatrix}x\\y\end{bmatrix}\right)=T\left(\begin{bmatrix}cx\\cy\end{bmatrix}\right)=\begin{bmatrix}-4cx-5cy\\2cx-4cy\end{bmatrix}$$

and

$$cT\left(\begin{bmatrix}x\\y\end{bmatrix}\right)=c\begin{bmatrix}-4x-5y\\2x-4y\end{bmatrix}=\begin{bmatrix}-4cx-5cy\\2cx-4cy\end{bmatrix}.$$

We see that $T\left(c\begin{bmatrix}x\\y\end{bmatrix}\right)=cT\left(\begin{bmatrix}x\\y\end{bmatrix}\right)$, so T preserves scalar multiplication.

Since T preserves both addition and scalar multiplication, we have proven that T is a linear transformation.

$\square$

Sample Exercises with Solutions

Example B.1.13 AT2.

1. Find the standard matrix for the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^4$ given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} -x+y \\ -x+3y-z \\ 7x+y+3z \\ 0 \end{bmatrix}.$$

2. Let $S : \mathbb{R}^4 \to \mathbb{R}^3$ be the linear transformation given by the standard matrix

$$\begin{bmatrix} 2 & 3 & 4 & 1 \\ 0 & 1 & -1 & -1 \\ 3 & -2 & -2 & 4 \end{bmatrix}.$$

Compute $S\left(\begin{bmatrix} -2 \\ 1 \\ 3 \\ 2 \end{bmatrix}\right)$.

Solution.

1. Since

$$T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ -1 \\ 7 \\ 0 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 3 \\ 1 \\ 0 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ -1 \\ 3 \\ 0 \end{bmatrix},$$

the standard matrix for T is $\begin{bmatrix} -1 & 1 & 0 \\ -1 & 3 & -1 \\ 7 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$.

2.

$$S\left(\begin{bmatrix} -2 \\ 1 \\ 3 \\ 2 \end{bmatrix}\right) = -2S(\vec{e}_1) + S(\vec{e}_2) + 3S(\vec{e}_3) + 2S(\vec{e}_4)$$

$$= -2\begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} + \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix} + 3\begin{bmatrix} 4 \\ -1 \\ -2 \end{bmatrix} + 2\begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix} = \begin{bmatrix} 13 \\ -4 \\ -6 \end{bmatrix}.$$

Sample Exercises with Solutions

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Sample Exercises with Solutions

Example B.1.14 AT3. Let $T : \mathbb{R}^4 \to \mathbb{R}^3$ be the linear transformation given by

$$T\left(\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}\right) = \begin{bmatrix} x+3y+2z-3w \\ 2x+4y+6z-10w \\ x+6y-z+3w \end{bmatrix}$$

1. Explain how to find the image of T and the kernel of T.
2. Explain how to find a basis of the image of T and a basis of the kernel of T.
3. Explain how to find the rank and nullity of T, and why the rank-nullity theorem holds for T.

Solution.

1. To find the image we compute

$$\operatorname{Im}(T) = T\left(\operatorname{span}\{\vec{e}_1, \vec{e}_2, \vec{e}_3, \vec{e}_4\}\right)$$

$$= \operatorname{span}\{T(\vec{e}_1), T(\vec{e}_2), T(\vec{e}_3), T(\vec{e}_4)\}$$

$$= \operatorname{span}\left\{\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 6 \end{bmatrix}, \begin{bmatrix} 2 \\ 6 \\ -1 \end{bmatrix}, \begin{bmatrix} -3 \\ -10 \\ 3 \end{bmatrix}\right\}.$$

2. The kernel is the solution set of the corresponding homogeneous system of equations, i.e.

$$\begin{aligned} x+3y+2z-\ 3w &= 0 \\ 2x+4y+6z-10w &= 0 \\ x+6y-\ z+\ 3w &= 0. \end{aligned}$$

So we compute

$$\operatorname{RREF}\left[\begin{array}{cccc|c} 1 & 3 & 2 & -3 & 0 \\ 2 & 4 & 6 & -10 & 0 \\ 1 & 6 & -1 & 3 & 0 \end{array}\right] = \left[\begin{array}{cccc|c} 1 & 0 & 5 & -9 & 0 \\ 0 & 1 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Then, letting $z = a$ and $w = b$ we have

$$\ker T = \left\{\begin{bmatrix} -5a+9b \\ a-2b \\ a \\ b \end{bmatrix} \,\middle|\, a, b \in \mathbb{R}\right\}.$$

3. Since $\operatorname{Im}(T) = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 6 \end{bmatrix}, \begin{bmatrix} 2 \\ 6 \\ -1 \end{bmatrix}, \begin{bmatrix} -3 \\ -10 \\ 3 \end{bmatrix} \right\}$, we simply need to find a linearly independent subset of these four spanning vectors. So we compute

$$\operatorname{RREF} \begin{bmatrix} 1 & 3 & 2 & -3 \\ 2 & 4 & 6 & -10 \\ 1 & 6 & -1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 5 & -9 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Since the first two columns are pivot columns, they form a linearly independent spanning set, so a basis for $\operatorname{Im} T$ is $\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 6 \end{bmatrix} \right\}$.

To find a basis for the kernel, note that

$$\ker T = \left\{ \begin{bmatrix} -5a + 9b \\ a - 2b \\ a \\ b \end{bmatrix} \middle| \, a, b \in \mathbb{R} \right\}$$

$$= \left\{ a \begin{bmatrix} -5 \\ 1 \\ 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} 9 \\ -2 \\ 0 \\ 1 \end{bmatrix} \middle| \, a, b \in \mathbb{R} \right\}$$

$$= \operatorname{span} \left\{ \begin{bmatrix} -5 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 9 \\ -2 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

so a basis for the kernel is

$$\left\{ \begin{bmatrix} -5 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 9 \\ -2 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

4. The dimension of the image (the rank) is 2, the dimension of the kernel (the nullity) is 2, and the dimension of the domain of T is 4, so we see $2 + 2 = 4$, which verifies that the sum of the rank and nullity of T is the dimension of the domain of T.

□

Example B.1.15 AT4. Let $T : \mathbb{R}^4 \to \mathbb{R}^3$ be the linear transformation given by the standard matrix $\begin{bmatrix} 1 & 3 & 2 & -3 \\ 2 & 4 & 6 & -10 \\ 1 & 6 & -1 & 3 \end{bmatrix}$.

1. Explain why T is or is not injective.

2. Explain why T is or is not surjective.

Solution. Compute

$$\text{RREF} \begin{bmatrix} 1 & 3 & 2 & -3 \\ 2 & 4 & 6 & -10 \\ 1 & 6 & -1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 5 & -9 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

1. Note that the third and fourth columns are non-pivot columns, which means $\ker T$ contains infinitely many vectors, so T is not injective.

2. Since there are only two pivots, the image (i.e. the span of the columns) is a 2-dimensional subspace (and thus does not equal $\mathbb{R}^3$), so T is not surjective.

□

Sample Exercises with Solutions

Example B.1.16 AT5. Let V be the set of all pairs of numbers (x, y) of real numbers together with the following operations:

$$\begin{aligned}(x_1, y_1) \oplus (x_2, y_2) &= (2x_1 + 2x_2, 2y_1 + 2y_2) \\ c \odot (x, y) &= (cx, c^2 y)\end{aligned}$$

1. Show that scalar multiplication distributes over vector addition:

$$c \odot ((x_1, y_1) \oplus (x_2, y_2)) = c \odot (x_1, y_1) \oplus c \odot (x_2, y_2)$$

2. Explain why V nonetheless is not a vector space.

Solution.

1. We compute both sides:

$$\begin{aligned}c \odot ((x_1, y_1) \oplus (x_2, y_2)) &= c \odot (2x_1 + 2x_2, 2y_1 + 2y_2) \\ &= (c(2x_1 + 2x_2), c^2(2y_1 + 2y_2)) \\ &= (2cx_1 + 2cx_2, 2c^2 y_1 + 2c^2 y_2)\end{aligned}$$

   and

$$\begin{aligned}c \odot (x_1, y_1) \oplus c \odot (x_2, y_2) &= (cx_1, c^2 y_1) \oplus (cx_2, c^2 y_2) \\ &= (2cx_1 + 2cx_2, 2c^2 y_1 + 2c^2 y_2)\end{aligned}$$

   Since these are the same, we have shown that the property holds.

2. To show V is not a vector space, we must show that it fails one of the 8 defining properties of vector spaces. We will show that scalar multiplication does not distribute over scalar addition, i.e., there are values such that

$$(c + d) \odot (x, y) \neq c \odot (x, y) \oplus d \odot (x, y)$$

   - (Solution method 1) First, we compute

$$\begin{aligned}(c + d) \odot (x, y) &= ((c + d)x, (c + d)^2 y) \\ &= ((c + d)x, (c^2 + 2cd + d^2)y).\end{aligned}$$

     Then we compute

$$\begin{aligned}c \odot (x, y) \oplus d \odot (x, y) &= (cx, c^2 y) \oplus (dx, d^2 y) \\ &= (2cx + 2dx, 2c^2 y + 2d^2 y).\end{aligned}$$

     Since $(c + d)x \neq 2cx + 2dy$ when $c, d, x, y = 1$, the property fails to hold.

- (Solution method 2) When we let $c, d, x, y = 1$, we may simplify both sides as follows.

$$\begin{aligned}(c+d) \odot (x, y) &= 2 \odot (1, 1) \\ &= (2 \cdot 1, 2^2 \cdot 1) \\ &= (2, 4)\end{aligned}$$

$$\begin{aligned}c \odot (x, y) \oplus d \odot (x, y) &= 1 \odot (1, 1) \oplus 1 \odot (1, 1) \\ &= (1 \cdot 1, 1^2 \cdot 1) \oplus (1 \cdot 1, 1^2 \cdot 1) \\ &= (1, 1) \oplus (1, 1) \\ &= (2 \cdot 1 + 2 \cdot 1, 2 \cdot 1 + 2 \cdot 1) \\ &= (4, 4)\end{aligned}$$

  Since these ordered pairs are different, the property fails to hold.

$\square$

Example B.1.17 AT6.

1. Given the set

$$\left\{x^3 - 2\,x^2 + x + 2, 2\,x^2 - 1, -x^3 + 3\,x^2 + 3\,x - 2, x^3 - 6\,x^2 + 9\,x + 5\right\}$$

   write a statement involving the solutions to a polynomial equation that's equivalent to each claim below.

   - The set of polynomials is linearly *independent*.
   - The set of polynomials is linearly *dependent*.

2. Explain how to determine which of these statements is true.

Solution. The set of polynomials

$$\left\{x^3 - 2\,x^2 + x + 2, 2\,x^2 - 1, -x^3 + 3\,x^2 + 3\,x - 2, x^3 - 6\,x^2 + 9\,x + 5\right\}$$

is linearly *independent* exactly when the polynomial equation

$$y_1\left(x^3 - 2\,x^2 + x + 2\right) + y_2\left(2\,x^2 - 1\right) + y_3\left(-x^3 + 3\,x^2 + 3\,x - 2\right) + y_4\left(x^3 - 6\,x^2 + 9\,x + 5\right) = 0$$

has no nontrivial (i.e. nonzero) solutions. The set is linearly *dependent* when this equation has a nontrivial (i.e. nonzero) solution.

To solve this equation, we distribute and then collect coefficients to obtain

$$(y_1 - y_3 + y_4)\,x^3 + (-2y_1 + 2y_2 + 3y_3 - 6y_4)\,x^2 + (y_1 + 3y_3 + 9y_4)\,x + (2y_1 - y_2 - 2y_3 + 5y_4) = 0.$$

These polynomials are equal precisely when their coefficients are equal, leading to the system

$$\begin{array}{ccccccccc} y_1 & & & - & y_3 & + & y_4 & = & 0 \\ -2y_1 & + & 2y_2 & + & 3y_3 & - & 6y_4 & = & 0 \\ y_1 & + & & + & 3y_3 & + & 9y_4 & = & 0 \\ 2y_1 & - & y_2 & - & 2y_3 & + & 5y_4 & = & 0 \end{array}.$$

To solve this, we compute

$$\mathrm{RREF}\left[\begin{array}{cccc|c} 1 & 0 & -1 & 1 & 0 \\ -2 & 2 & 3 & -6 & 0 \\ 1 & 0 & 3 & 9 & 0 \\ 2 & -1 & -2 & 5 & 0 \end{array}\right] = \left[\begin{array}{cccc|c} 1 & 0 & 0 & 3 & 0 \\ 0 & 1 & 0 & -3 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The system has (infintely many) nontrivial solutions, so we that the set of polynomials is linearly *dependent*. □

Example B.1.18 MX1. Of the following three matrices, only two may be multiplied.

$$A = \begin{bmatrix} -1 & 3 & -2 & -3 \\ 1 & -4 & 2 & 3 \end{bmatrix} \quad B = \begin{bmatrix} 1 & -6 & -1 \\ 0 & 1 & 0 \end{bmatrix} \quad C = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -2 \\ -2 & 4 & -1 \\ -2 & 3 & -1 \end{bmatrix}$$

(a) Explain which two can be multiplied and why.

Solution. C is the 4×3 standard matrix for a transformation of $\mathbb{R}^3$ vectors into $\mathbb{R}^4$ vectors, and A is the 2×4 matrix for a transformation of $\mathbb{R}^4$ vectors into $\mathbb{R}^2$ vectors, so AC will be the 2×3 standard matrix for their composition, a transformation of $\mathbb{R}^3$ vectors into $\mathbb{R}^2$ vectors.

(b) Find their product using technology.

Solution.

```
A = [
-1 3 -2 -3
1  4  2  3
]
C = [
1 -1 -1
0  1 -2
-2 4 -1
-2 3 -1
]
A*C
```

```
A =

-1 3 -2 -3
1 4 2 3

C =

1 -1 -1
0 1 -2
-2 4 -1
-2 3 -1

ans =

9 -13 0
-9 20 -14
```

(c) Show how to find this product without technology.

Solution. We may compute each $AC\vec{e}_i$ to obtain each column of AC:

$$AC\vec{e}_1 = A\begin{bmatrix} 1 \\ 0 \\ -2 \\ -2 \end{bmatrix} = 1\begin{bmatrix} -1 \\ 1 \end{bmatrix} + 0\begin{bmatrix} -1 \\ 1 \end{bmatrix} - 2\begin{bmatrix} -1 \\ 1 \end{bmatrix} - 2\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 9 \\ -9 \end{bmatrix}$$

$$AC\vec{e}_2 = A\begin{bmatrix} -1 \\ 1 \\ 4 \\ 3 \end{bmatrix} = -1\begin{bmatrix} -1 \\ 1 \end{bmatrix} + 1\begin{bmatrix} -1 \\ 1 \end{bmatrix} + 4\begin{bmatrix} -1 \\ 1 \end{bmatrix} + 3\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -13 \\ 12 \end{bmatrix}$$

$$AC\vec{e}_3 = A\begin{bmatrix} -1 \\ -2 \\ -1 \\ -1 \end{bmatrix} = -1\begin{bmatrix} -1 \\ 1 \end{bmatrix} - 1\begin{bmatrix} -1 \\ 1 \end{bmatrix} - 2\begin{bmatrix} -1 \\ 1 \end{bmatrix} - 1\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

□

Sample Exercises with Solutions

Example B.1.19 MX2. Consider each of the following matrices.

(a)

$$D = \begin{bmatrix} 2 & 1 & 3 & 1 \\ -1 & 0 & -2 & -2 \\ -1 & 0 & -1 & -3 \\ 2 & 1 & 5 & 0 \end{bmatrix}$$

(i) Explain why this matrix is or is not invertible by discussing its corresponding linear transformation.

Solution. First, we calculate RREF(D):

$$\text{RREF} \begin{bmatrix} 2 & 1 & 3 & 1 \\ -1 & 0 & -2 & -2 \\ -1 & 0 & -1 & -3 \\ 2 & 1 & 5 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Since RREF(D) $= I$, we conclude that D is invertible.

(ii) If the matrix is invertible, use technology to find its inverse.

Solution. Using technology, its inverse is $\begin{bmatrix} 4 & -7 & 6 & -4 \\ -3 & 9 & -7 & 4 \\ -1 & 1 & -1 & 1 \\ -1 & 2 & -2 & 1 \end{bmatrix}$.

(iii) If the matrix is invertible, explain and demonstrate how to find the 4th column of this inverse using a technique that could be performed without technology (though you may use technology for this exercise).

Solution. We find the 4th column of the inverse matrix by solving the equation $D\vec{x} = \vec{e}_4$. Since

$$\text{RREF} \left[\begin{array}{cccc|c} 2 & 1 & 3 & 1 & 0 \\ -1 & 0 & -2 & -2 & 0 \\ -1 & 0 & -1 & -3 & 0 \\ 2 & 1 & 5 & 0 & 1 \end{array}\right] = \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -4 \\ 0 & 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right],$$

the 4th column of the inverse is $\begin{bmatrix} -4 \\ 4 \\ 1 \\ 1 \end{bmatrix}$.

(iv) If the matrix is invertible, explain how to use it with technology to solve the vector equation

$$x_1 \begin{bmatrix} 2 \\ -1 \\ -1 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ -2 \\ -1 \\ 5 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ -2 \\ -3 \\ 0 \end{bmatrix} = \begin{bmatrix} -18 \\ 6 \\ 0 \\ -28 \end{bmatrix}.$$

Solution. To solve the equation $D\vec{x} = \begin{bmatrix} -18 \\ 6 \\ 0 \\ -28 \end{bmatrix}$, we left-multiply by D^{-1} to get

$$\vec{x} = D^{-1} \begin{bmatrix} -18 \\ 6 \\ 0 \\ -28 \end{bmatrix}.$$

Since

$$\begin{bmatrix} 4 & -7 & 6 & -4 \\ -3 & 9 & -7 & 4 \\ -1 & 1 & -1 & 1 \\ -1 & 2 & -2 & 1 \end{bmatrix} \begin{bmatrix} -18 \\ 6 \\ 0 \\ -28 \end{bmatrix} = \begin{bmatrix} -2 \\ -4 \\ -4 \\ 2 \end{bmatrix},$$

the solution is $\vec{x} = \begin{bmatrix} -2 \\ -4 \\ -4 \\ 2 \end{bmatrix}$.

(b)

$$N = \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -3 \\ 5 & -10 & 1 & 12 \\ -2 & 4 & -1 & -3 \end{bmatrix}$$

(i) Explain why this matrix is or is not invertible by discussing its corresponding linear transformation.

Solution.

$$\text{RREF} \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -3 \\ 5 & -10 & 1 & 12 \\ -2 & 4 & -1 & -3 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 0 & 3 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

N is not invertible.

(ii) If the matrix is invertible, use technology to find its inverse.

Solution. N/A

(iii) If the matrix is invertible, explain and demonstrate how to find the column of this inverse using a technique that could be performed without technology (though you may use technology for this exercise).

Solution. N/A

(iv) If the matrix is invertible, explain how to use it with technology to solve the vector equation

$$x_1 \begin{bmatrix} 1 \\ 0 \\ 5 \\ -2 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 0 \\ -10 \\ 4 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ -3 \\ 12 \\ -3 \end{bmatrix} = \begin{bmatrix} 2 \\ -10 \\ 50 \\ -14 \end{bmatrix}.$$

Solution. N/A

□

Example B.1.20 MX3. Let $\mathcal{B} = \left\{ \begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} \right\}$, and $\vec{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

(a) Explain and demonstrate how to verify that $\mathcal{B}$ is a basis of $\mathbb{R}^3$ and how to calculate $M_{\mathcal{B}}$, the change-of-basis matrix from the standard basis of $\mathbb{R}^3$ to $\mathcal{B}$.

Solution. We can accomplish both tasks by calculating the RREF of the following matrix:

$$\text{RREF} \left[\begin{array}{ccc|ccc} -2 & -1 & 1 & 1 & 0 & 0 \\ -2 & -2 & 3 & 0 & 1 & 0 \\ 1 & -1 & 2 & 0 & 0 & 1 \end{array} \right] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 1 \\ 0 & 1 & 0 & -7 & 5 & -4 \\ 0 & 0 & 1 & -4 & 3 & -2 \end{array} \right].$$

The fact that the matrix to the left of the vertical bar is the identity matrix tells that $\mathcal{B}$ is a basis. The matrix on the right hand side of the bar is equal to the change-of-basis matrix:

$$M_{\mathcal{B}} = \begin{bmatrix} 1 & -1 & 1 \\ -7 & 5 & -4 \\ -4 & 3 & -2 \end{bmatrix}.$$

(b) Explain and demonstrate how to use $M_{\mathcal{B}}$ to express $\vec{v}$ in terms of $\mathcal{B}$-basis vectors.

Solution. By definition of the change of basis matrix, if $\vec{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, then the coordinates of $\vec{v}$ with respect to $\mathcal{B}$ are given by:

$$M_{\mathcal{B}}\vec{v} = M_{\mathcal{B}} = \begin{bmatrix} 1 & -1 & 1 \\ -7 & 5 & -4 \\ -4 & 3 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ -9 \\ -4 \end{bmatrix}.$$

It follows that:

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 2 \begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix} - 9 \begin{bmatrix} -1 \\ -2 \\ -1 \end{bmatrix} - 4 \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}.$$

□

Example B.1.21 MX4.

(a) Give a 3×3 matrix C that may be used to perform the row operation $-5R_1 \to R_1$.

Answer.

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \overset{-5R_1 \to R_1}{\sim} \begin{bmatrix} -5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = C$$

(b) Give a 3×3 matrix M that may be used to perform the row operation $R_1 \leftrightarrow R_3$.

Answer.

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \overset{R_1 \leftrightarrow R_3}{\sim} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = M$$

(c) Give a 3×3 matrix P that may be used to perform the row operation $R_3 - 2R_2 \to R_3$.

Answer.

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \overset{R_3 - 2R_2 \to R_3}{\sim} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} = P$$

(d) Give a 3×3 matrix that may be used to first apply $-5R_1 \to R_1$, then $R_3 - 2R_2 \to R_3$, and finally $R_1 \leftrightarrow R_3$ (note the order).

Answer. $MPC = \begin{bmatrix} 0 & -2 & 1 \\ 0 & 1 & 0 \\ -5 & 0 & 0 \end{bmatrix}$

(e) Show how to manually apply those row operations to $A = \begin{bmatrix} 1 & -3 & -3 \\ 2 & -6 & -5 \\ 0 & 0 & 2 \end{bmatrix}$, then use technology to verify that your matrix in the previous task gives the same result.

Answer.

$$\begin{bmatrix} 1 & -3 & -3 \\ 2 & -6 & -5 \\ 0 & 0 & 2 \end{bmatrix} \sim \begin{bmatrix} -5 & 15 & 15 \\ 2 & -6 & -5 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\sim \begin{bmatrix} -5 & 15 & 15 \\ 2 & -6 & -5 \\ -4 & 12 & 12 \end{bmatrix} \sim \begin{bmatrix} -4 & 12 & 12 \\ 2 & -6 & -5 \\ -5 & 15 & 15 \end{bmatrix}$$

Sample Exercises with Solutions

```
C = [
     -5  0  0
     0  1  0
     0  0  1
]

M = [
     0  0  1
     0  1  0
     1  0  0
]

P = [
     1  0  0
     0  1  0
     0  -2  1
]

M*P*C

A = [
     1  -3  -3
     2  -6  -5
     0  0  2
]

M*P*C*A
```

```
C =

-5  0  0
0  1  0
0  0  1

M =

0  0  1
0  1  0
1  0  0

P =

1  0  0
0  1  0
0  -2  1

ans =

0  -2  1
0  1  0
-5  0  0

A =

1  -3  -3
2  -6  -5
0  0  2

ans =

-4  12  12
2  -6  -5
-5  15  15
```

□

Sample Exercises with Solutions

Example B.1.22 GT1. Let A be *any* 4×4 matrix with determinant 4.

(a) Let Q be the matrix obtained from A by applying the row operation $R_4 \leftrightarrow R_1$. Explain and demonstrate how to find $\det Q$ without knowing the terms of A.

Solution. We apply this row operation to the identity matrix

$$I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} = R$$

so that $Q = RA$. It follows that

$$\det Q = \det(RA) = \det(R)\det(A) = (-1)(4) = -4.$$

(b) Let N be the matrix obtained from A by applying the row operation $3R_4 \to R_4$. Explain and demonstrate how to find $\det N$ without knowing the terms of A.

Solution. We apply this row operation to the identity matrix

$$I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} = R$$

so that $N = RA$. It follows that

$$\det N = \det(RA) = \det(R)\det(A) = (3)(4) = 12.$$

(c) Let M be the matrix obtained from A by applying the row operation $R_4 - 3R_2 \to R_4$. Explain and demonstrate how to find $\det M$ without knowing the terms of A.

Solution. We apply this row operation to the identity matrix

$$I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -3 & 0 & 1 \end{bmatrix} = R$$

so that $M = RA$. It follows that

$$\det M = \det(RA) = \det(R)\det(A) = (1)(4) = 4.$$

□

Sample Exercises with Solutions

Example B.1.23 GT2. Show how to compute the determinant of the matrix

$$A = \begin{bmatrix} 1 & 3 & 0 & -1 \\ 1 & 1 & 2 & 4 \\ 1 & 1 & 1 & 3 \\ -3 & 1 & 2 & -5 \end{bmatrix}$$

Solution. Here is one possible solution, first applying a single row operation, and then performing Laplace/cofactor expansions to reduce the determinant to a linear combination of 2×2 determinants:

$$\begin{aligned} \det \begin{bmatrix} 1 & 3 & 0 & -1 \\ 1 & 1 & 2 & 4 \\ 1 & 1 & 1 & 3 \\ -3 & 1 & 2 & -5 \end{bmatrix} &= \det \begin{bmatrix} 1 & 3 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 3 \\ -3 & 1 & 2 & -5 \end{bmatrix} = (-1) \det \begin{bmatrix} 1 & 3 & -1 \\ 1 & 1 & 3 \\ -3 & 1 & -5 \end{bmatrix} + (1) \det \begin{bmatrix} 1 & 3 & 0 \\ 1 & 1 & 1 \\ -3 & 1 & 2 \end{bmatrix} \\ &= (-1) \left((1) \det \begin{bmatrix} 1 & 3 \\ 1 & -5 \end{bmatrix} - (1) \det \begin{bmatrix} 3 & -1 \\ 1 & -5 \end{bmatrix} + (-3) \det \begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix} \right) + \\ &\quad (1) \left((1) \det \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} - (3) \det \begin{bmatrix} 1 & 1 \\ -3 & 2 \end{bmatrix} \right) \\ &= (-1)(-8 + 14 - 30) + (1)(1 - 15) \\ &= 10 \end{aligned}$$

Here is another possible solution, using row and column operations to first reduce the determinant to a 3×3 matrix and then applying a formula:

$$\begin{aligned} \det \begin{bmatrix} 1 & 3 & 0 & -1 \\ 1 & 1 & 2 & 4 \\ 1 & 1 & 1 & 3 \\ -3 & 1 & 2 & -5 \end{bmatrix} &= \det \begin{bmatrix} 1 & 3 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 3 \\ -3 & 1 & 2 & -5 \end{bmatrix} = \det \begin{bmatrix} 1 & 3 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 2 \\ -3 & 1 & 2 & -7 \end{bmatrix} \\ &= -\det \begin{bmatrix} 1 & 3 & 0 & -1 \\ 1 & 1 & 1 & 2 \\ 0 & 0 & 1 & 0 \\ -3 & 1 & 2 & -7 \end{bmatrix} = -\det \begin{bmatrix} 1 & 3 & -1 \\ 1 & 1 & 2 \\ -3 & 1 & -7 \end{bmatrix} \\ &= -((-7 - 18 - 1) - (3 + 2 - 21)) \\ &= 10 \end{aligned}$$

□

Example B.1.24 GT3. Explain how to find the eigenvalues of the matrix $\begin{bmatrix} -2 & -2 \\ 10 & 7 \end{bmatrix}$.

Solution. Compute the characteristic polynomial:

$$\det(A - \lambda I) = \det \begin{bmatrix} -2-\lambda & -2 \\ 10 & 7-\lambda \end{bmatrix}$$

$$= (-2-\lambda)(7-\lambda) + 20 = \lambda^2 - 5\lambda + 6 = (\lambda - 2)(\lambda - 3)$$

The eigenvalues are the roots of the characteristic polynomial, namely 2 and 3. □

Example B.1.25 GT4. Explain how to find a basis for the eigenspace associated to the eigenvalue 3 in the matrix

$$\begin{bmatrix} -7 & -8 & 2 \\ 8 & 9 & -1 \\ \frac{13}{2} & 5 & 2 \end{bmatrix}.$$

Solution. The eigenspace associated to 3 is the kernel of $A - 3I$, so we compute

$$\text{RREF}(A-3I) = \text{RREF} \begin{bmatrix} -7-3 & -8 & 2 \\ 8 & 9-3 & -1 \\ \frac{13}{2} & 5 & 2-3 \end{bmatrix} =$$

$$\text{RREF} \begin{bmatrix} -10 & -8 & 2 \\ 8 & 6 & -1 \\ \frac{13}{2} & 5 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -\frac{3}{2} \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus we see the kernel is

$$\left\{ \begin{bmatrix} -a \\ \frac{3}{2}a \\ a \end{bmatrix} \middle| a \in \mathbb{R} \right\}$$

which has a basis of $\left\{ \begin{bmatrix} -1 \\ \frac{3}{2} \\ 1 \end{bmatrix} \right\}$. □

B.2 Definitions

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