Precalculus for Team-Based Inquiry Learning

2025 Edition

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TBIL Community

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[1] tbil.org

[2] github.com/TeamBasedInquiryLearning/library/blob/main/LICENSE.md

Contents

Chapter 1

Equations, Inequalities, and Applications (EQ)

Objectives

How do we find solutions of equations?
By the end of this chapter, you should be able to...

1. Solve linear equations in one variable. Solve linear inequalities in one variable and express the solution graphically and using interval notation.
2. Solve application problems involving linear equations.
3. Given two points, determine the distance between them and the midpoint of the line segment connecting them.
4. Solve linear equations involving an absolute value. Solve linear inequalities involving absolute values and express the answers graphically and using interval notation.
5. Solve quadratic equations using factoring, the square root property, completing the square, and the quadratic formula and express these answers in exact form.

1.1 Linear Equations and Inequalities (EQ1)

Objectives

- Solve linear equations in one variable. Solve linear inequalities in one variable and express the solution graphically and using interval notation.

Linear Equations and Inequalities (EQ1)

Remark 1.1.1 Recall that when solving a linear equation, you use addition, subtraction, multiplication and division to isolate the variable.

Activity 1.1.2 Solve the linear equations.

(a) $3x - 8 = 5x + 2$

A. $x = 2$

B. $x = 5$

C. $x = -5$

D. $x = -2$

(b) $5(3x - 4) = 2x - (x + 3)$

A. $x = \frac{17}{14}$

B. $x = \frac{14}{17}$

C. $x = \frac{23}{14}$

D. $x = \frac{14}{23}$

Activity 1.1.3 Solve the linear equation.

$$\frac{2}{3}x - 8 = \frac{5x+1}{6}$$

(a) Which equation is equivalent to $\frac{2}{3}x - 8 = \frac{5x+1}{6}$ but does not contain any fractions?

A. $12x - 48 = 15x + 3$

B. $3x - 24 = 10x + 2$

C. $4x - 8 = 5x + 1$

D. $4x - 48 = 5x + 1$

(b) Use the simplified equation from part (a) to solve $\frac{2}{3}x - 8 = \frac{5x+1}{6}$.

A. $x = -17$

B. $x = -\frac{26}{7}$

C. $x = -9$

D. $x = -49$

Activity 1.1.4 It is not always the case that a linear equation has exactly one solution. Consider the equation

$$3 + 5x = 5(x + 2) - 7.$$

(a) Solve the linear equation. Which of the following is your last step?

A. $3 + 5x = 5x + 3$

B. $0 = 0$

C. $3 = 3$

D. $3 = 10$

(b) Which of the following best describes the statement you got in part (a)?

A. always true

B. sometimes true

C. never true

(c) What do you think this means about the number of solutions?

Remark 1.1.5 In Activity 1.1.4, we saw that a linear equation can have infinitely many solutions. Linear equations can also have a unique solution or no solution!

Activity 1.1.6 For each part in this activity, consider the conditions for when a linear equation has a unique solution, no solution, or infinitely many solutions.

(a) Which of these equations has one unique solution?

A. $4(x-2) = 4x+6$

B. $4(x-1) = 4x-4$

C. $4(x-1) = x+4$

(b) Which of these equations has no solutions?

A. $4(x-2) = 4x+6$

B. $4(x-1) = 4x-4$

C. $4(x-1) = x+4$

(c) Which of these equations has many solutions?

A. $4(x-2) = 4x+6$

B. $4(x-1) = 4x-4$

C. $4(x-1) = x+4$

(d) What happens to the x variable when a linear equation has no solution or many solutions?

Definition 1.1.7 A linear equation with one unique solution is a **conditional equation.** A linear equation that is true for all values of the variable is an **identity equation**. A linear equation with no solutions is an **inconsistent equation.** ◊

Definition 1.1.8 The remainder of this section will focus on **linear inequalities**. A linear inequality is an inequality that can be written in one of the following forms:

- $$ax + b > 0$$
- $$ax + b < 0$$
- $$ax + b \geq 0$$
- $$ax + b \leq 0$$

where a and b are real numbers and $a \neq 0$. $\Diamond$

Linear Equations and Inequalities (EQ1)

Activity 1.1.9 In this activity, we explore the relationship between a linear equation and a linear inequality, as well as how to express solutions on a number line and in interval notation.

(a) What is the solution to the linear equation $3x - 1 = 5$?

(b) Which of these values are solutions of the inequality $3x - 1 \geq 5$?

A. $x = 0$

B. $x = 2$

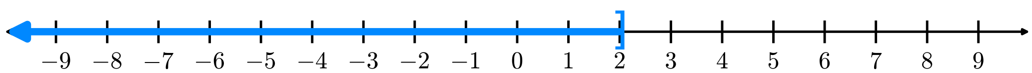
C. $x = 4$

D. $x = 10$

(c) There are more solutions to the inequality than the ones found in part (b). How would you characterize all of them?

(d) Draw the solution to the inequality on a number line.

A.



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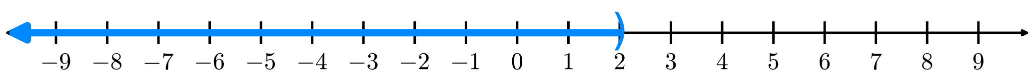

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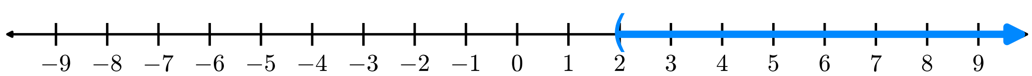

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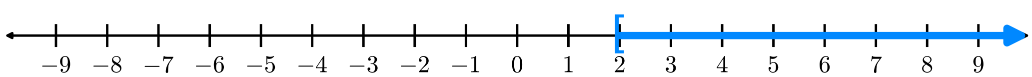


(e) Express the solution of the inequality $3x - 1 \geq 5$ in interval notation.

A. $(-\infty, 2]$

B. $(-\infty, 2)$

C. $(2, \infty)$

D. $[2, \infty)$

Hint. Use the graph from part (d) to help you write the solution in interval notation.

Linear Equations and Inequalities (EQ1)

Activity 1.1.10 Let's consider what happens to the inequality when the variable has a negative coefficient.

(a) Which of these values is a solution of the inequality $-x < 8$?

A. $x = -10$

B. $x = -8$

C. $x = 4$

D. $x = 10$

(b) Solve the linear inequality $-x < 8$. How does your solution compare to the values chosen in part (a)?

(c) Expression the solution of the inequality $-x < 8$ in interval notation.

A. $(-\infty, -8]$

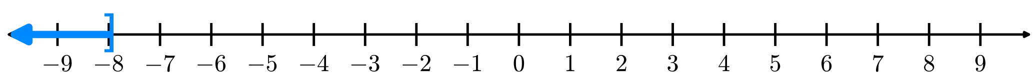
B. $(-\infty, -8)$

C. $(-8, \infty)$

D. $[-8, \infty)$

(d) Draw the solution to the inequality on a number line.

A.



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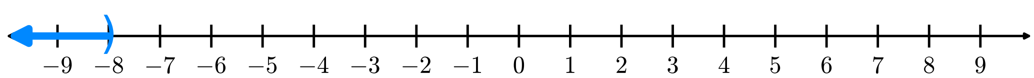
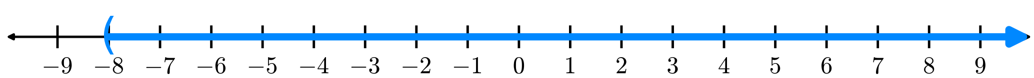


C.



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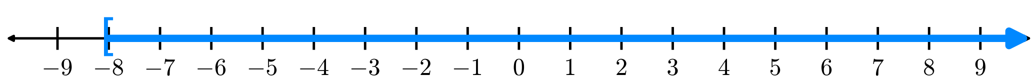


Remark 1.1.11 You can treat solving linear inequalities, just like solving an equation. The one exception is when you multiply or divide by a negative value, reverse the inequality symbol.

Linear Equations and Inequalities (EQ1)

Activity 1.1.12 Solve the following inequalities. Express your solution in interval notation and graphically on a number line.

(a) $-3x - 1 \leq 5$

(b) $3(x + 4) > 2x - 1$

(c) $-\frac{1}{2}x \geq -\frac{3}{4} + \frac{5}{4}x$

Definition 1.1.13 A **compound inequality** includes multiple inequalities in one statement.
◊

Linear Equations and Inequalities (EQ1)

Activity 1.1.14 Consider the statement $3 \leq x < 8$. This really means that $3 \leq x$ and $x < 8$.

(a) Which of the following inequalities are equivalent to the compound inequality $3 \leq 2x - 3 < 8$?

A. $3 \leq 2x - 3$

B. $3 \geq 2x - 3$

C. $2x - 3 < 8$

D. $2x - 3 > 8$

(b) Solve the inequality $3 \leq 2x - 3$.

A. $x \leq 0$

B. $x \geq 0$

C. $x \leq 3$

D. $x \geq 3$

(c) Solve the inequality $2x - 3 < 8$.

A. $x > \frac{11}{2}$

B. $x < \frac{11}{2}$

C. $x > \frac{5}{2}$

D. $x < \frac{5}{2}$

(d) Which compound inequality describes how the two solutions overlap?

A. $0 \leq x < \frac{11}{2}$

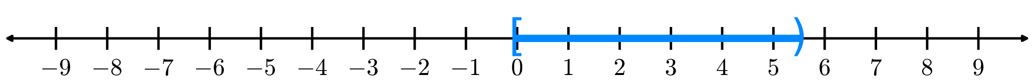
B. $0 \leq x < \frac{5}{2}$

C. $\frac{5}{2} < x \leq 3$

D. $3 \leq x < \frac{11}{2}$

(e) Draw the solution to the inequality on a number line.

A.



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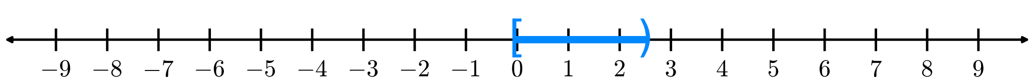

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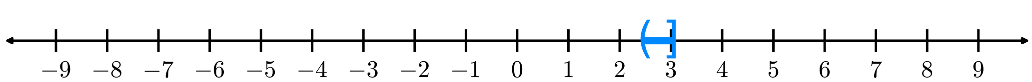
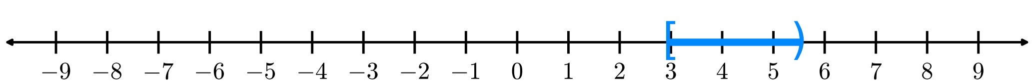


D.



Remark 1.1.15 Solving a compound linear inequality, uses the same methods as a single linear inequality ensuring that you perform the same operations on all three parts. Alternatively, you can break the compound inquality up into two and solve separately.

Activity 1.1.16 Solve the following inequalities. Express your solution in interval notation and graphically on a number line.

(a) $8 < -3x - 1 \leq 11$

(b) $-6 \leq \dfrac{x-12}{4} < -2$

1.2 Applications of Linear Equations (EQ2)

Objectives

- Solve application problems involving linear equations.

Applications of Linear Equations (EQ2)

Observation 1.2.1 Linear equations can be used to solve many types of real-world applications. We'll investigate some of those in this section.

Remark 1.2.2 Distance, rate, and time problems are a standard example of an application of a linear equation. For these, it's important to remember that

$$d = rt$$

where d is distance, r is the rate (or speed), and t is time.

Often we will have more than one moving object, so it is helpful to denote which object's distance, rate, or time we are referring to. One way we can do this is by using a subscript. For example, if we are describing an eastbound train (as we will in the first example), it may be helpful to denote its distance, rate, and time as d_E, r_E, and t_E respectively. Notice that the subscript E is a label reminding us that we are referring to the eastbound train.

Applications of Linear Equations (EQ2)

Activity 1.2.3 Two trains leave a station at the same time. One is heading east at a speed of 75 mph, while the other is heading west at a speed of 85 mph. After how long will the trains be 400 miles apart?

(a) How fast is each train traveling?

A. $r_E = 85$ mph, $r_W = 75$ mph

B. $r_E = 75$ mph, $r_W = 85$ mph

C. $r_E = 400$ mph, $r_W = 400$ mph

D. $r_E = 75$ mph, $r_W = 400$ mph

E. $r_E = 400$ mph, $r_W = 85$ mph

(b) Which of the statements describes how the times of the eastbound and westbound train are related?

A. The eastbound train is slower than the westbound train, so $75 + t_E = 85 + t_W$.

B. The eastbound train left an hour before the westbound train, so if we let $t_E = t$, then $t_W = t - 1$.

C. Both trains have been traveling the same amount of time, so $t_E = t_W$. Since they are the same, we can just call them both t.

D. We don't know how the times relate to each other, so we must denote them separately as t_E and t_W.

E. Since the trains are traveling at different speeds, we need the proportion $\dfrac{r_E}{r_W} = \dfrac{t_E}{t_W}$.

(c) Fill in the following table using the information you've just determined about the trains' rates and times since they left the station. Some values are there to help you get started.

	rate	**time**	**distance from station**
eastbound train			$75t$
westbound train		t	

(d) At the moment in question, the trains are 400 miles apart. How does that total distance relate to the distance each train has traveled?

A. The 400 miles is irrelevant. They've been traveling the same amount of time so they must be the same distance away from the station. That tells us $d_E = d_W$.

B. The 400 miles is the difference between the distance each train traveled, so $d_E - d_W = 400$.

C. The 400 miles represents the sum of the distances that each train has traveled, so $d_E + d_W = 400$.

D. The 400 miles is the product of the distance each train traveled, so $(d_E)(d_W) = 400$.

(e) Now plug in the expressions from your table for d_E and d_w. What equation do you get?

A. $75t = 85t$

B. $75t - 85t = 400$

C. $75t + 85t = 400$

D. $(75t)(85t) = 400$

(f) Notice that we now have a linear equation in one variable, t. Solve for t, and put that answer in context of the problem.

A. The trains are 400 miles apart after 2 hours.

B. The trains are 400 miles apart after 2.5 hours.

C. The trains are 400 miles apart after 3 hours.

D. The trains are 400 miles apart after 3.5 hours.

E. The trains are 400 miles apart after 4 hours.

Remark 1.2.4 In Activity 1.2.3, we examined the motion of two objects moving at the same time in opposite directions. In Activity 1.2.5, we will examine a different perspective, but still apply $d = rt$ to solve.

Activity 1.2.5 Jalen needs groceries, so decides to ride his bike to the store. It takes him half an hour to get there. After finishing his shopping, he sees his friend Alex who offers him a ride home. He takes the same route home as he did to the store, but this time it only takes one-fifth of an hour. If his average speed was 18 mph faster on the way home, how far away does Jalen live from the grocery store?

We'll use the subscript b to refer to variables relating to Jalen's trip to the store while riding his *b*ike and the subscript c to refer to variables relating to Jalen's trip home while riding in his friend's *c*ar.

(a) How long does his bike trip from home to the store and his car trip from the store back home take?

A. $t_b = 18$ hours, $t_c = 18$ hours

B. $t_b = \frac{1}{5}$ of an hour, $t_c = \frac{1}{2}$ of an hour

C. $t_b = \frac{1}{2}$ of an hour, $t_c = \frac{1}{5}$ of an hour

D. $t_b = 2$ hours, $t_c = 5$ hours

E. $t_b = 5$ hours, $t_c = 2$ hours

(b) Which of the statements describes how the speed (rate) of the bike trip and the car trip are related?

A. Both the trip to the store and the trip home covered the same distance, so $r_b = r_c$. Since they are the same, we can just call them both r.

B. We don't know how the two rates relate to each other, so cannot write an equation comparing them and must leave them as separate variables r_b and r_c.

C. Jalen's rate on the trip home in the car was 18 mph faster than his trip to the store on his bike, so if we let $r_b = r$, then $r_c = r - 18$.

D. Jalen's rate on the trip home in the car was 18 mph faster than his trip to the store on his bike, so if we let $r_b = r$, then $r_c = r + 18$.

(c) Fill in the following table using the information you've just determined about the Jalen's rates and times on each leg of his grocery store trip. Then fill in the distance column based on how distance relates to rate and time in each case.

	rate	**time**	**distance covered**
bike trip (to the store)			
car trip (going back home)			

(d) Our goal is to figure out how far away Jalen lives from the store. To help us get there, write an equation relating d_b and d_c.

A. The distance he traveled by bike is the same as the distance he traveled by car, so $d_b = d_c$.

B. The distance he traveled by bike took longer than the distance he traveled by car, so $d_b + \frac{1}{2} = d_c + \frac{1}{5}$.

C. The distance, d, between his house and the grocery store is sum of the distance he traveled on his bike and the distance he traveled in the car, so $d_b + d_c = d$.

D. The distance, d, between his house and the grocery store is sum of the difference he traveled on his bike and the distance he traveled in the car, so $d_b - d_c = d$.

(e) Now plug in the expressions from your table for d_b and d_c into the equation you just found. Notice that it is a linear equation in one variable, r. Solve for r.

(f) Our goal was to determine the distance between Jalen's house and the grocery store. Solving for r did not tell us that distance, but it did get us one step closer. Use that value to help you determine the distance between his house and the store, and write your answer using the context of the problem.

A. The grocery store is 6 miles away from Jalen's house.

B. The grocery store is 8 miles away from Jalen's house.

C. The grocery store is 10 miles away from Jalen's house.

D. The grocery store is 12 miles away from Jalen's house.

E. The grocery store is 14 miles away from Jalen's house.

Hint. Can you find an expression involving r that we made that represents that distance?

Applications of Linear Equations (EQ2)

Remark 1.2.6 Another type of application of linear equations is called a mixture problem. In these we will mix together two things, like two types of candy in a candy store or two solutions of different concentrations of alcohol.

Applications of Linear Equations (EQ2)

Activity 1.2.7 Ammie's favorite snack to share with friends is candy salad, which is a mixture of different types of candy. Today she chooses to mix Nerds Gummy Clusters, which cost \$8.38 per pound, and Starburst Jelly Beans, which cost \$7.16 per pound. If she makes seven pounds of candy salad and spends a total of \$55.61, how many pounds of each candy did she buy?

(a) There are two "totals" in this situation: the total weight (in pounds) of candy Ammie bought and the total amount of money (in dollars) Ammie spent. Let's begin with the total weight. If we let N represent the pounds of Nerds Gummy Clusters and S represent the pounds of Starburst Jelly Beans, which of the following equations can represent the total weight?

A. $N - S = 7$

B. $NS = 7$

C. $N + S = 7$

D. $\dfrac{N}{S} = 7$

(b) Which expressions represent the amount she spent on each candy? Again, we will let N represent the pounds of Nerds Gummy Clusters and S represent the pounds of Starburst Jelly Beans.

A. N spent on Nerds Gummy Clusters; S spent on Starburst Jelly Beans

B. $8.38N$ spent on Nerds Gummy Clusters; $7.16S$ spent on Starburst Jelly Beans

C. $8.38 + N$ spent on Nerds Gummy Clusters; $7.16 + S$ spent on Starburst Jelly Beans

D. $8.38 - N$ spent on Nerds Gummy Clusters; $7.16 - S$ spent on Starburst Jelly Beans

(c) Now we focus on the total cost. Which of the following equations can represent the total amount she spent?

A. $N + S = 55.61$

B. $8.38N + 7.16S = 55.61$

C. $8.38 + N + 7.16 + S = 55.61$

D. $8.38 - N + 7.16 - S = 55.61$

(d) We are almost ready to solve, but we have two variables in our weight equation and our cost equation. We will get the cost equation to one variable by using the weight equation as a substitution. Which of the following is a way to express one variable in terms of the other?

A. If N is the total weight of the Nerds Gummy Clusters, then $7-N$ could represent the weight of the Starburst Jelly Beans.

B. If N is the total weight of the Nerds Gummy Clusters, then $7+N$ could represent the weight of the Starburst Jelly Beans.

C. If S is the total weight of the Starburst Jelly Beans, then $7-S$ could represent the weight of the Nerds Gummy Clusters.

D. If S is the total weight of the Starburst Jelly Beans, then $7+S$ could represent the weight of the Nerds Gummy Clusters.

Hint. More than one answer may be correct here!

(e) Plug your expressions in to the total cost equation.

A. $8.38N + 7.16(7 - N) = 55.61$

B. $8.38S + 7.16(7 - S) = 55.61$

C. $8.38(7 - N) + 7.16N = 55.61$

D. $8.38(7 - S) + 7.16S = 55.61$

Hint. More than one of these may be correct!

(f) Now solve for N and S, and put your answer in the context of the problem.

A. Ammie bought 2.5 lbs of Nerds Gummy Clusters and 4.5 lbs of Starburst Jelly Beans.

B. Ammie bought 3.5 lbs of Nerds Gummy Clusters and 3.5 lbs of Starburst Jelly Beans.

C. Ammie bought 4.5 lbs of Nerds Gummy Clusters and 2.5 lbs of Starburst Jelly Beans.

D. Ammie bought 5.5 lbs of Nerds Gummy Clusters and 1.5 lbs of Starburst Jelly Beans.

Applications of Linear Equations (EQ2)

Activity 1.2.8 A chemist needs to mix two solutions to create a mixture consisting of 30% alcohol. She uses 20 liters of the first solution, which has a concentration of 21% alcohol. How many liters of the second solution (that is 45% alcohol) should she add to the first solution to create the mixture that is 30% alcohol?

Applications of Linear Equations (EQ2)

Remark 1.2.9 In the next couple of activities, we will explore linear applications in geometry.

Applications of Linear Equations (EQ2)

Activity 1.2.10 A rectangular field is ten times as long as it is wide. If the perimeter of the field is 1,870 feet, what are the dimensions of the field?

(a) Let's first think about the relationship between the length and width of the rectangular field. How can we express the length of the field (l) if we let w represent the width of the field?

A. $l = \frac{1}{10}w$

B. $w = 10l$

C. $l = 10w$

D. $w = \frac{1}{10}l$

(b) Now that we can represent both the length and the width of the field, how can we represent the perimeter of the field in terms of w?

A. $P = 10w + w + 10w + w$

B. $P = 20w + 2w$

C. $P = 11w$

D. $P = 22w$

E. $P = 10w^2$

(c) Use your expression from part (b) and the fact that we know the perimeter is 1,870 feet to create an equation. Then solve for w to find the width of the field.

A. $w = 850$ feet

B. $w = 14$ feet

C. $w = 13$ feet

D. $w = 85$ feet

(d) Use the value you got in part (c) to find the length of the field.

A. $l = 850$ feet

B. $l = 14$ feet

C. $l = 13$ feet

D. $l = 85$ feet

Applications of Linear Equations (EQ2)

Activity 1.2.11 Two sides of a triangle are 5 and 7 inches longer than the third side. If the perimeter measures 21 inches, find the length of each side.

Remark 1.2.12 As you can see, there are many applications of linear equations! This section only highlights some of the more common types of applications - have you seen other types of linear applications in your life?

1.3 Distance and Midpoint (EQ3)

Objectives

- Given two points, determine the distance between them and the midpoint of the line segment connecting them.

Distance and Midpoint (EQ3)

Activity 1.3.1 The points A and B are shown in the graph below. Use the graph to answer the following questions:

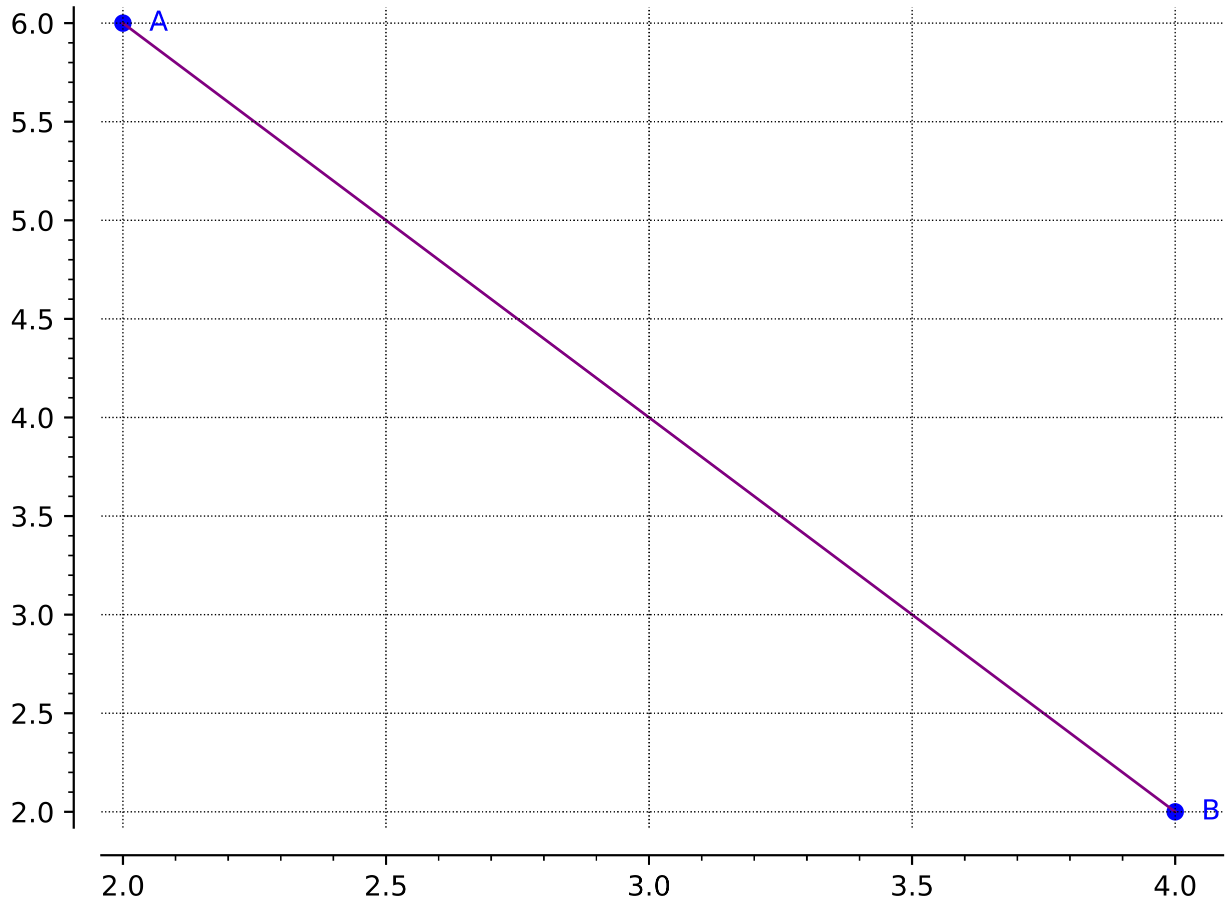


Figure 1.3.2

(a) Draw a right triangle so that the hypotenuse is the line segment between points A and B. Label the third point of the triangle C.

(b) Find the lengths of line segments AC and BC.

(c) Now that you know the lengths of AC and BC, how can you find the length of AB? Find the length of AB.

Distance and Midpoint (EQ3)

Remark 1.3.3 Using the **Pythagorean Theorem** ($a^2 + b^2 = c^2$) can be helpful in finding the distance of a line segment (as long as you create a right triangle!).

Distance and Midpoint (EQ3)

Activity 1.3.4 Suppose you are given two points (x_1, y_1) and (x_2, y_2). Let's investigate how to find the length of the line segment that connects these two points!

(a) Draw a sketch of a right triangle so that the hypotenuse is the line segment between the two points.

(b) Find the lengths of the legs of the right triangle.

(c) Find the length of the line segment that connects the two original points.

Distance and Midpoint (EQ3)

Definition 1.3.5 The distance, d, between two points, (x_1, y_1) and (x_2, y_2), can be found by using the **distance formula**:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Notice that the distance formula is an application of the Pythagorean Theorem! ◊

Distance and Midpoint (EQ3)

Activity 1.3.6 Apply Definition 1.3.5 to calculate the distance between the given points.

(a) What is the distance between $(4, 6)$ and $(9, 15)$?

A. 10.2

B. 10.3

C. $\sqrt{106}$

D. $\sqrt{56}$

(b) What is the distance between $(-2, 5)$ and $(-7, -1)$?

A. $\sqrt{11}$

B. 7.8

C. 3.3

D. $\sqrt{61}$

(c) Suppose the line segment AB has one endpoint, A, at the origin. For which coordinate of B would make the line segment AB the longest?

A. $(3, 7)$

B. $(2, -8)$

C. $(-6, 4)$

D. $(-5, -5)$

Remark 1.3.7 Notice in Activity 1.3.6, you can give a distance in either exact form (leaving it with a square root) or as an approximation (as a decimal). Make sure you can give either form as sometimes one form is more useful than another!

Distance and Midpoint (EQ3)

Remark 1.3.8 A **midpoint** refers to the point that is located in the middle of a line segment. In other words, the midpoint is the point that is halfway between the two endpoints of a given line segment.

Activity 1.3.9 Two line segments are shown in the graph below. Use the graph to answer the following questions:

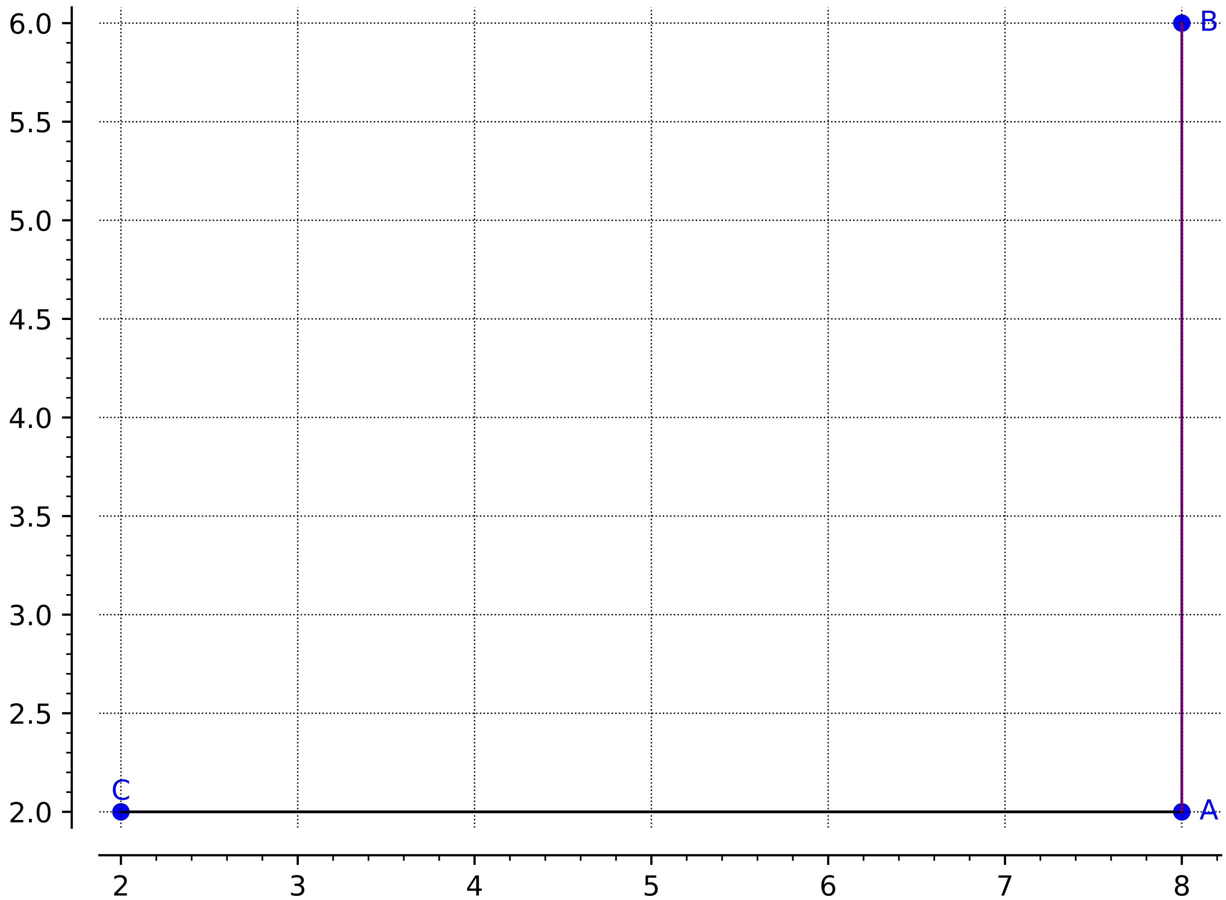


Figure 1.3.10

(a) What is the midpoint of the line segment AB?

A. $(16, 4)$

B. $(8, 4)$

C. $(8, 8)$

D. $(10, 2)$

(b) What is the midpoint of the line segment AC?

A. $(6, 0)$

B. $(4, 4)$

C. $(6, 4)$

D. $(5, 2)$

(c) Suppose we connect the two endpoints of the two line segments together, to create the new line segment, BC. Can you make an educated guess to where the midpoint of BC is?

A. $(10, 8)$

B. $(6, 4)$

C. $(5, 4)$

D. $(5, 2)$

(d) How can you test your conjecture? Is there a mathematical way to find the midpoint of any line segment?

Distance and Midpoint (EQ3)

Definition 1.3.11 The midpoint of a line segment with endpoints (x_1, y_1) and (x_2, y_2), can be found by taking the average of the x and y values. Mathematically, the **midpoint formula** states that the midpoint of a line segment can be found by:

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

$\diamondsuit$

Distance and Midpoint (EQ3)

Activity 1.3.12 Apply Definition 1.3.11 to calculate the midpoint of the following line segments.

(a) What is the midpoint of the line segment with endpoints $(-4, 5)$ and $(-2, -3)$?

A. $(3, 1)$

B. $(-3, 1)$

C. $(1, 1)$

D. $(1, 4)$

(b) What is the midpoint of the line segment with endpoints $(2, 6)$ and $(-6, -8)$?

A. $(-3, -1)$

B. $(-2, 0)$

C. $(-2, -1)$

D. $(4, 7)$

(c) Suppose C is the midpoint of AB and is located at $(9, 8)$. The coordinates of A are $(10, 10)$. What are the coordinates of B?

A. $(9.5, 9)$

B. $(11, 12)$

C. $(18, 16)$

D. $(8, 6)$

Distance and Midpoint (EQ3)

Activity 1.3.13 On a map, your friend Sarah's house is located at $(-2,5)$ and your other friend Austin's house is at $(6,-2)$.

(a) How long is the direct path from Sarah's house to Austin's house?

(b) Suppose your other friend, Micah, lives in the middle between Sarah and Austin. What is the location of Micah's house on the map?

1.4 Absolute Value Equations and Inequalities (EQ4)

Objectives

- Solve linear equations involving an absolute value. Solve linear inequalities involving absolute values and express the answers graphically and using interval notation.

Absolute Value Equations and Inequalities (EQ4)

Remark 1.4.1 An absolute value, written $|x|$, is the non-negative value of x. If x is a positive number, then $|x| = x$. If x is a negative number, then $|x| = -x$.

Absolute Value Equations and Inequalities (EQ4)

Activity 1.4.2 Let's consider how to solve an equation when an absolute value is involved.

(a) Which values are solutions to the absolute value equation $|x| = 2$?

A. $x = 2$

B. $x = 0$

C. $x = -1$

D. $x = -2$

(b) Which values are solutions to the absolute value equation $|x - 7| = 2$?

A. $x = 9$

B. $x = 7$

C. $x = 5$

D. $x = -9$

(c) Which values are solutions to the absolute value equation $3|x - 7| + 5 = 11$? It may be helpful to rewrite the equation to isolate the absolute value.

A. $x = 7$

B. $x = -9$

C. $x = 5$

D. $x = 9$

Absolute Value Equations and Inequalities (EQ4)

Activity 1.4.3 Absolute value represents the distance a value is from 0 on the number line. So, $|x - 7| = 2$ means that the expression $x - 7$ is 2 units away from 0.

(a) What values on the number line could $x - 7$ equal?

A. $x = -7$

B. $x = -2$

C. $x = 0$

D. $x = 2$

E. $x = 7$

(b) This gives us two separate equations to solve. What are those two equations?

A. $x - 7 = -7$

B. $x - 7 = -2$

C. $x - 7 = 0$

D. $x - 7 = 2$

E. $x - 7 = 7$

(c) Solve each equation for x.

Absolute Value Equations and Inequalities (EQ4)

Remark 1.4.4 When solving an absolute value equation, begin by isolating the absolute value expression. Then rewrite the equation into two linear equations and solve. If $c > 0$,

$$|ax + b| = c$$

becomes the following two equations

$$ax + b = c \quad \text{and} \quad ax + b = -c$$

Absolute Value Equations and Inequalities (EQ4)

Activity 1.4.5 Solve the following absolute value equations.

(a) $|3x+4|=10$

A. $\{-2,2\}$

B. $\left\{-\dfrac{14}{3},2\right\}$

C. $\{-10,10\}$

D. No solution

(b) $3|x-7|+5=11$

A. $\{-2,2\}$

B. $\{-9,9\}$

C. $\{5,9\}$

D. No solution

(c) $2|x+1|+8=4$

A. $\{-4,4\}$

B. $\{-6,6\}$

C. $\{5,7\}$

D. No solution

Remark 1.4.6 Since the absolute value represents a distance, it is always a positive number. Whenever you encounter an isolated absolute value equation equal to a negative value, there will be no solution.

Absolute Value Equations and Inequalities (EQ4)

Activity 1.4.7 Just as with linear equations and inequalities, we can consider absolute value inequalities from equations.

(a) Which values are solutions to the absolute value inequality $|x-7| \leq 2$?

A. $x = 9$

B. $x = 7$

C. $x = 5$

D. $x = -9$

(b) Rewrite the absolute value inequality $|x-7| \leq 2$ as a compound inequality.

A. $0 \leq x - 7 \leq 2$

B. $-2 \leq x - 7 \leq 2$

C. $-2 \leq x - 7 \leq 0$

D. $2 \leq x \leq 7$

(c) Solve the compound inequality that is equivalent to $|x-7| \leq 2$ found in part (b). Write the solution in interval notation.

A. $[7, 9]$

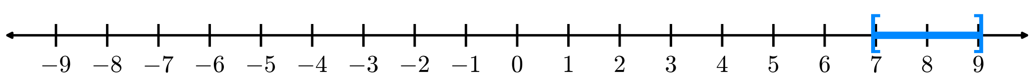
B. $[5, 9]$

C. $[5, 7]$

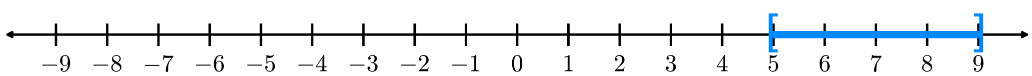
D. $[2, 7]$

(d) Draw the solution to $|x-7| \leq 2$ on the number line.

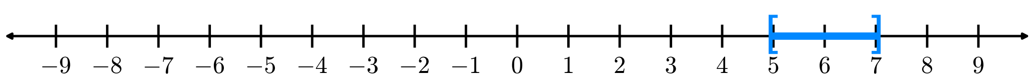
A.



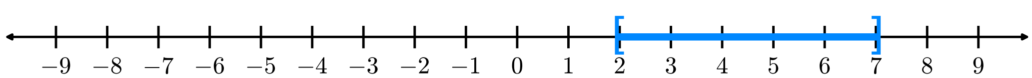
B.



C.



D.



Absolute Value Equations and Inequalities (EQ4)

Activity 1.4.8 Now let's consider another type of absolute value inequality.

(a) Which values are solutions to the absolute value inequality $|x - 7| \geq 2$?

A. $x = 9$

B. $x = 7$

C. $x = 5$

D. $x = -9$

(b) Which two of the following inequalities are equivalent to $|x - 7| \geq 2$.

A. $x - 7 \leq 2$

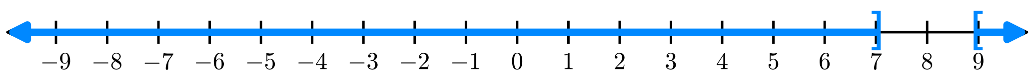
B. $x - 7 \leq -2$

C. $x - 7 \geq 2$

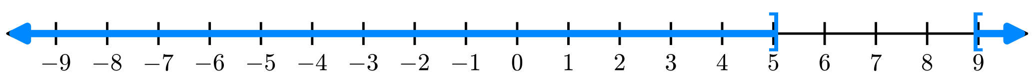
D. $x - 7 \geq -2$

(c) Solve the two inequalities found in part (b). Write the solution in interval notation and graph on the number line.

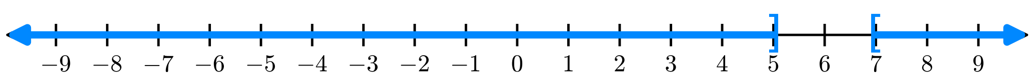
A. $(-\infty, 7] \cup [9, \infty)$



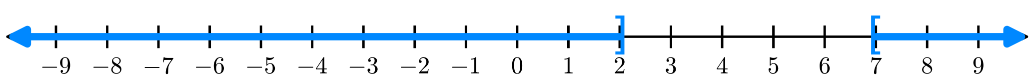
B. $(-\infty, 5] \cup [9, \infty)$



C. $(-\infty, 5] \cup [7, \infty)$



D. $(-\infty, 2] \cup [7, \infty)$



Absolute Value Equations and Inequalities (EQ4)

Definition 1.4.9 When solving an absolute value inequality, rewrite it as compound inequalities. Assume k is positive. $|x| < k$ becomes $-k < x < k$. $|x| > k$ becomes $x > k$ or $x < -k$.

$\Diamond$

Absolute Value Equations and Inequalities (EQ4)

Activity 1.4.10 Solve the following absolute value inequalities. Write your solution in interval notation and graph on a number line.

(a) $|3x + 4| < 10$

(b) $3|x - 7| + 5 > 11$

1.5 Quadratic Equations (EQ5)

Objectives

- Solve quadratic equations using factoring, the square root property, completing the square, and the quadratic formula and express these answers in exact form.

Quadratic Equations (EQ5)

Definition 1.5.1 A **quadratic equation** is an equation that can be written in the form

$$ax^2 + bx + c = 0,$$

where a, b, and c are real numbers and $a \neq 0$).

There are multiple ways to write a quadratic equation. When it is in the form above, we call it **standard form**. $\diamond$

Quadratic Equations (EQ5)

Activity 1.5.2 Which of the following is a quadratic equation? Select all that apply.

A. $6 - x^2 = 3x$

B. $(2x - 1)(x + 3) = 0$

C. $4(x - 3) + 7 = 0$

D. $(x - 4)^2 + 1 = 0$

E. $5x^2 - 3x = 17 - 4x$

Quadratic Equations (EQ5)

Definition 1.5.3 To solve a quadratic equation, one method we can use is called the **zero product property**. It states that if $a \cdot b = 0$, then either $a = 0$ or $b = 0$. In other words, you can only have a product of 0 if one (or both!) of the factors is 0. $\Diamond$

Quadratic Equations (EQ5)

Activity 1.5.4 In this activity, we will look at how to apply the zero product property when solving quadratic equations.

(a) In which of the following equations can you apply the zero product property (Definition 1.5.3) as your first step in solving?

A. $2x^2 - 3x + 1 = 0$

B. $(2x-1)(x-1) = 0$

C. $3x^2 - 4 = 6x$

D. $x(3x+5) = 4$

(b) Suppose you are given the quadratic equation, $(2x-1)(x-1) = 0$. Applying Definition 1.5.3 would give you:

A. $2x^2 - 3x + 1 = 0$

B. $(2x+1) = 0$ and $(x+1) = 0$

C. $(2x-1) = 0$ and $(x-1) = 0$

(c) After applying the zero product property, what are the solutions to the quadratic equation $(2x-1)(x-1) = 0$?

A. $x = -\dfrac{1}{2}$ and $x = 1$

B. $x = \dfrac{1}{2}$ and $x = -1$

C. $x = -\dfrac{1}{2}$ and $x = -1$

D. $x = \dfrac{1}{2}$ and $x = 1$

Remark 1.5.5 Notice in Activity 1.5.2 and Activity 1.5.4, that not all equations are set up "nicely." You will need to do some manipulation to get everything on one side (AND in factored form!) and 0 on the other ∗before∗ applying the zero product property.

Quadratic Equations (EQ5)

Activity 1.5.6 Suppose you want to solve the equation $x^2 - 3x - 10 = 0$, which is NOT in factored form.

(a) Which of the following is the correct factored form of $x^2 - 3x - 10 = 0$?

A. $(x + 2)(x + 5) = 0$

B. $(x - 2)(x - 5) = 0$

C. $(x - 2)(x + 5) = 0$

D. $(x + 2)(x - 5) = 0$

(b) After applying Definition 1.5.3, which of the following will be a solution to $x^2 - 3x - 10 = 0$?

A. $x = 2$ and $x = 5$

B. $x = -2$ and $x = -5$

C. $x = 2$ and $x = -5$

D. $x = -2$ and $x = 5$

Quadratic Equations (EQ5)

Activity 1.5.7 Solve each of the following quadratic equations:

(a) $(2x - 5)(x + 7) = 0$

(b) $3x(4x - 1) = 0$

(c) $x^2 - 2x - 24 = 0$

(d) $3x^2 - 14x - 5 = 0$

(e) $6 - x^2 = 5x$

Quadratic Equations (EQ5)

Activity 1.5.8 Suppose you are given the equation, $x^2 = 9$:

(a) How many solutions does this equation have?

A. 0 B. 1 C. 2 D. 3

(b) What are the solutions to this equation?

A. $x = 0$ B. $x = 3$ C. $x = 9, -9$ D. $x = 3, -3$

(c) How is this quadratic equation different than the equations we've solved thus far?

Quadratic Equations (EQ5)

Definition 1.5.9 The **square root property** states that a quadratic equation of the form

$$x^2 = k^2$$

(where k is a nonzero number) will give solutions $x = k$ and $x = -k$. In other words, if we have an equation with a perfect square on one side and a number on the other side, we can take the square root of both sides to solve the equation. $\Diamond$

Quadratic Equations (EQ5)

Activity 1.5.10 Suppose you are given the equation, $3x^2 - 8 = 4$:

(a) What would be the first step in solving $3x^2 - 8 = 4$?

A. Divide by 3 on both sides

B. Subtract 4 on both sides

C. Add 8 on both sides

D. Multiply by 3 on both sides

(b) Isolate the x^2 term and apply Definition 1.5.9 to solve for x.

(c) What are the solution(s) to $3x^2 - 8 = 4$?

A. $x = 6, -6$

B. $x = 2, -2$

C. $x = 0$

D. $x = 2$

Quadratic Equations (EQ5)

Activity 1.5.11 Solve the following quadratic equations by applying the square root property (Definition 1.5.9).

(a) $3x^2 + 1 = 28$

(b) $5x^2 + 7 = 47$

(c) $2x^2 = -144$

Hint. Recall that when you have a negative number under a square root, that gives an imaginary number ($\sqrt{-1} = i$).

(d) $(x + 2)^2 + 3 = 19$

Hint. Isolate the binomial $(x + 2)$ first.

(e) $3(x - 4)^2 = 15$

Quadratic Equations (EQ5)

Remark 1.5.12 Not all quadratic equations can be factored or can be solved by using the square root property. In the next few activities, we will learn two additional methods in solving quadratics.

Quadratic Equations (EQ5)

Definition 1.5.13 Another method for solving a quadratic equation is known as **completing the square**. With this method, we add or subtract terms to both sides of an equation until we have a perfect square trinomial on one side of the equal sign and a constant on the other side. We then apply the square root property. Note: A perfect square trinomial is a trinomial that can be factored into a binomial squared. For example, $x^2 + 4x + 4$ is a perfect square trinomial because it can be factored into $(x + 2)(x + 2)$ or $(x + 2)^2$. $\diamondsuit$

Quadratic Equations (EQ5)

Activity 1.5.14 Let's work through an example together to solve $x^2 + 6x = 4$. (Notice that the methods of factoring and the square root property do not work with this equation.)

(a) In order to apply Definition 1.5.13, we first need to have a perfect square trinomial on one side of the equal sign. Which of the following number(s) could we add to the left side of the equation to create a perfect square trinomial?

A. 4

B. 9

C. -9

D. 2

(b) Add your answer from part *a* to the right side of the equation as well (i.e. whatever you do to one side of an equation you must do to the other side too!) and then factor the perfect square trinomial on the left side. Which equation best represents the equation now?

A. $(x + 3)^2 = -5$

B. $(x - 3)^2 = 13$

C. $(x + 3)^2 = 13$

D. $(x - 3)^2 = -5$

(c) Apply the square root property (Definition 1.5.9) to both sides of the equation to determine the solution(s). Which of the following is the solution(s) of $x^2 + 6x = 4$?

A. $3 + \sqrt{13}$ and $3 - \sqrt{13}$

B. $-3 + \sqrt{13}$ and $-3 - \sqrt{13}$

C. $3 + \sqrt{5}$ and $3 - \sqrt{5}$

D. $-3 + \sqrt{5}$ and $-3 - \sqrt{5}$

Remark 1.5.15 To complete the square, the leading coefficient, a (i.e., the coefficient of the x^2 term), must equal 1. If it does not, then factor the entire equation by a and then follow similar steps as in Activity 1.5.14.

Quadratic Equations (EQ5)

Activity 1.5.16 Let's solve the equation $2x^2 + 8x - 6 = 0$ by completing the square.

(a) Rewrite the equation so that all the terms with the variable x is on one side of the equation and a constant is on the other.

(b) Notice that the coefficient of the x^2 term is not 1. What could we factor the left side of the equation by so that the coefficient of the x^2 is 1?

(c) Once you factor the left side, what equation represents the equation you now have?

A. $2(x^2 - 8x) = -6$

B. $2(x^2 - 4x) = -6$

C. $2(x^2 + 4x) = 6$

D. $2(x^2 + 8x) = 6$

(d) Just like in Activity 1.5.14, let's now try and create the perfect square trinomial (inside the parentheses) on the left side of the equation. Which of the following number(s) could we add to the left side of the equation to create a perfect square trinomial?

A. 4 B. 8 C. -8 D. 2

(e) What would we need to add to the right-hand side of the equation to keep the equation balanced?

A. 4 B. 8 C. -8 D. 2

(f) Which of the following equations represents the quadratic equation you have now?

A. $2(x + 2)^2 = 9$

B. $2(x - 2)^2 = 2$

C. $2(x + 2)^2 = 14$

D. $2(x - 2)^2 = 14$

(g) Apply the square root property and solve the quadratic equation.

Quadratic Equations (EQ5)

Activity 1.5.17 Solve the following quadratic equations by completing the square.

(a) $x^2 - 12x = -11$

(b) $x^2 + 2x - 33 = 0$

(c) $5x^2 + 29x = 6$

Quadratic Equations (EQ5)

Definition 1.5.18 Another method for solving quadratic equations is called the **quadratic formula** - a formula that will solve all quadratic equations!

A quadratic equation of the form $ax^2+bx+c=0$ can be solved by the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

where a, b, and c are real numbers and $a \neq 0$. ◊

Quadratic Equations (EQ5)

Activity 1.5.19 Use the quadratic formula (Definition 1.5.18) to solve $x^2 + 4x = -3$.

(a) When written in standard form, what are the values of a, b, and c?

Hint. Rewrite the equation so that your a-value is positive.

(b) When applying the quadratic formula, what would you get when you substitute a, b, and c?

A. $x = \dfrac{-4 \pm \sqrt{4^2 - 4(1)(3)}}{2(1)}$

B. $x = \dfrac{4 \pm \sqrt{(-4)^2 - 4(1)(3)}}{2(1)}$

C. $x = \dfrac{-4 \pm \sqrt{(-4)^2 - 4(1)(-3)}}{2(1)}$

D. $x = \dfrac{4 \pm \sqrt{4^2 - 4(1)(-3)}}{2(1)}$

(c) What is the solution(s) to $x^2 + 4x = -3$?

A. $x = -1, 3$ B. $x = 1, 3$ C. $x = -1, -3$ D. $x = 1, -3$

Quadratic Equations (EQ5)

Activity 1.5.20 Use the quadratic formula (Definition 1.5.18) to solve $2x^2 - 13 = 7x$.

(a) When written in standard form, what are the values of a, b, and c?

Hint. Rewrite the equation so that your a-value is positive.

(b) When applying the quadratic formula, what would you get when you substitute a, b, and c?

A. $x = \dfrac{-7 \pm \sqrt{7^2 - 4(2)(13)}}{2(1)}$

B. $x = \dfrac{7 \pm \sqrt{(-7)^2 - 4(2)(-13)}}{2(2)}$

C. $x = \dfrac{-7 \pm \sqrt{(-7)^2 - 4(1)(-13)}}{2(1)}$

D. $x = \dfrac{7 \pm \sqrt{7^2 - 4(2)(-13)}}{2(2)}$

(c) What is the solution(s) to $2x^2 - 13 = 7x$?

A. $x = \dfrac{7 + \sqrt{73}}{4}$ and $\dfrac{7 - \sqrt{73}}{4}$

B. $x = \dfrac{7 + 3\sqrt{17}}{4}$ and $\dfrac{7 - 3\sqrt{17}}{4}$

C. $x = \dfrac{-7 + \sqrt{55}}{4}$ and $\dfrac{-7 - \sqrt{55}}{4}$

D. $x = \dfrac{-7 + \sqrt{155}}{4}$ and $\dfrac{-7 - \sqrt{155}}{4}$

Quadratic Equations (EQ5)

Activity 1.5.21 Use the quadratic formula (Definition 1.5.18) to solve $x^2 = 6x - 12$.

(a) When written in standard form, what are the values of a, b, and c?

Hint. Rewrite the equation so that your a-value is positive.

(b) When applying the quadratic formula, what would you get when you substitute a, b, and c?

A. $x = \dfrac{-6 \pm \sqrt{6^2 - 4(1)(12)}}{2(1)}$

B. $x = \dfrac{6 \pm \sqrt{(-6)^2 - 4(1)(12)}}{2(1)}$

C. $x = \dfrac{-6 \pm \sqrt{(-6)^2 - 4(1)(-12)}}{2(1)}$

D. $x = \dfrac{6 \pm \sqrt{6^2 - 4(1)(-12)}}{2(1)}$

(c) Notice that the number under the square root is a negative. Recall that when you have a negative number under a square root, that gives an imaginary number ($\sqrt{-1} = i$). What is the solution(s) to $x^2 = 6x - 12$?

A. $x = 3 + i\sqrt{3}$ and $3 - i\sqrt{3}$

B. $x = 6 + i\sqrt{3}$ and $6 - i\sqrt{3}$

C. $x = -3 + i\sqrt{3}$ and $-3 - i\sqrt{3}$

D. $x = 3 + 2i\sqrt{3}$ and $3 - 2i\sqrt{3}$

Quadratic Equations (EQ5)

Activity 1.5.22 Solve the following quadratic equations by applying the quadratic formula (Definition 1.5.18).

(a) $2x^2 - 3x = 5$

(b) $4x^2 - 1 = -8x$

(c) $2x^2 - 7x - 13 = -10$

(d) $2x^2 - 4x + 3 = 0$

Quadratic Equations (EQ5)

Activity 1.5.23 Now that you have seen all the different ways to solve a quadratic equation, you will need to know WHEN to use which method. Are some methods better than others?

(a) Which is the best method to use to solve $5x^2 = 80$?

A. Factoring and Zero Product Property
B. Square Root Property
C. Completing the Square
D. Quadratic Formula

(b) Which is the best method to use to solve $5x^2 + 9x = -4$?

A. Factoring and Zero Product Property
B. Square Root Property
C. Completing the Square
D. Quadratic Formula

(c) Which is the best method to use to solve $3x^2 + 9x = 0$?

A. Factoring and Zero Product Property
B. Square Root Property
C. Completing the Square
D. Quadratic Formula

(d) Go back to parts a, b, and c and solve each of the quadratic equations. Would you still use the same method?

Chapter 2

Functions (FN)

Objectives

How do we express relationships between two quantities?
By the end of this chapter, you should be able to...

1. Determine if a relation, equation, or graph defines a function using the definition as well as the vertical line test.
2. Use and interpret function notation to evaluate a function for a given input value and find a corresponding input value given an output value.
3. Use the graph of a function to find the domain and range in interval notation, the x- and y-intercepts, the maxima and minima, and where it is increasing and decreasing using interval notation.
4. Apply transformations including horizontal and vertical shifts, stretches, and reflections to a function. Express the result of these transformations graphically and algebraically.
5. Find the sum, difference, product, quotient, and composition of two or more functions and evaluate them.
6. Find the inverse of a one-to-one function.

2.1 Introduction to Functions (FN1)

Objectives

- Determine if a relation, equation, or graph defines a function using the definition as well as the vertical line test.

Introduction to Functions (FN1)

Definition 2.1.1 A **relation** is a relationship between sets of values. Relations in mathematics are usually represented as ordered pairs: (input, output) or (x, y). When observing relations, we often refer to the x-values as the **domain** and the y-values as the **range**. $\Diamond$

Definition 2.1.2 Mapping Notation (also known as an arrow diagram) is a way to show relationships visually between sets. For example, suppose you are given the following ordered pairs: $(3,-8),(4,6)$, and $(2,-1)$. Each of the x-values "map onto" a y-value and can be visualized in the following way:

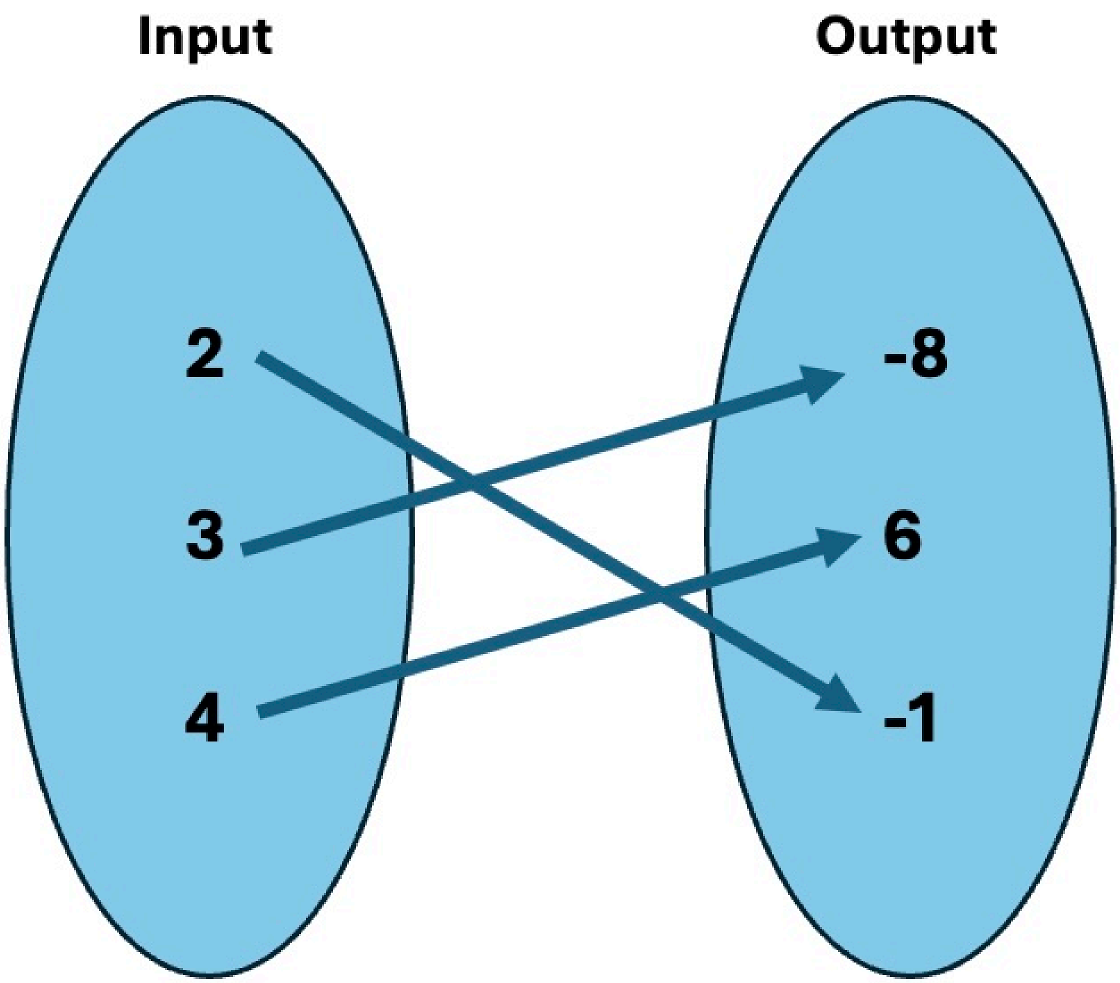


Figure 2.1.3 Every x-value from the ordered pair list is listed in the input set and every y-value is listed in the output set. An arrow is drawn from every x-value to its corresponding y-value.

Notice that an arrow is used to indicate which x-value is mapped onto its corresponding y-value. ◊

Introduction to Functions (FN1)

Activity 2.1.4 Use mapping notation to create a visual representation of the following relation.

$$(-1, 5), (2, 6), (4, -2)$$

(a) What is the domain?

A. $\{5, 6, -2\}$

B. $\{-1, 2, 4\}$

C. $\{-2, -1, 2, 4, 5, 6\}$

(b) What is the range?

A. $\{5, 6, -2\}$

B. $\{-1, 2, 4\}$

C. $\{-2, -1, 2, 4, 5, 6\}$

Activity 2.1.5 Use mapping notation to create a visual representation of the following relation.

$$(6,4),(3,4),(6,5)$$

(a) What is the domain?

A. $\{3,6\}$

B. $\{6,3,6\}$

C. $\{3,4,5,6\}$

D. $\{4,5\}$

(b) What is the range?

A. $\{3,6\}$

B. $\{6,3,6\}$

C. $\{3,4,5,6\}$

D. $\{4,5\}$

Activity 2.1.6 Use mapping notation to create a visual representation of the following relation.

$$(1, 2), (-5, 2), (-7, 2)$$

(a) What is the domain?

A. $\{2, 2, 2\}$
B. $\{-7, -5, 1, 2\}$
C. $\{-7, -5, 1\}$
D. $\{2\}$

(b) What is the range?

A. $\{2, 2, 2\}$
B. $\{-7, -5, 1, 2\}$
C. $\{-7, -5, 1\}$
D. $\{2\}$

Activity 2.1.7 Use mapping notation to create a visual representation of the following relation.

$$(3, -2), (-4, -1), (3, 5)$$

(a) What is the domain?

A. $\{-4, -2, -1, 3, 5\}$

B. $\{-4, 3\}$

C. $\{-2, -1, 5\}$

D. $\{-4, 3, 3\}$

(b) What is the range?

A. $\{-4, -2, -1, 3, 5\}$

B. $\{-4, 3\}$

C. $\{-2, -1, 5\}$

D. $\{-4, 3, 3\}$

Activity 2.1.8 Now that you have created a visual representation for each of Activity 2.1.4, Activity 2.1.5, Activity 2.1.6, and Activity 2.1.7, what similarities do you see? What differences?

Activity 2.1.9 Consider the menus shown below for two different fast food restaurants. We'll consider the Items as inputs (x-values) and the Cost as outputs (y-values).

McRonald's Item	**Cost**
Nuggets	\$5
Burger	\$6
Fries	\$3
Nachos	\$6

Burger Queen's Item	**Cost**
Nuggets	\$4
Burger	\$8
Fries	\$2
Nuggets	\$7

(a) How much would it cost to get nuggets and fries at McRonald's?

A. \$8 B. \$9 C. \$11 D. \$12

(b) How much would it cost to get nuggets and fries at Burger Queen?

A. \$6 B. \$9 C. \$10 D. \$12

(c) Notice that burgers and nachos cost the same amount at McRonald's. Is that an issue? Explain your reasoning.

(d) Notice on the Burger Queen menu that nuggets cost \$4 in one spot but \$7 in another spot. Is that an issue? Explain your reasoning.

Introduction to Functions (FN1)

Definition 2.1.10 A **function** is a relation where every input (or x-value) is mapped onto *exactly one* output (or y-value).

Note that all functions are relations but not all relations are functions! $\Diamond$

Remark 2.1.11 We see that in Activity 2.1.9, the McRonald's menu is a function. There is no confusion in determining the cost of a given menu item. However, Burger Queen's menu is not a function. There is a discrepancy in the cost of nuggets, and the menu could cause confusion.

Introduction to Functions (FN1)

Activity 2.1.12 Relations can be expressed in multiple ways (ordered pairs, tables, and verbal descriptions).

(a) Let's revisit some of the sets of ordered pairs we've previously explored in Activity 2.1.4, Activity 2.1.5, and Activity 2.1.6. Which of the following sets of ordered pairs represent a function?

A. $(-1, 5), (2, 6), (4, -2)$

B. $(6, 4), (3, 4), (6, 5)$

C. $(1, 2), (-5, 2), (-7, 2)$

D. $(-1, 2), (-1, 9), (1, 9)$

(b) Note that relations can be expressed in a table. A table of values is shown below. Is this an example of a function? Why or why not?

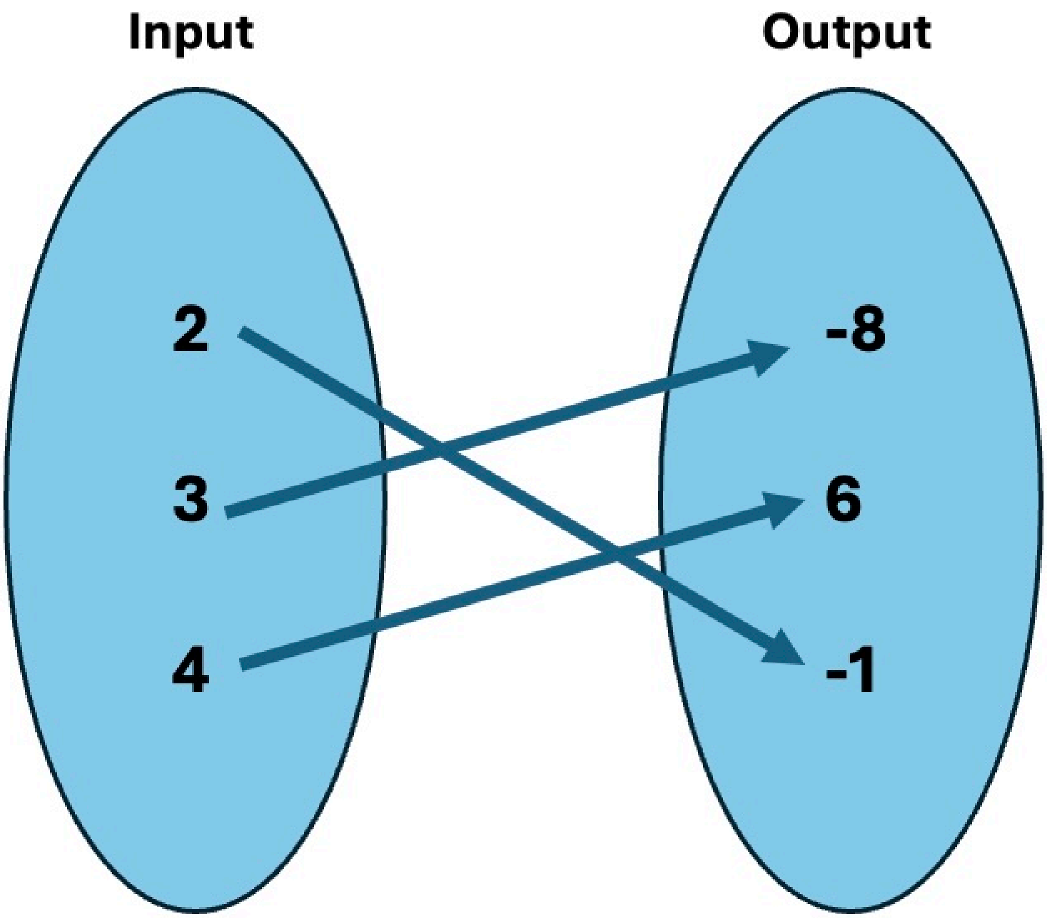
x	y
-5	-2
-4	-5
-2	8
8	-4
8	1

(c) Relations can also be expressed in words. Suppose you are looking at the amount of time you spend studying versus the grade you earn in your Algebra class. Is this an example of a function? Why or why not?

Remark 2.1.13 Notice that when trying to determine if a relation is a function, we often have to rely on looking at the domain and range values. Thus, it is important to be able to idenfity the domain and range of any relation!

Activity 2.1.14 For each of the given functions, determine the domain and range.

(a) $(-4, 3), (-1, 8), (7, 4), (1, 9)$



(b)

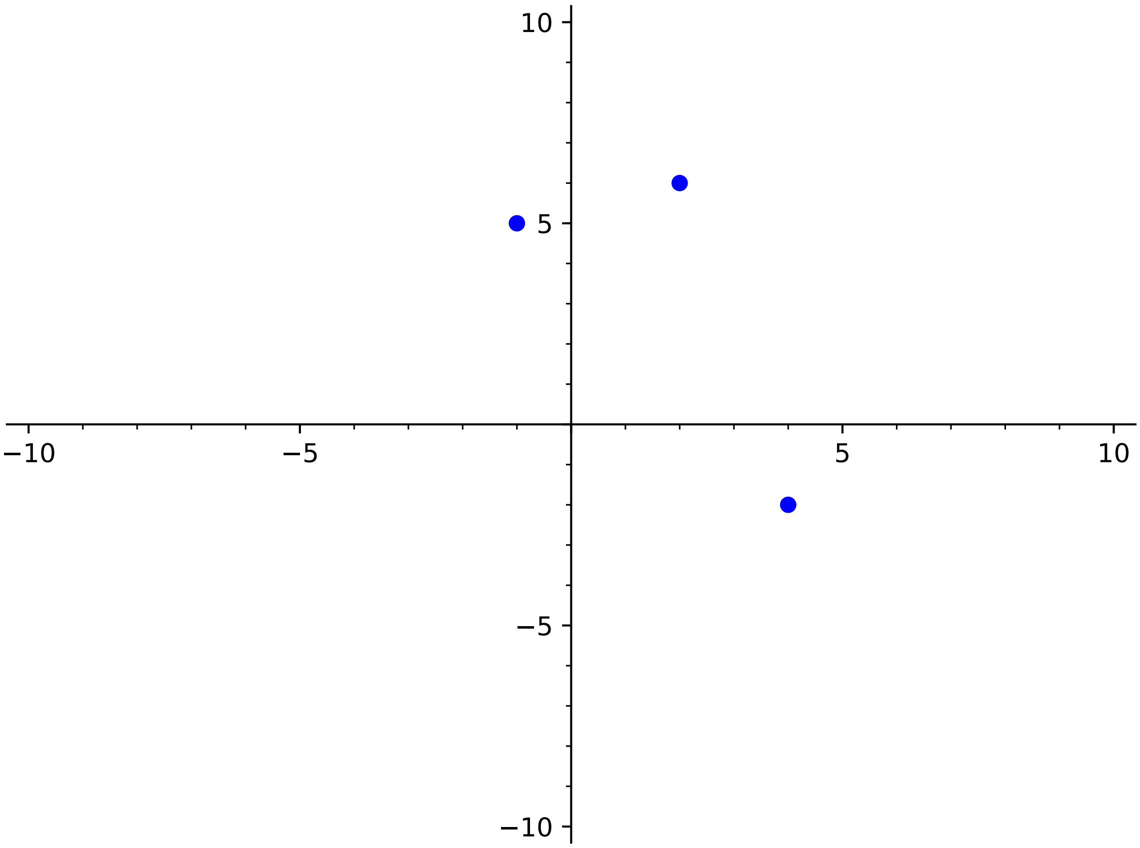
(c)

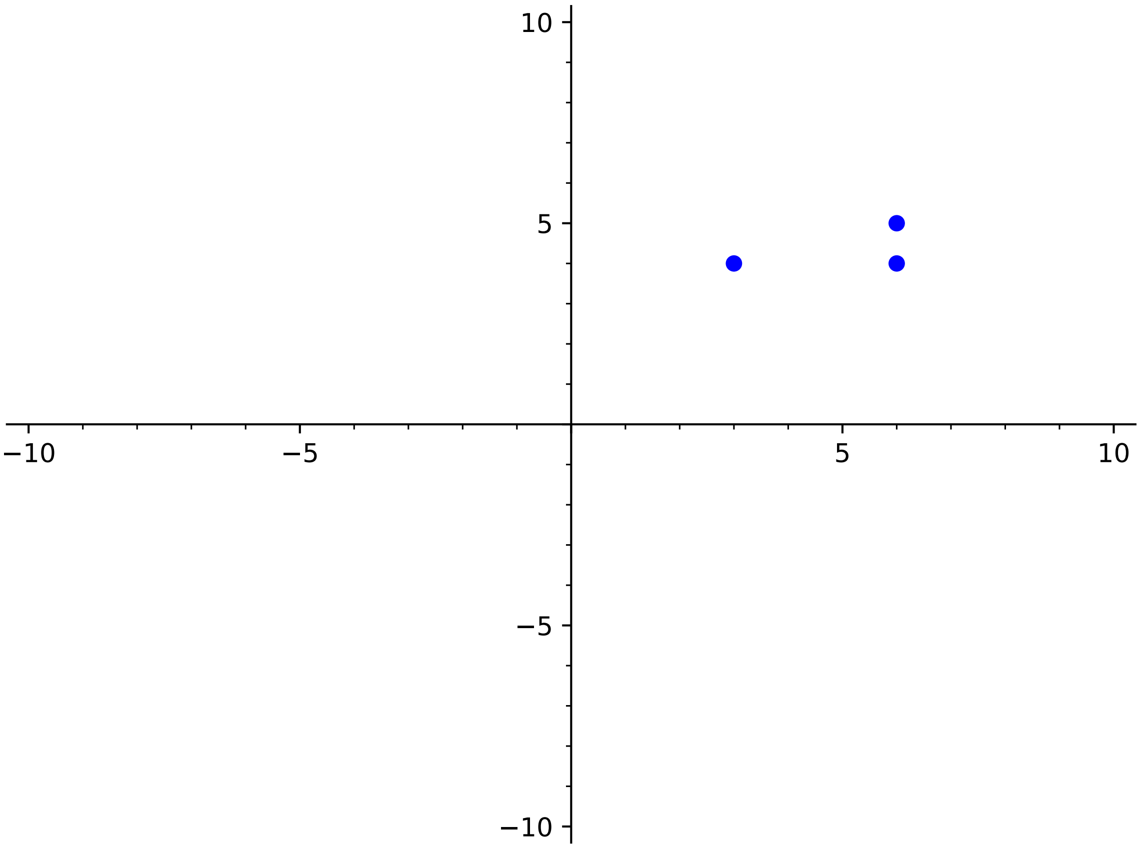
x	y
-2	5
0	4
3	6
8	1

(d) The amount of time you spend studying versus the grade you earn in your Algebra class.

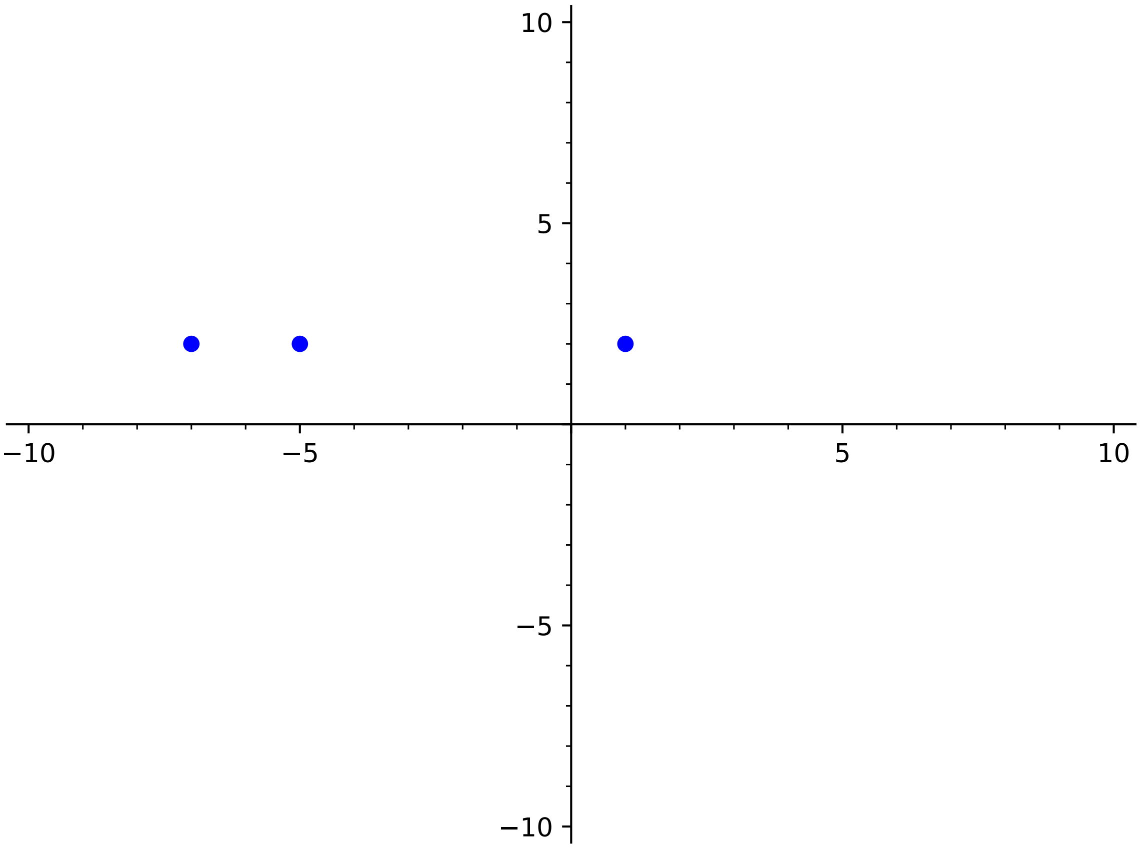
Activity 2.1.15 Determine whether each of the following relations is a function.

(a)





(b)



(c)

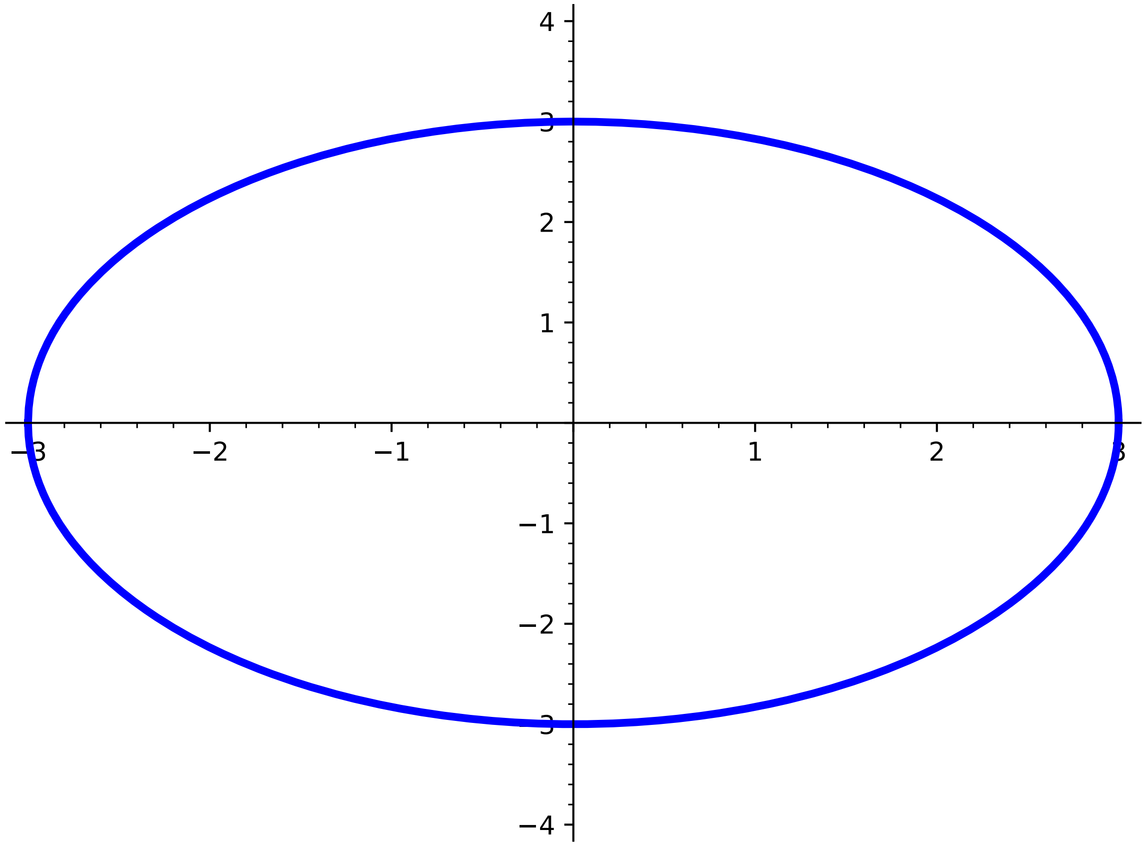
Remark 2.1.16 You probably noticed (in Activity 2.1.15) that when the graph has points that ”line up” or are on top of each other, they have the same x-values. When this occurs, this shows that the same x-value has two different outputs (y-values) and that the relation is not a function.

Definition 2.1.17 The **vertical line test** is a method used to determine whether a relation on a graph is a function.

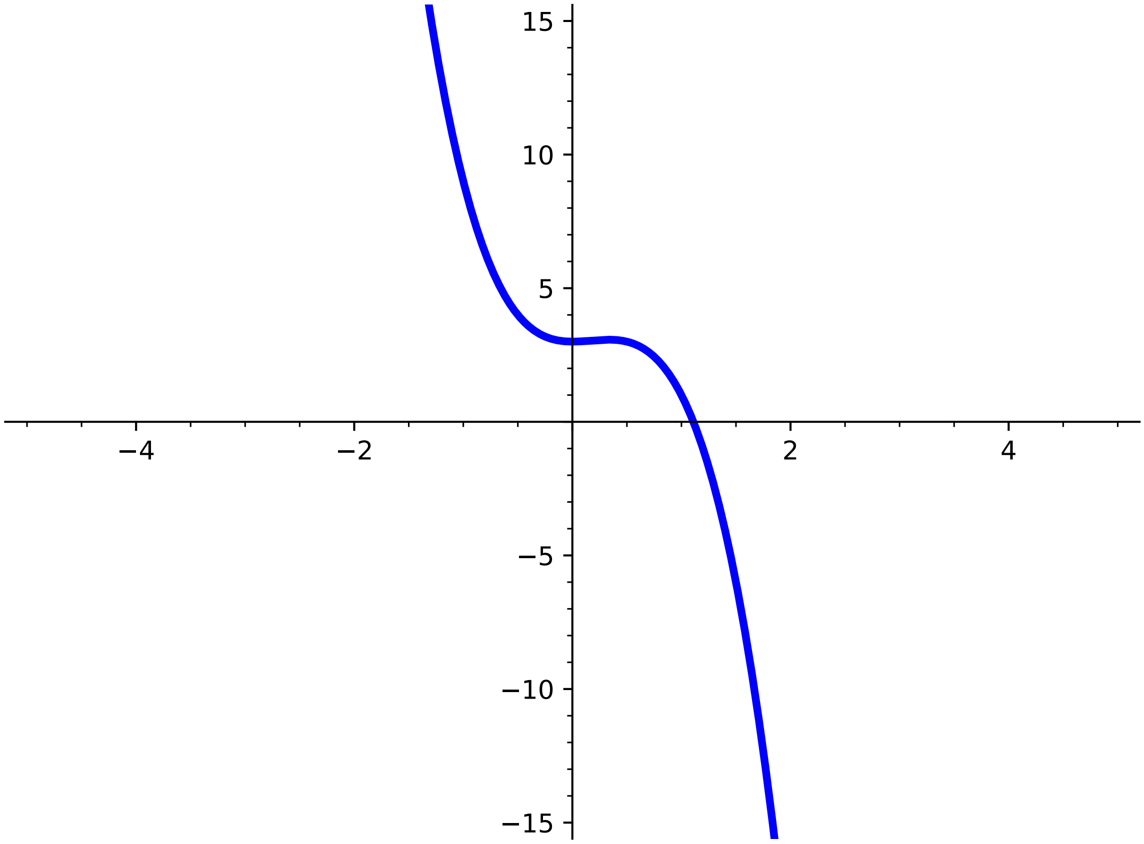
Start by drawing a vertical line anywhere on the graph and observe the number of times the relation on the graph intersects with the vertical line. If every possible vertical line intersects the graph at only one point, then the relation is a function. If, however, the graph of the relation intersects a vertical line more than once (anywhere on the graph), then the relation is not a function. ♢

Activity 2.1.18 Use the vertical line test (Definition 2.1.17) to determine whether each graph of a relation represents a function.

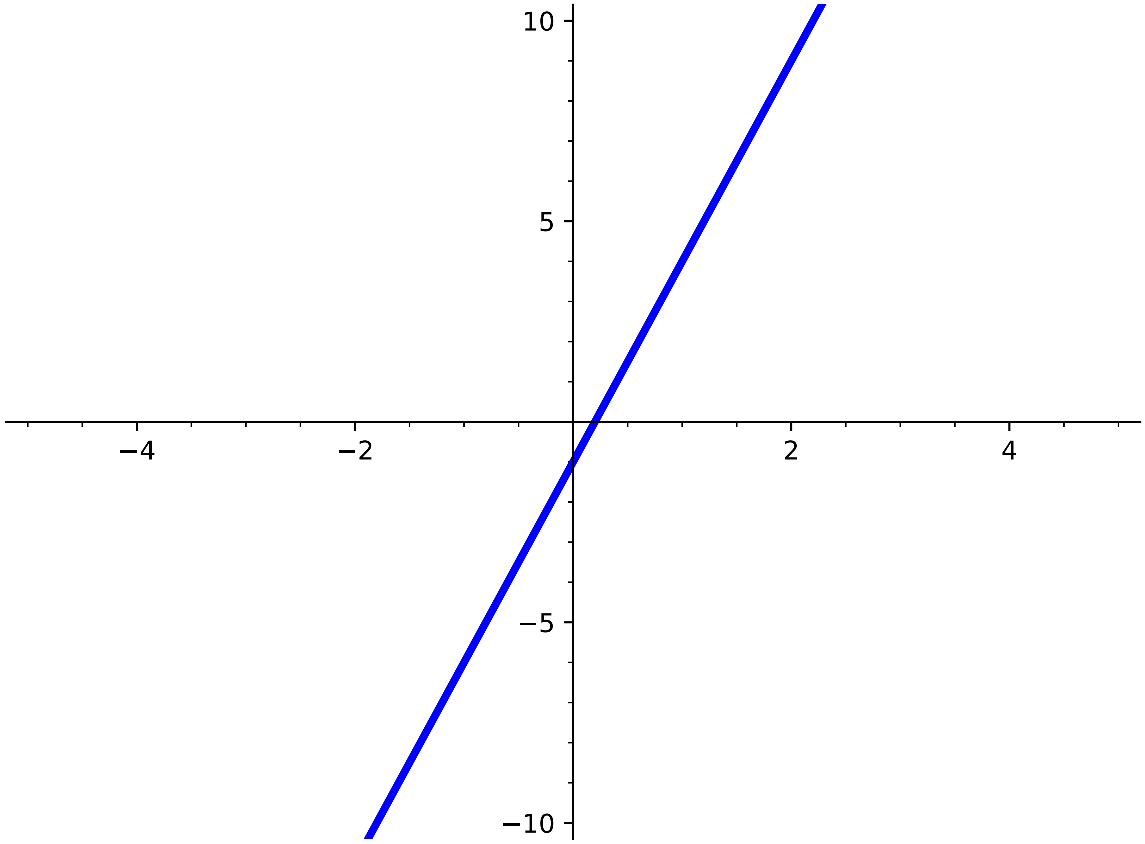
(a)



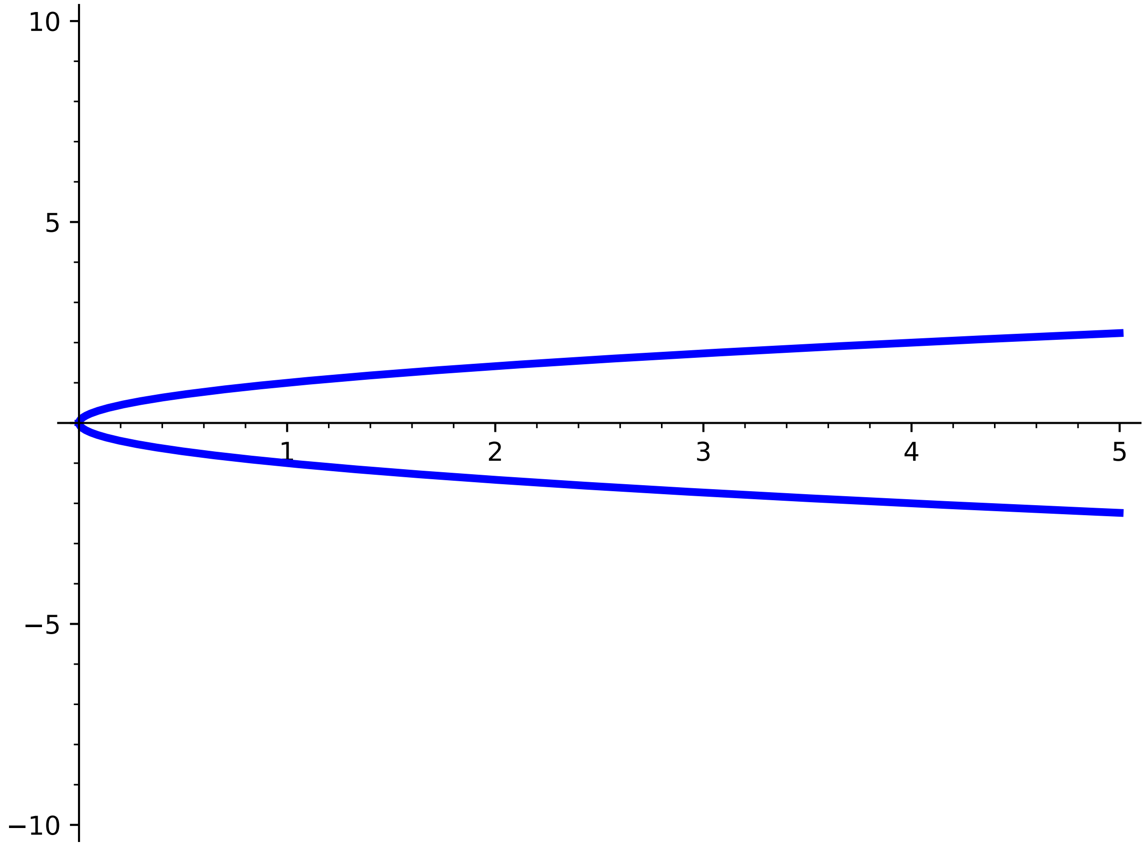
(b)



(c)



(d)



Introduction to Functions (FN1)

Activity 2.1.19 Let's explore how to determine whether an equation represents a function.

(a) Suppose you are given the equation $x = y^2$.

- If $x = 4$, what kind of y-values would you get for $x = y^2$?
- Based on this information, do you think $x = y^2$ is a function?

(b) Suppose you are given the equation $y = 3x^2 + 2$.

- If $x = 4$, what kind of y-values would you get for $y = 3x^2 + 2$?
- Based on this information, do you think $y = 3x^2 + 2$ is a function?

(c) Suppose you are given the equation $x^2 + y^2 = 25$.

- If $x = 4$, what kind of y-values would you get for $x^2 + y^2 = 25$?
- Based on this information, do you think $x^2 + y^2 = 25$ is a function?

(d) Suppose you are given the equation $y = -4x - 3$.

- If $x = 4$, what kind of y-values would you get for $y = -4x - 3$?
- Based on this information, do you think $y = -4x - 3$ is a function?

(e) How can you look at an equation to determine whether or not it is a function?

Remark 2.1.20 Notice that Activity 2.1.19 shows that equations with a y^2 term generally do not define functions. This is because to solve for a squared variable, you must consider both positive and negative inputs. For example, both $2^2 = 4$ and $(-2)^2 = 4$.

Introduction to Functions (FN1)

Activity 2.1.21 It's important to be able to determine the domain of any equation, especially when thinking about functions. Answer the following questions given the equation $y = \sqrt{x-2}$.

(a) Which of the following values of x would cause y to be undefined (if any)?

A. -2

B. 0

C. 2

D. 4

E. none of the above

(b) Based on this information, for which of the following values of x would y be defined?

A. -2

B. 0

C. 2

D. 4

E. none of the above

(c) There are many more x- values than just those found above that, when plugged in, give a defined value for y. How can we represent the domain of this equation in interval notation?

A. $(-\infty, 2)$

B. $(-\infty, 2]$

C. $[2, \infty)$

D. $(2, \infty)$

E. $(-\infty, \infty)$

Activity 2.1.22 Answer the following questions given the equation $y = -5x + 1$.

(a) Which of the following values of x would cause y to be undefined (if any)?

A. -2

B. 0

C. 4

D. -5

E. none of the above

(b) Based on this information, for which of the following values of x would y be defined?

A. -2

B. 0

C. 4

D. -5

E. none of the above

(c) How can we represent the domain of this equation in interval notation?

A. $(-\infty, 0)$

B. $(0, \infty)$

C. $(-5, 1)$

D. $(-\infty, \infty)$

Introduction to Functions (FN1)

Activity 2.1.23 Answer the following questions given the equation $y = \dfrac{3}{x-5}$.

(a) Which of the following values of x would cause y to be undefined (if any)?

A. -3

B. 0

C. -4

D. 5

E. none of the above

(b) Based on this information, for which of the following values of x would y be defined?

A. -3

B. 0

C. -4

D. 5

E. none of the above

(c) How can we represent the domain of this equation in interval notation?

A. $(-\infty, 5)$

B. $(5, \infty)$

C. $(-5, 5)$

D. $(-\infty, 5)\text{U}(5, \infty)$

Activity 2.1.24 Find the domain of each of the following functions. Write your answer in interval notation.

(a) $f(x) = \dfrac{x+3}{(x-2)(x+5)}$

(b) $f(x) = \sqrt{2x-5}$

(c) $f(x) = \dfrac{2}{\sqrt{4-x}}$

Hint. Notice that this function has both a denominator and a root to consider!

Remark 2.1.25 When determining the domain of an equation, it is often easier to first find values of x that make the function undefined. Once you have those values, then you know that x can be any value but those.

2.2 Function Notation (FN2)

Objectives

- Use and interpret function notation to evaluate a function for a given input value and find a corresponding input value given an output value.

Function Notation (FN2)

Remark 2.2.1 As we saw in the last section, we can represent functions in many ways, like using a set of ordered pairs, a graph, a description, or an equation. When describing a function with an equation, we will often use function notation.

If y is written as a function of x, like in the equation

$$y = x + 5,$$

we can replace the y with $f(x)$ and get the function notation

$$f(x) = x + 5.$$

The x is the input variable, and $f(x)$ is the y-value or output that corresponds to x.

Generally, we use the letter f for functions. Other letters are okay as well; $g(x)$ and $h(x)$ are common. If we are using multiple functions at one time, we often denote them with different letters so we can refer to one without any confusion as to which function we mean.

Function Notation (FN2)

Activity 2.2.2 Rewrite the following equations using function notation. In each case, assume y is a function of the variable x.

(a) $y = 2x + 14$

(b) $y + x = 3x^2 - 5$

(c) $\frac{2}{x} - x^4 = y - 5$

Function Notation (FN2)

Activity 2.2.3 Let $f(x) = 3x^2 - 4x + 1$. Find the value of $f(x)$ for the given values of x.

Table 2.2.4

x	$f(x)$
-5	
$-\frac{1}{2}$	
0	
2	
10	

Function Notation (FN2)

Remark 2.2.5 If we are asked to find the value of $f(x)$ for a certain x-value, say $x = 5$, we use the notation $f(5)$ to indicate that.

Function Notation (FN2)

Activity 2.2.6 A projectile is fired into the air from an initial height of 144 feet. The height $h(t)$ after t seconds is defined by the function

$$h(t) = -16t^2 + 128t + 144.$$

(a) Find $h(4)$.

A. 400

B. 912

C. $-64\,t^2 + 512\,t + 576$

D. $-64t^2 + 128t + 144$

(b) Describe what you've just found using the context of the situation.

A. After 400 seconds, the height of the projectile is 4 feet.

B. After 4 seconds, the height of the projectile is 400 feet.

C. After 4 feet, the height of the projectile is 400 seconds.

D. After 400 feet, the height of the projectile is 4 seconds.

Function Notation (FN2)

Activity 2.2.7 Let $f(x)$, $g(x)$, and $h(x)$ be defined as shown.

$$
\begin{aligned}
f(x) &= 3x^2 - 4x + 1 \\
g(x) &= \sqrt{13 - x^2} \\
h(x) &= \frac{x^2 - 6x + 8}{x^2 - 4x + 3}
\end{aligned}
$$

Find the following, if they exist.

(a) $f(-4)$, $f(0)$, and $f(2)$

(b) $g(0)$, $g(2)$, and $g(8)$

(c) $h(3)$, $h(4)$, and $h(10)$

Remark 2.2.8 Sometimes functions are made up of multiple functions put together. We call these **piecewise functions**. Each piece is defined for only a certain interval, and these intervals do not overlap. When evaluating a piecewise function at a given x-value, we first need to find the interval that includes the x-value, and then plug in to the corresponding function piece.

Function Notation (FN2)

Activity 2.2.9 Let $f(x)$ be a piecewise function as shown below.

$$f(x) = \begin{cases} x^2 + 3, & x < 5 \\ 9 - 2x, & x \geq 5 \end{cases}$$

(a) On which interval from the piecewise function does the value $x = 1$ belong?

A. $x < 5$ B. $x \leq 5$ C. $x > 5$ D. $x \geq 5$

(b) Find $f(1)$.

A. 3 B. 4 C. 5 D. 6 E. 7

(c) On which interval from the piecewise function does the value $x = 5$ belong?

A. $x < 5$ B. $x \leq 5$ C. $x > 5$ D. $x \geq 5$

(d) Find $f(5)$.

A. -10 B. -5 C. -1 D. 17 E. 28

Function Notation (FN2)

Remark 2.2.10 We've been practicing evaluating functions at specific numeric values. It's also possible to evaluate a function given an expression involving variables.

Function Notation (FN2)

Activity 2.2.11 Let $g(x) = x^2 - 3x$.

(a) Find $g(a)$.

A. $(ax)^2 - 3ax$

B. $a^2 - 3a$

C. $a(x^2 - 3x)$

D. $ax^2 - 3ax$

E. $a - 3$

(b) Find $g(x + h)$.

A. $x^2 - 3x + h$

B. $(x + h)^2 - 3x$

C. $(x + h)^2 - 3(x + h)$

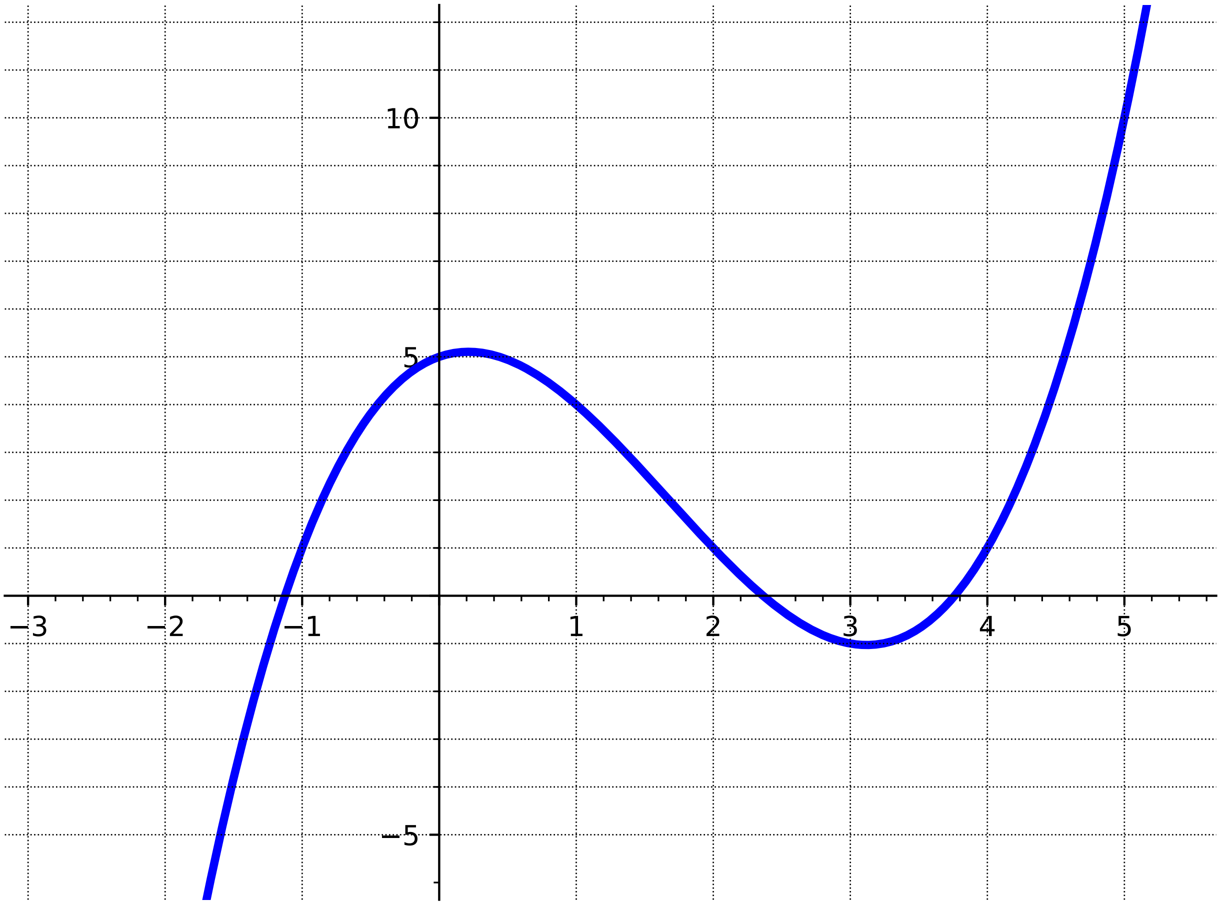
D. $x^2 - 3(x + h)$

Function Notation (FN2)

Remark 2.2.12 We should also be able to look at a graph of a function and evaluate it for different values of x. The next activity explores that.

Function Notation (FN2)

Activity 2.2.13 Let $f(x)$ be the function graphed below.



(a) Find $f(1)$.

A. -4 B. -2 C. 0 D. 2 E. 4

(b) Find $f(3)$.

A. -1 B. 0 C. 1 D. 2 E. 3 F. 4

(c) For which x-value(s) does $f(x) = 1$?

A. -1 B. 0 C. 1 D. 2 E. 3 F. 4

(d) For which x-value(s) does $f(x) = 4$. Estimate as needed!

Function Notation (FN2)

Activity 2.2.14 In these activities, we are flipping the question around. This time we know what the function equals at some x-value, and we want to recover that x-value (or values!).

(a) Let $h(x) = 5x + 7$. Find the x-value(s) such that $h(x) = -13$.

(b) Let $f(x) = x^2 - 3x - 9$. Find the x-value(s) such that $f(x) = 9$.

Function Notation (FN2)

Activity 2.2.15 Ellie has \$13 in her piggy bank, and she gets an additional \$1.50 each week for her allowance. Assuming she does not spend any money, the total amount of allowance, $A(w)$, she has after w weeks can be modeled by the function

$$A(w) = 13 + 1.50w.$$

(a) How much money will be in her piggy bank after 5 weeks?

(b) After how many weeks will she have \$40 in her piggy bank?

2.3 Characteristics of a Function's Graph (FN3)

Objectives

- Use the graph of a function to find the domain and range in interval notation, the x- and y-intercepts, the maxima and minima, and where it is increasing and decreasing using interval notation.

Characteristics of a Function's Graph (FN3)

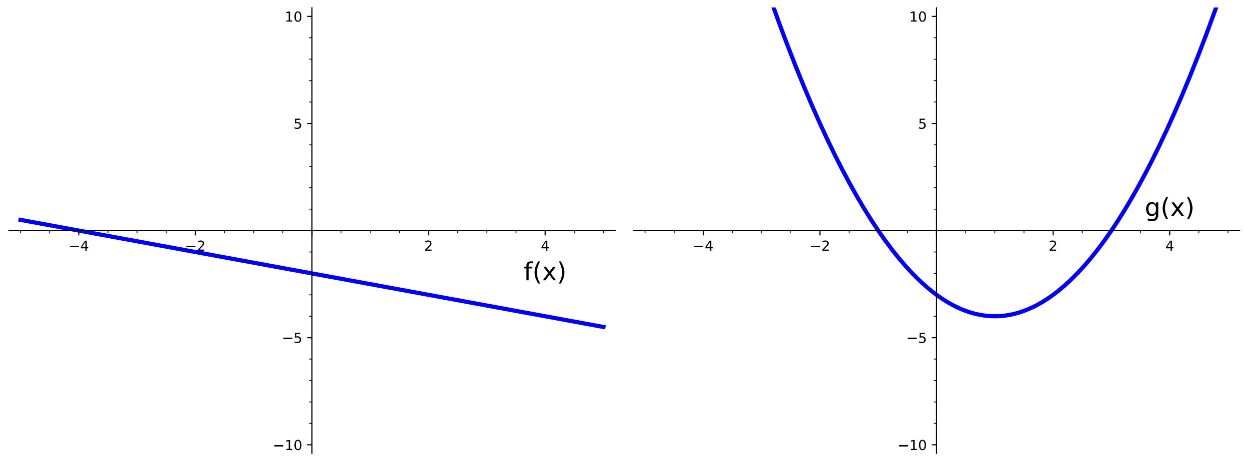
Remark 2.3.1 In this section, we will be looking at different kinds of graphs and will identify various characteristics. These ideas can span all kinds of functions, so you will see these come up multiple times!

Characteristics of a Function's Graph (FN3)

Definition 2.3.2 One of the easiest things to identify from a graph are the **intercepts**, which are points at which the graph crosses the axes. An x**-intercept** is a point at which the graph crosses the x-axis and a y**-intercept** is a point at which the graph crosses the y-axis. Because intercepts are points, they are typically written as an ordered pair: (x, y). $\Diamond$

Characteristics of a Function's Graph (FN3)

Activity 2.3.3 Use the following graphs to answer the questions.



(a) What are the x-intercept(s) of $f(x)$?

A. $(0, -4)$ B. $(-2, 0)$ C. $(-4, 0)$ D. $(0, -2)$

(b) What are the x-intercept(s) of $g(x)$?

A. $(0, -3)$ B. $(-1, 0)$ C. $(3, 0)$ D. $(-3, 0)$

(c) What are the y-intercept(s) of $f(x)$?

A. $(0, -4)$ B. $(-2, 0)$ C. $(-4, 0)$ D. $(0, -2)$

(d) What are the y-intercept(s) of $g(x)$?

A. $(0, -3)$ B. $(-1, 0)$ C. $(3, 0)$ D. $(-3, 0)$

(e) Sketch a graph of a function with the following intercepts:

- x-intercepts: $(-2, 0)$ and $(6, 0)$
- y-intercept: $(0, 4)$

(f) Sketch a graph of a function with the following intercepts:

- x-intercept: $(-1, 0)$
- y-intercept: $(0, 6)$ and $(0, -2)$

Remark 2.3.4 Notice in Activity 2.3.3, that a function can have multiple x-intercepts, but only one y-intercept. Having more than one y-intercept would create a graph that is not a function!

Characteristics of a Function's Graph (FN3)

Definition 2.3.5 The **domain** refers to the set of possible input values and the **range** refers to the set of possible output values. If given a graph, however, it would be impossible to list out all the values for the domain and range so we use interval notation to represent the set of values.

Recall that the terms **domain** and **range** were first introduced in Definition 2.1.1.

◊

Activity 2.3.6 Use the following graph to answer the questions below.

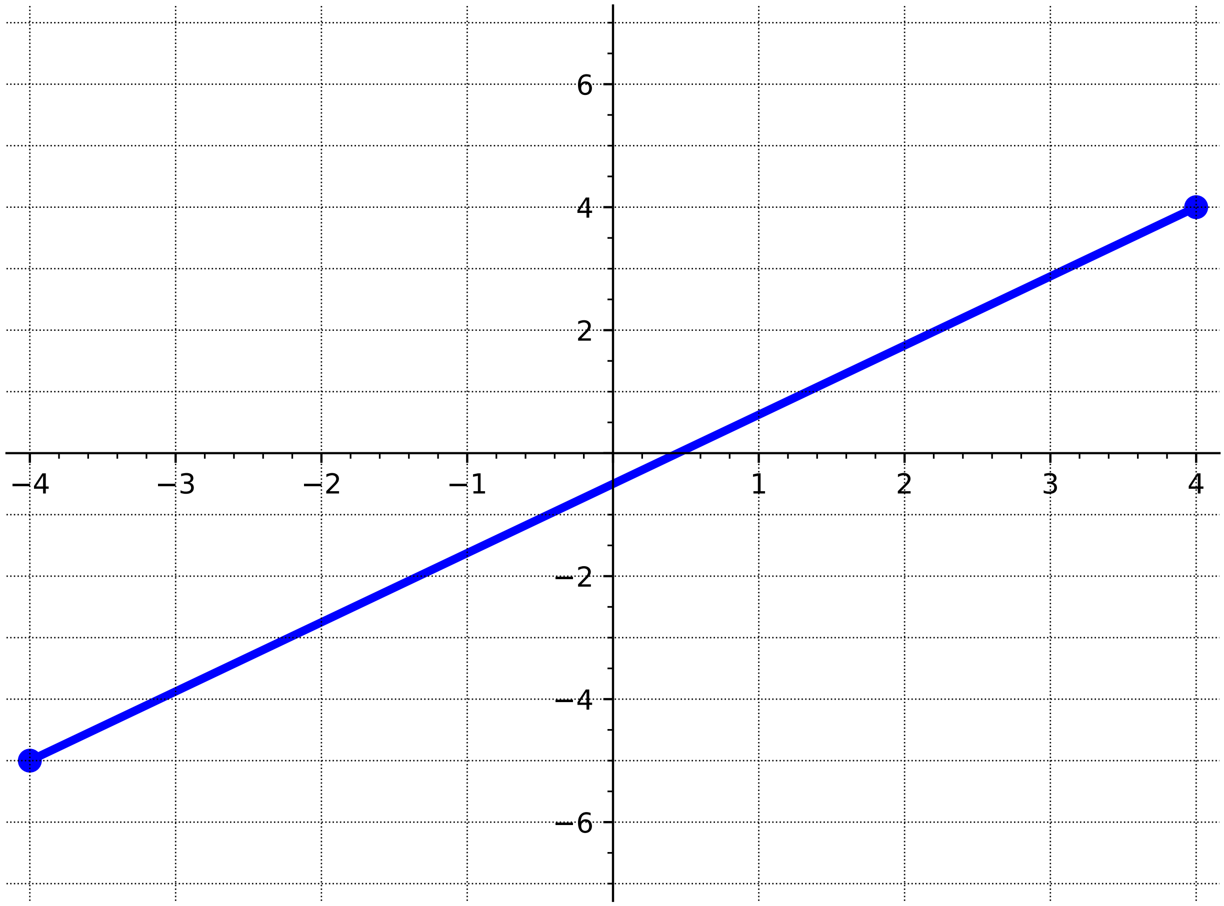


Figure 2.3.7

(a) Draw on the x-axis all the values in the domain.

(b) What interval represents the domain you drew in part (a)?

A. $[4, -4]$ B. $[-4, 4]$ C. $(-4, 4)$ D. $(4, -4)$

(c) Draw on the y-axis all the values in the range.

(d) What interval represents the range you drew in part (c)?

A. $(-5, 4)$ B. $[-4, 4]$ C. $[-5, 4]$ D. $(4, -5)$

Activity 2.3.8 Use the following graph to answer the questions below.

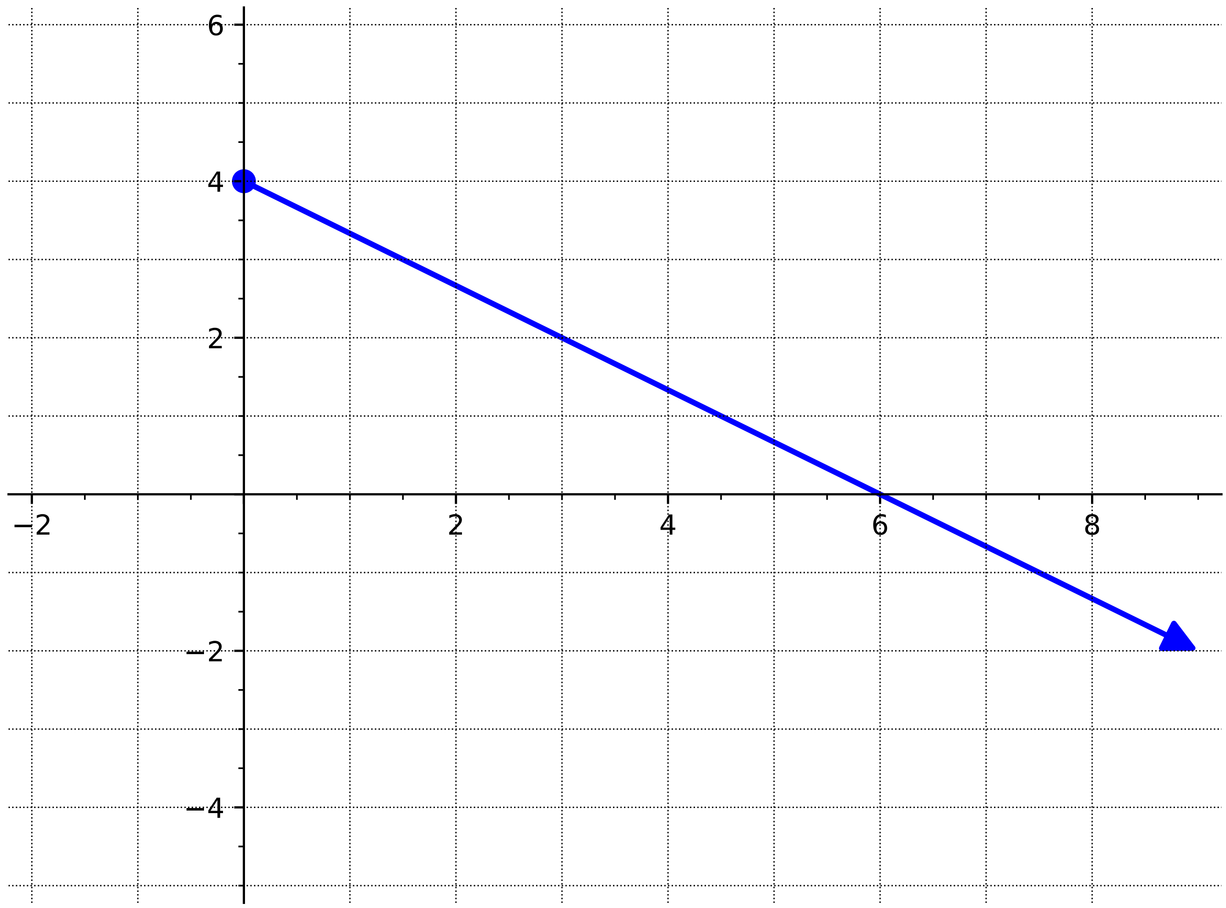


Figure 2.3.9

(a) What is the domain of this graph?

A. $[4, \infty)$ B. $(-\infty, 0]$ C. $(-\infty, 4]$ D. $[0, \infty)$

(b) What is the range of this graph?

A. $[4, \infty)$ B. $(-\infty, 0]$ C. $(-\infty, 4]$ D. $[0, \infty)$

Remark 2.3.10 When writing your intervals for domain and range, notice that you will need to write them from the smallest values to the highest values. For example, we wouldn't write $[4, -\infty)$ as an interval because $-\infty$ is smaller than 4.

For domain, read the graph from left to right. For range, read the graph from bottom to top.

Activity 2.3.11 Use the following graph to answer the questions below.

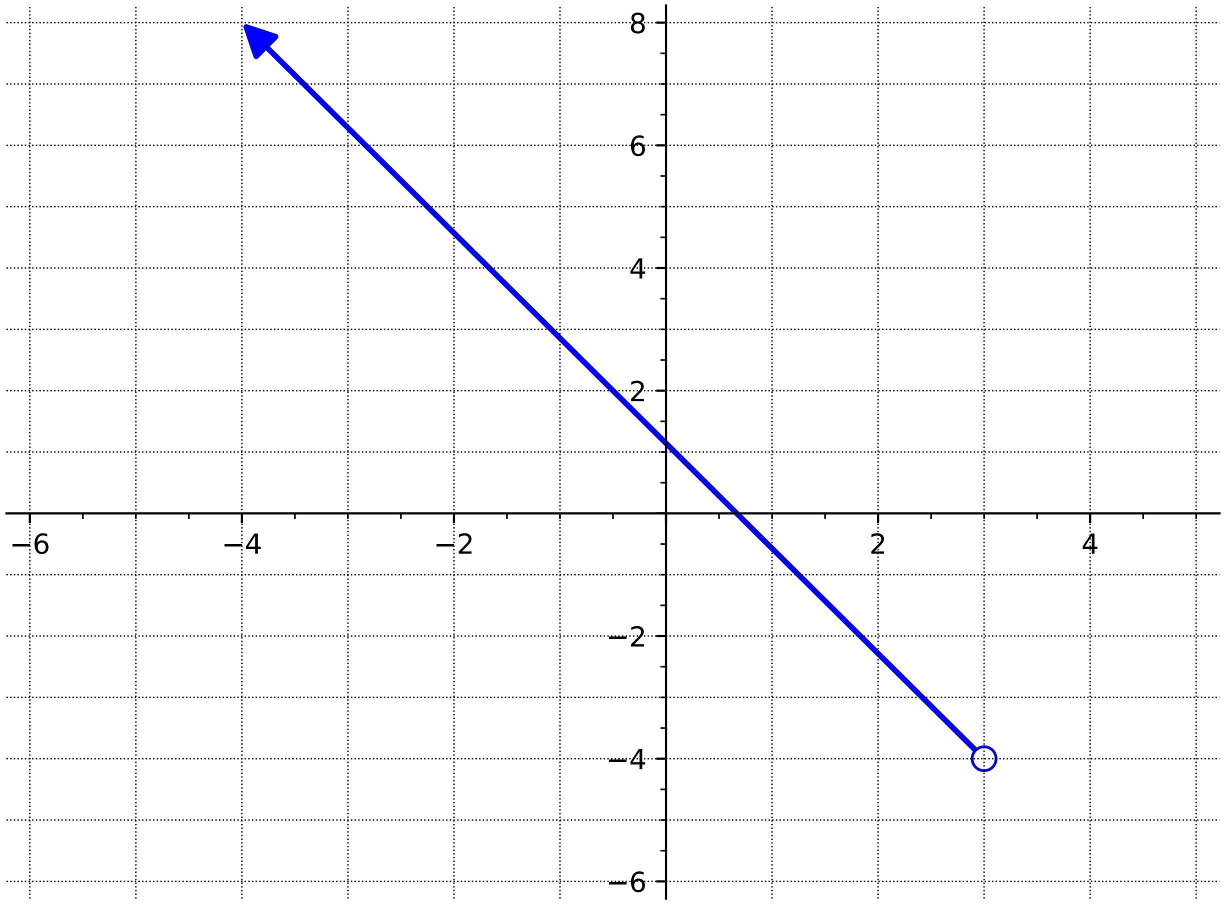


Figure 2.3.12

(a) What is the domain of this graph?

A. $(-\infty, 3)$ B. $(\infty, -4]$ C. $(-4, \infty)$ D. $(-\infty, 3]$

(b) What is the range of this graph?

A. $(-\infty, 3)$ B. $(\infty, -4]$ C. $(-4, \infty)$ D. $(-\infty, 3]$

Activity 2.3.13 Use the following graph to answer the questions below.

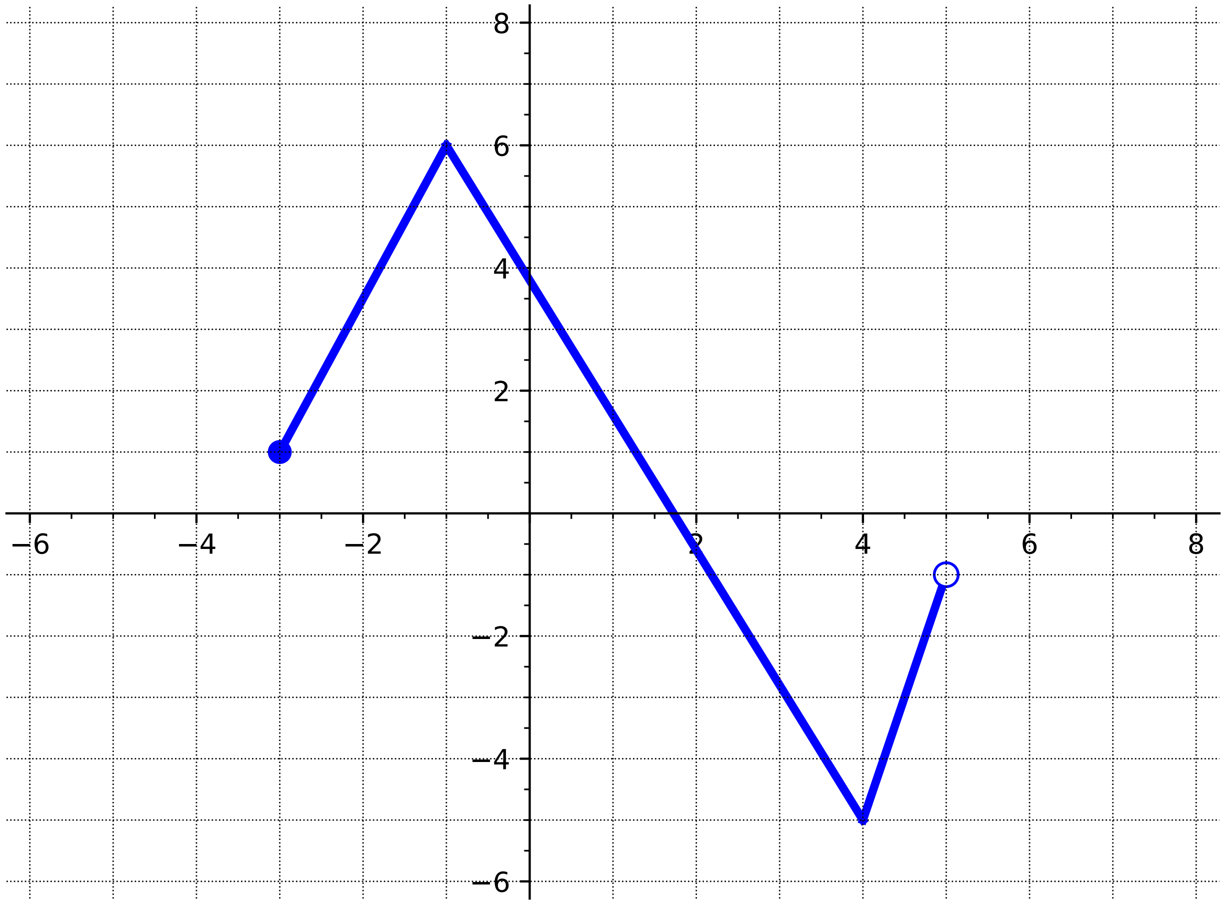


Figure 2.3.14

(a) What is the domain of this graph?

A. $(-3, 5)$ B. $(-5, 7)$ C. $[-5, 7]$ D. $[-3, 5)$

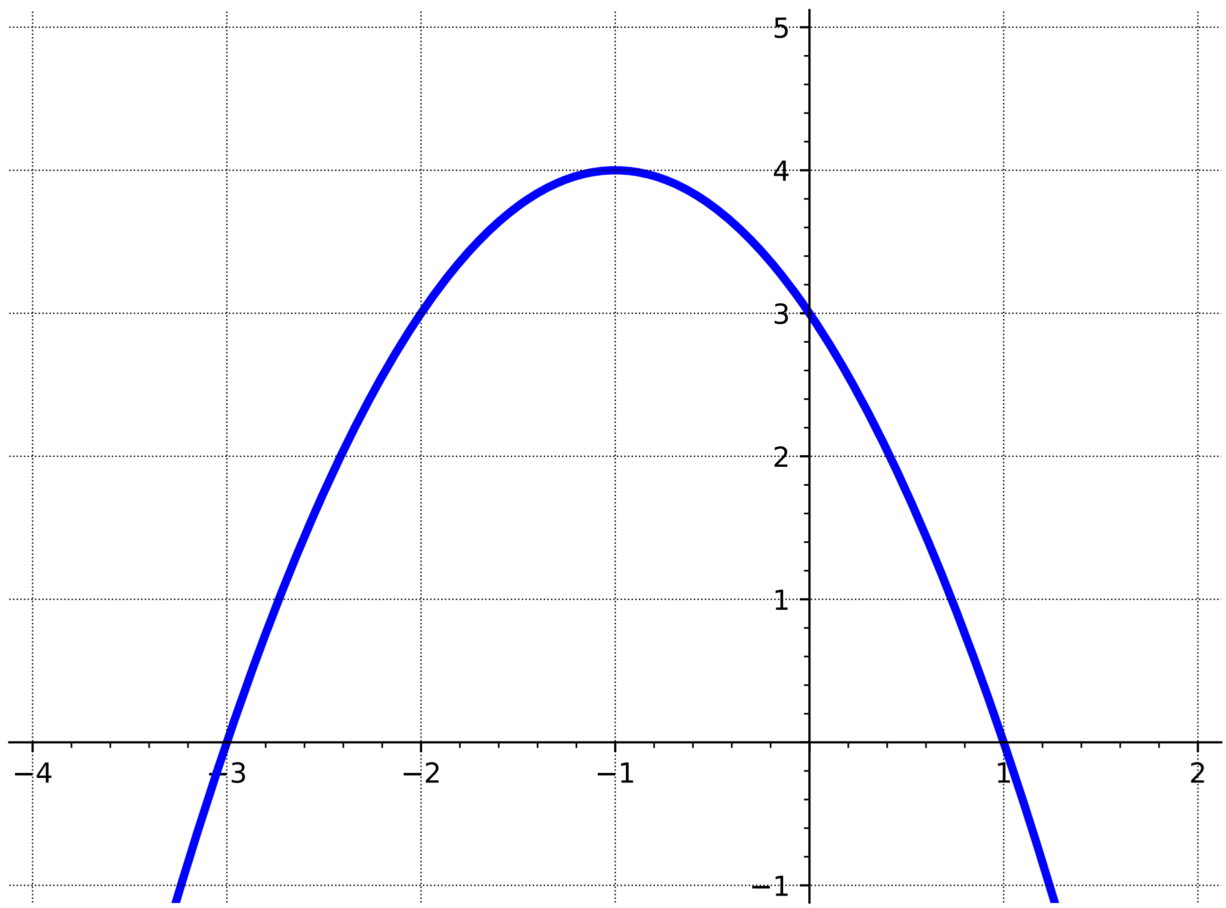
(b) What is the range of this graph?

A. $(-3, 5)$ B. $(-5, 6)$ C. $[-5, 6]$ D. $[-3, 5)$

Remark 2.3.15 Notice that finding the domain and range can be tricky! Be sure to pay attention to the x- and y-values of the entire graph - not just the endpoints!

Characteristics of a Function's Graph (FN3)

Activity 2.3.16 In this activity, we will look at where the function is increasing and decreasing. Use the following graph to answer the questions below.



(a) Where do you think the graph is increasing?

(b) Which interval best represents where the function is increasing?

A. $(-\infty, -1]$ B. $(-\infty, -1)$ C. $(-1, \infty)$ D. $[-1, \infty)$

(c) Where do you think the graph is decreasing?

(d) Which interval best represents where the function is decreasing?

A. $(-\infty, -1]$ B. $(-\infty, -1)$ C. $(-1, \infty)$ D. $[-1, \infty)$

(e) Based on what you see on the graph, do you think this graph has any maxima or minima?

Characteristics of a Function's Graph (FN3)

Definition 2.3.17 As you noticed in Activity 2.3.16, functions can increase or decrease (or even remain constant!) for a period of time. The **interval of increase** is when the y-values of the function increase as the x-values increase. The **interval of decrease** is when the y-values of the function decrease as the x-values increase. The function is constant when the y-values remain constant as x-values increase (also known as the **constant interval**).

The easiest way to identify these intervals is to read the graph from left to right and look at what is happening to the y-values. ◊

Characteristics of a Function's Graph (FN3)

Definition 2.3.18 The **maximum**, or **global maximum**, of a graph is the point where the y-coordinate has the largest value. The **minimum**, or **global minimum** is the point on the graph where the y-coordinate has the smallest value.

Graphs can also have **local maximums** and **local minimums**. A local maximum point is a point where the function value (i.e, y-value) is larger than all others in some neighborhood around the point. Similarly, a local minimum point is a point where the function value (i.e, y-value) is smaller than all others in some neighborhood around the point. ◊

Remark 2.3.19 Global extrema are sometimes referred to as absolute extrema, while local extrema are sometimes referred to as relative extrema.

Activity 2.3.20 Use the following graph to answer the questions below.

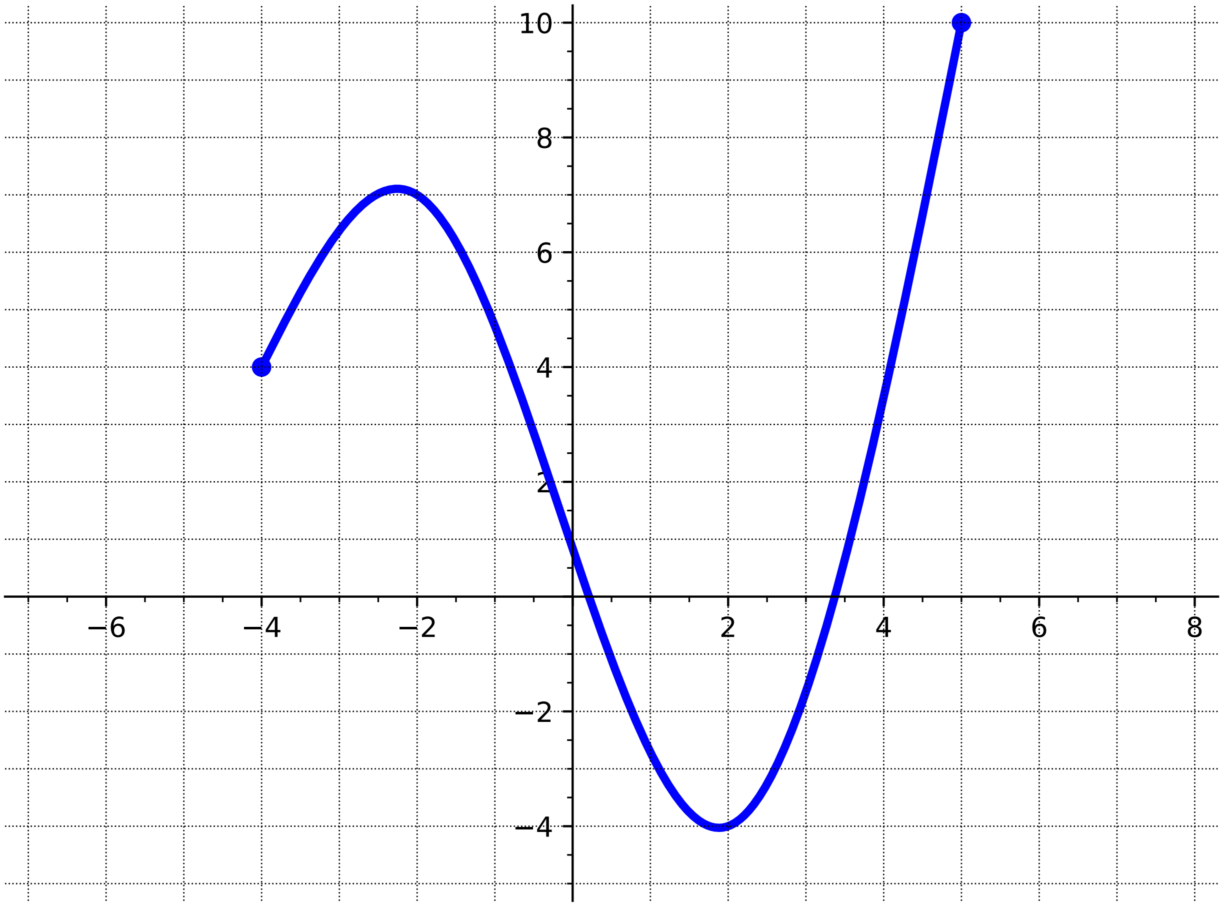


Figure 2.3.21

(a) At what value of x is there a global maximum?

A. $x = -4$ B. $x = -2$ C. $x = 2$ D. $x = 5$

(b) What is the global maximum value?

A. 10 B. 7 C. 4 D. -4

(c) At approximately what value of x is there a global minimum?

A. $x \approx -4$ B. $x \approx -2$ C. $x \approx 2$ D. $x \approx 5$

(d) What is the global minimum value?

A. 10 B. 7 C. 4 D. -4

(e) At approximately what value of x is there a local maximum?

A. $x \approx -4$ B. $x \approx -2$ C. $x \approx 2$ D. $x \approx 5$

Characteristics of a Function's Graph (FN3)

(f) What is the local maximum value?

A. 10 B. 7 C. 4 D. -4

(g) At approximately what value of x is there a local minimum?

A. $x \approx -4$ B. $x \approx -2$ C. $x \approx 2$ D. $x \approx 5$

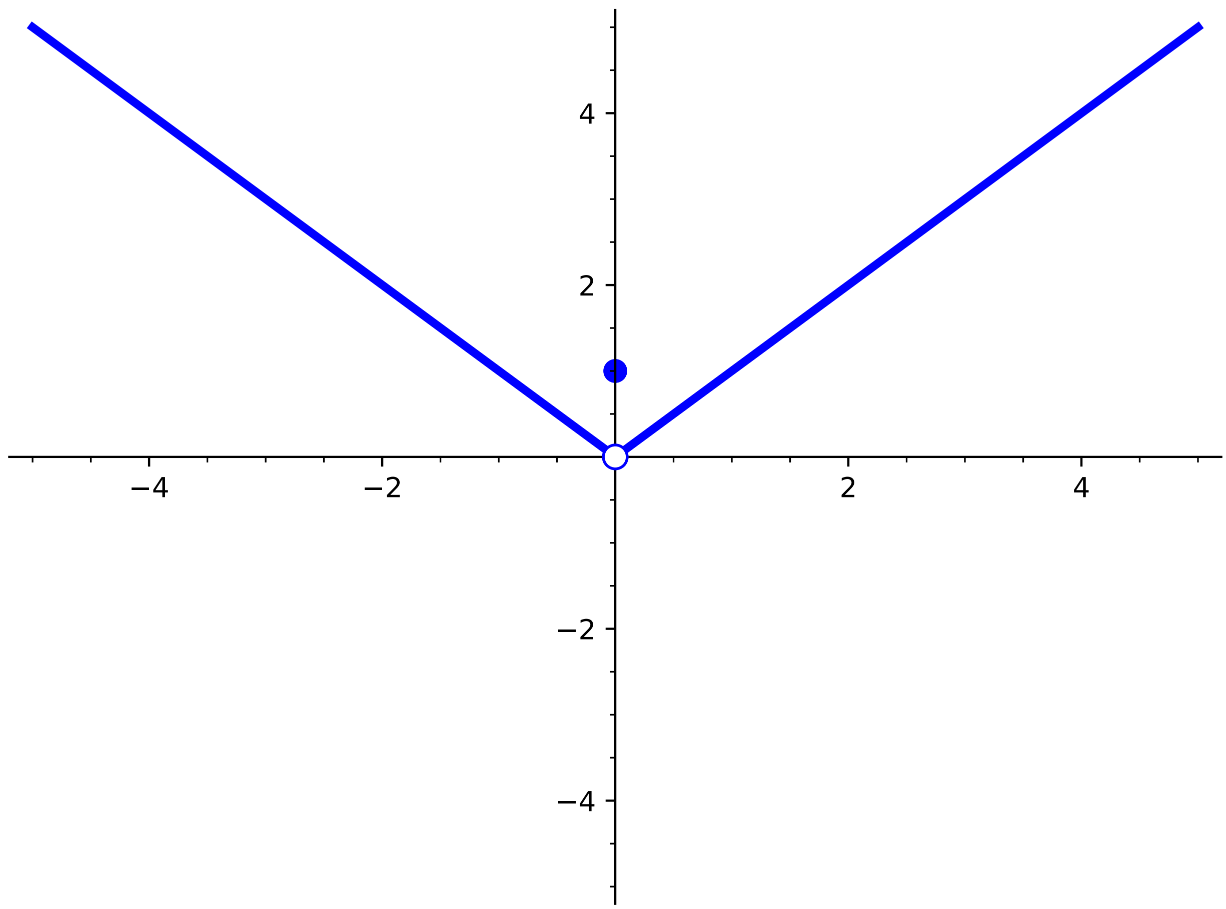
(h) What is the local minimum value?

A. 10 B. 7 C. 4 D. -4

Remark 2.3.22 Notice that in Activity 2.3.20, there are two ways we talk about max and min. We might want to know the location of where the max or min are (i.e., determining at which x-value the max or min occurs at) or we might want to know what the max or min values are (i.e., the y-value).

Also, note that in Activity 2.3.20, a local minimum is also a global minimum.

Activity 2.3.23 Sometimes, it is not always clear what the maxima or minima are or if they exist. Consider the following graph of $f(x)$:



(a) What is the value of $f(0)$?

A. 1 B. 0 C. $f(0)$ does not exist

(b) What is the local minimum value of $f(x)$?

A. 1 B. 0 C. There is no local minimum

(c) What is the global minimum value of $f(x)$?

A. 1 B. 0 C. There is no global minimum

Characteristics of a Function's Graph (FN3)

Activity 2.3.24 Use the following graph to answer the questions below.

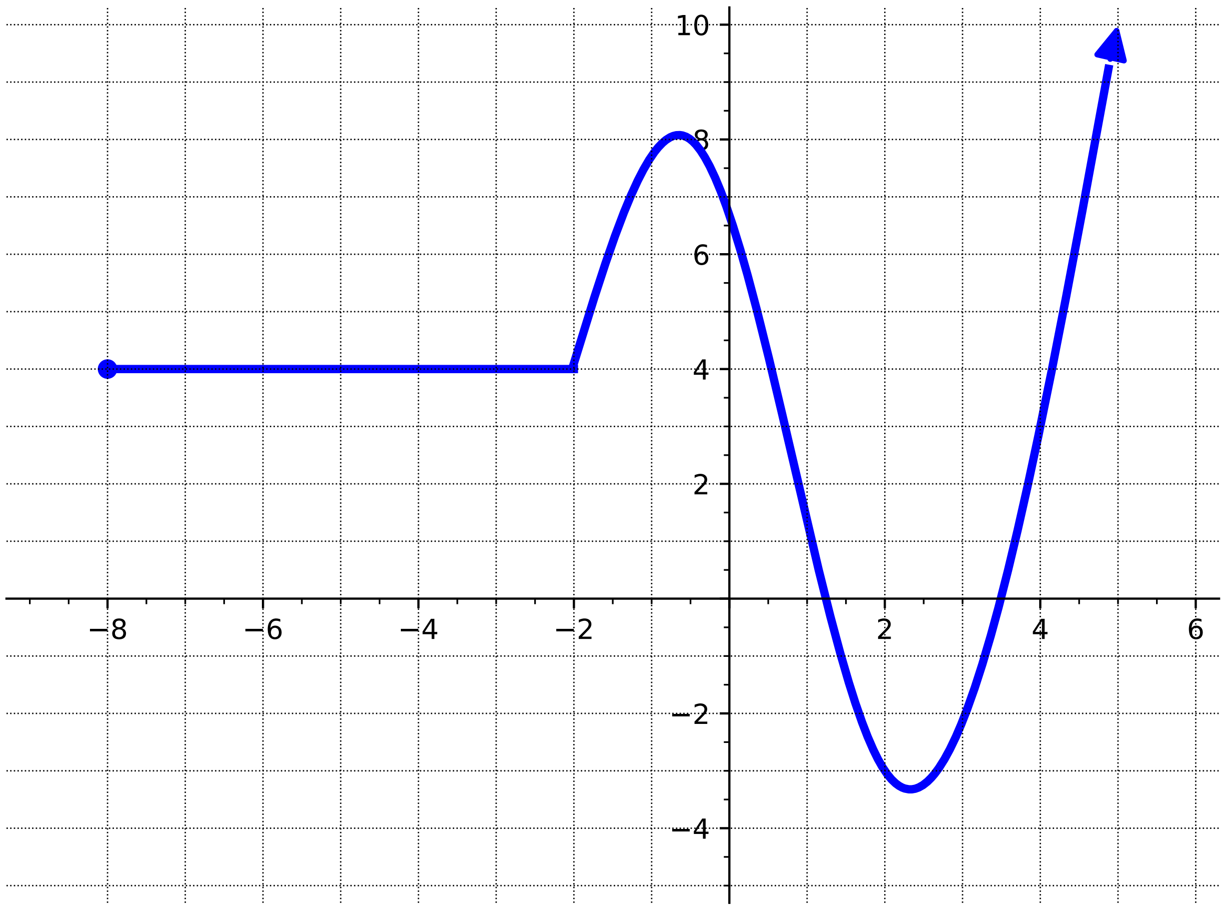


Figure 2.3.25

(a) What is the domain?

(b) What is the range?

(c) What is the x-intercept(s)?

(d) What is the y-intercept?

(e) Where is the function increasing?

(f) Where is the function decreasing?

(g) Where is the constant interval?

(h) At what x-values do the local maxima occur?

(i) At what x-values do the local minima occur?

(j) What are the global max and min?

2.4 Transformation of Functions (FN4)

Objectives

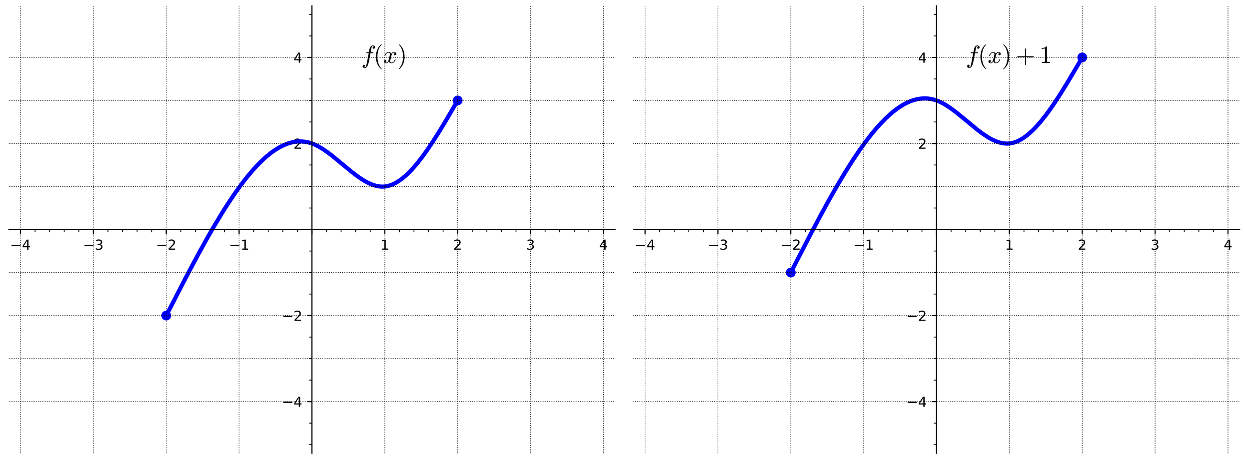
- Apply transformations including horizontal and vertical shifts, stretches, and reflections to a function. Express the result of these transformations graphically and algebraically.

Transformation of Functions (FN4)

Remark 2.4.1 Informally, a transformation of a given function is an algebraic process by which we change the function to a related function that has the same fundamental shape, but may be shifted, reflected, and/or stretched in a systematic way.

Transformation of Functions (FN4)

Activity 2.4.2 Consider the following two graphs.



(a) How is the graph of $f(x) + 1$ related to that of $f(x)$?

A. Shifted up 1 unit

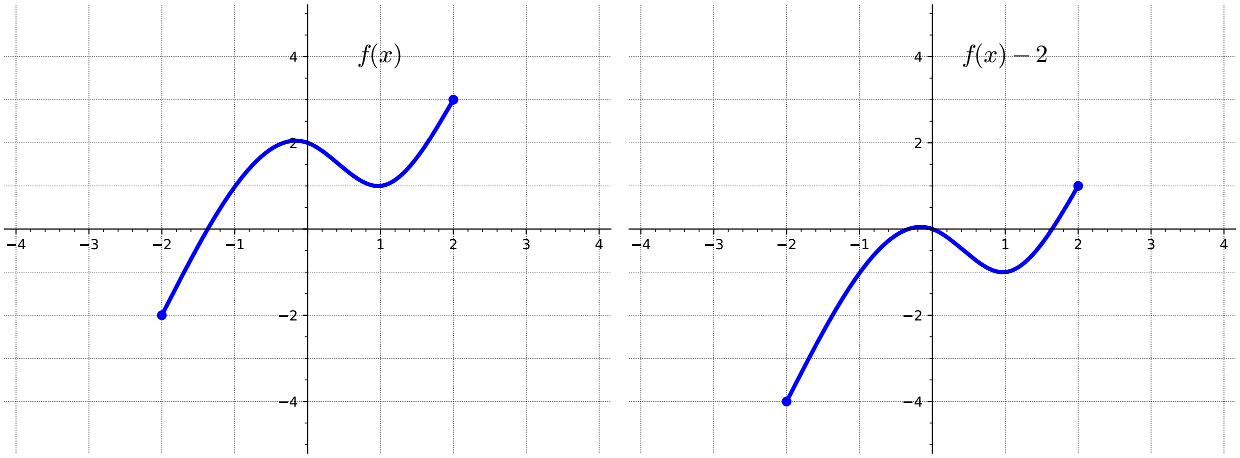
B. Shifted left 1 unit

C. Shifted down 1 unit

D. Shifted right 1 unit

Transformation of Functions (FN4)

Activity 2.4.3 Consider the following two graphs.



(a) How is the graph of $f(x) - 2$ related to that of $f(x)$?

A. Shifted up 2 units

B. Shifted left 2 units

C. Shifted down 2 units

D. Shifted right 2 units

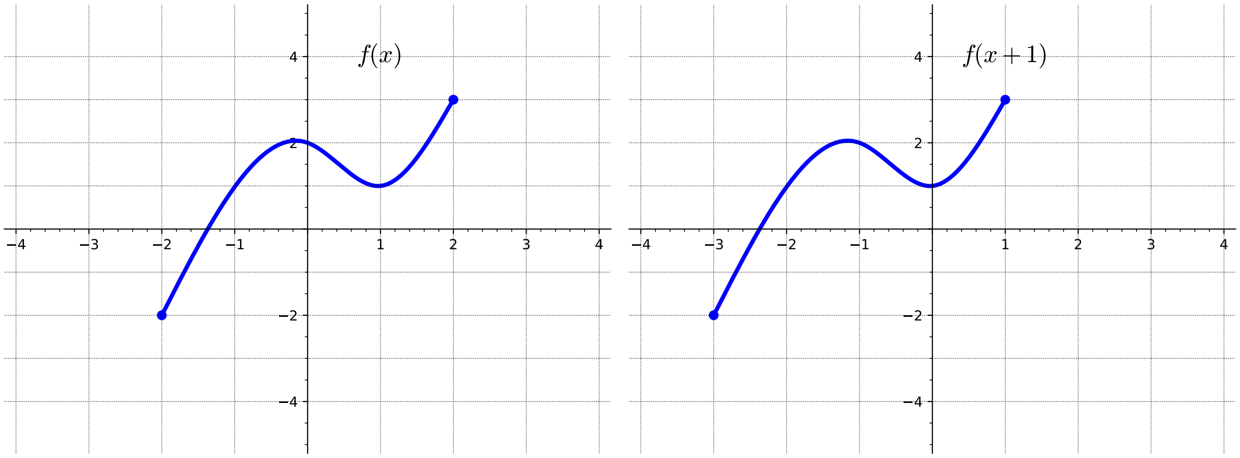
Remark 2.4.4 Notice that in Activity 2.4.2 and Activity 2.4.3, the y-values of the transformed graph are changed while the x-values remain the same.

Transformation of Functions (FN4)

Definition 2.4.5 Given a function $f(x)$ and a constant c, the transformed function $g(x) = f(x) + c$ is a **vertical translation** of the graph of $f(x)$. That is, all the outputs change by c units. If c is positive, the graph will shift up. If c is negative, the graph will shift down. $\Diamond$

Transformation of Functions (FN4)

Activity 2.4.6 Consider the following two graphs.



(a) How is the graph of $f(x+1)$ related to that of $f(x)$?

A. Shifted up by 1 unit

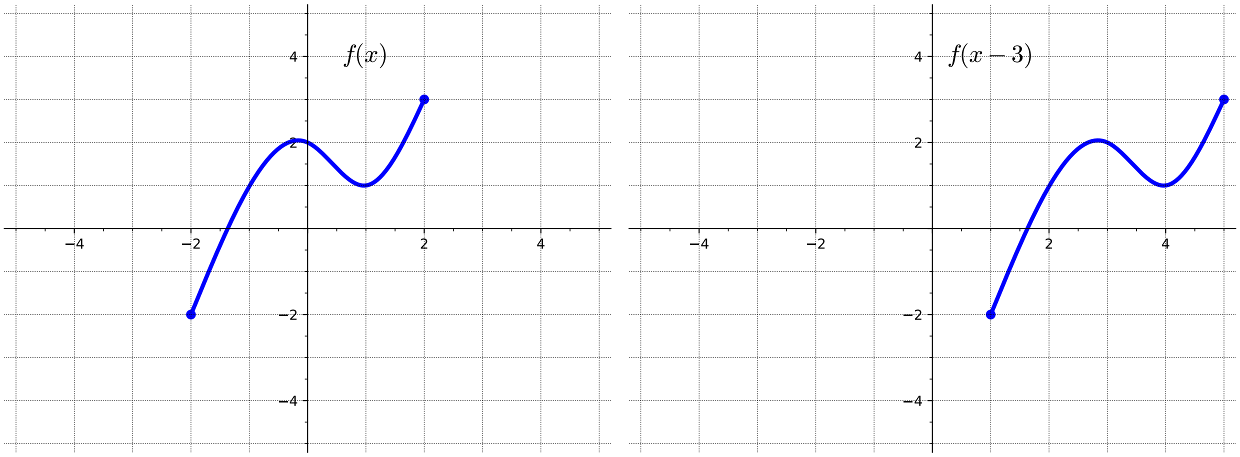
B. Shifted left 1 unit

C. Shifted down 1 unit

D. Shifted right 1 unit

Transformation of Functions (FN4)

Activity 2.4.7 Consider the following two graphs.



(a) How is the graph of $f(x-3)$ related to that of $f(x)$?

A. Shifted up by 3 units

B. Shifted left 3 units

C. Shifted down 3 units

D. Shifted right 3 units

Remark 2.4.8 Notice that in Activity 2.4.6 and Activity 2.4.7, the x-values of the transformed graph are changed while the y-values remain the same.

Transformation of Functions (FN4)

Definition 2.4.9 Given a function $f(x)$ and a constant c, the transformed function $g(x) = f(x+c)$ is a **horizontal translation** of the graph of $f(x)$. If c is positive, the graph will shift left. If c is negative, the graph will shift right. $\Diamond$

Transformation of Functions (FN4)

Activity 2.4.10 Describe how the graph of the function is a transformation of the graph of the original function f.

(a) $f(x-4)+1$

A. Shifted down 4 units

B. Shifted left 4 units

C. Shifted down 1 unit

D. Shifted right 4 units

E. Shifted up 1 unit

(b) $f(x+3)-2$

A. Shifted down 2 units

B. Shifted left 3 units

C. Shifted up 3 units

D. Shifted right 3 units

E. Shifted up 2 units

Transformation of Functions (FN4)

Activity 2.4.11 For each of the following, use the information given to find another point on the graph.

(a) If $(2, 3)$ is a point on the graph of $f(x)$, what point must be on the graph of $f(x) + 2$?

A. $(4, 3)$

B. $(2, 5)$

C. $(4, 5)$

D. $(2, 1)$

(b) If $(-1, 6)$ is a point on the graph of $g(x)$, what point must be on the graph of $g(x-4)$?

A. $(-1, 10)$

B. $(3, 6)$

C. $(-1, 2)$

D. $(-5, 6)$

(c) If $(-2, -5)$ is a point on the graph of $h(x)$, what point must be on the graph of $h(x+1) - 5$?

A. $(-3, -10)$

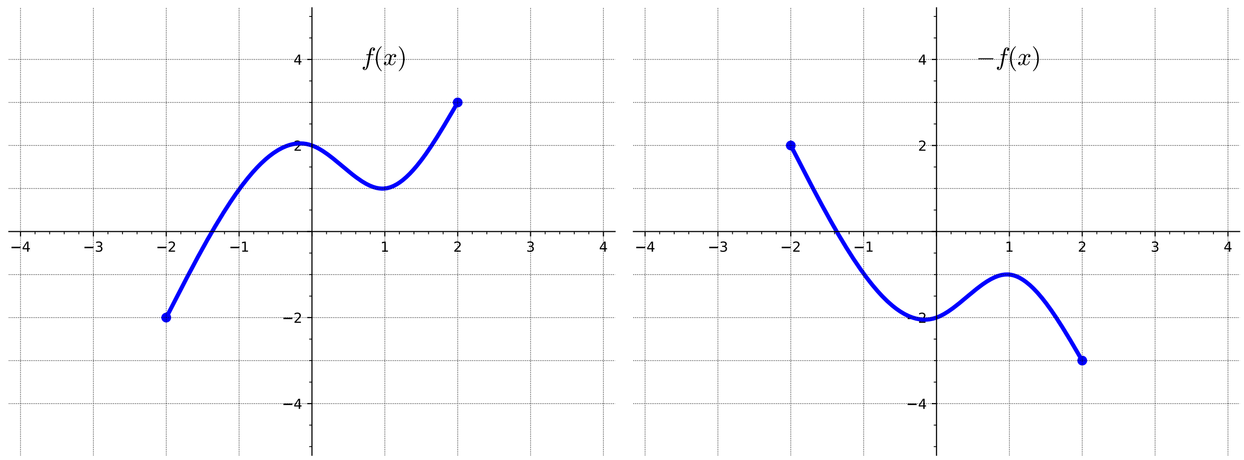
B. $(-3, 0)$

C. $(-1, -10)$

D. $(-1, 0)$

Transformation of Functions (FN4)

Activity 2.4.12 Consider the following two graphs.



(a) How is the graph of $-f(x)$ related to that of $f(x)$?

A. Shifted down 2 units

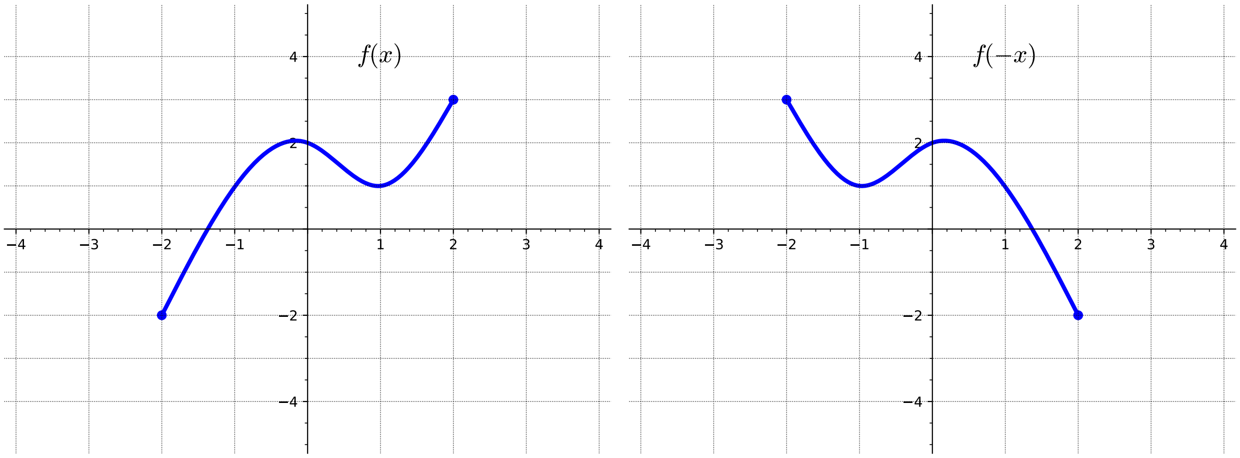
B. Reflected over the x-axis

C. Reflected over the y-axis

D. Shifted right 2 units

Transformation of Functions (FN4)

Activity 2.4.13 Consider the following two graphs.



(a) How is the graph of $f(-x)$ related to that of $f(x)$?

A. Shifted down 2 units

B. Reflected over the x-axis

C. Reflected over the y-axis

D. Shifted left 2 units

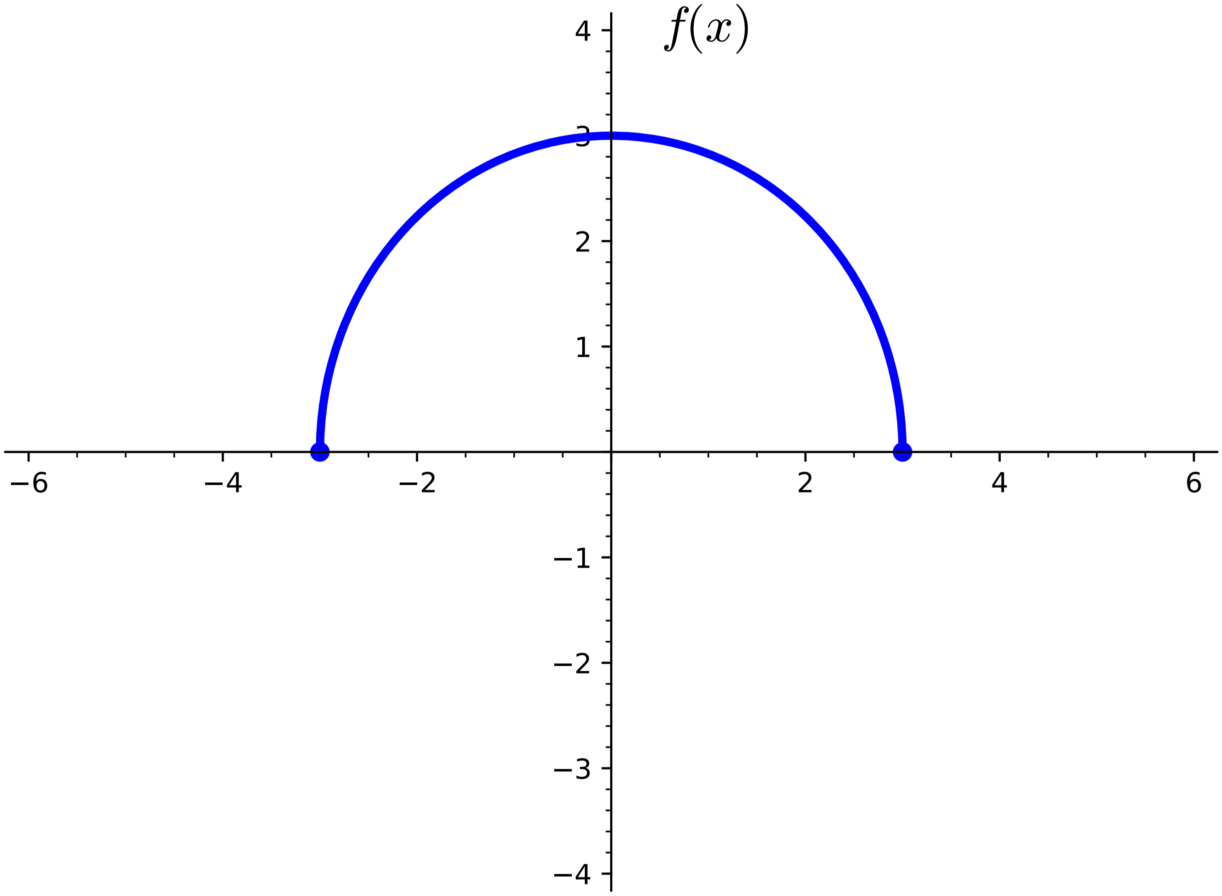
Remark 2.4.14 Notice that in Activity 2.4.12, the y-values of the transformed graph are changed while the x-values remain the same. While in Activity 2.4.13, the x-values of the transformed graph are changed while the y-values remain the same.

Definition 2.4.15 Given a function $f(x)$, the transformed function $g(x) = -f(x)$ is a **vertical reflection** of the graph of $f(x)$. That is, all the outputs are multiplied by -1. The new graph is a reflection of the old graph about the x-axis. $\Diamond$

Transformation of Functions (FN4)

Definition 2.4.16 Given a function $f(x)$, the transformed function $y = g(x) = f(-x)$ is a **horizontal reflection** of the graph of $f(x)$. That is, all the inputs are multiplied by -1. The new graph is a reflection of the old graph about the y-axis. $\Diamond$

Activity 2.4.17 Consider the following graph of the function $f(x)$.



(a) How is the graph of $-f(x+2)+3$ related to that of $f(x)$?

A. Shifted up 2 units

B. Shifted up 3 units

C. Reflected over the x-axis

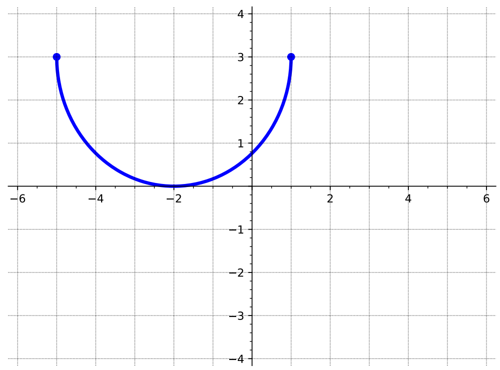
D. Reflected over the y-axis

E. Shifted left 3 units

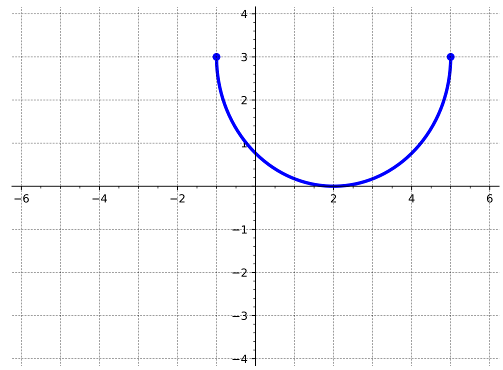
F. Shifted left 2 units

(b) Which of the following represents the graph of the transformed function $g(x) = -f(x+2)+3$?

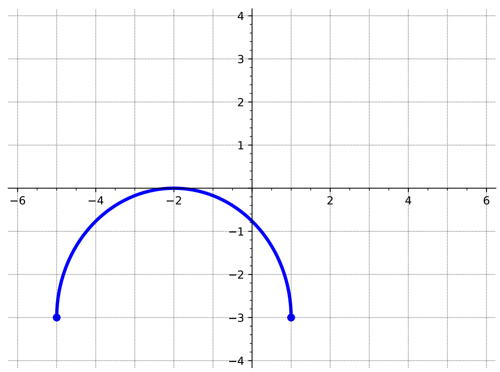
Transformation of Functions (FN4)



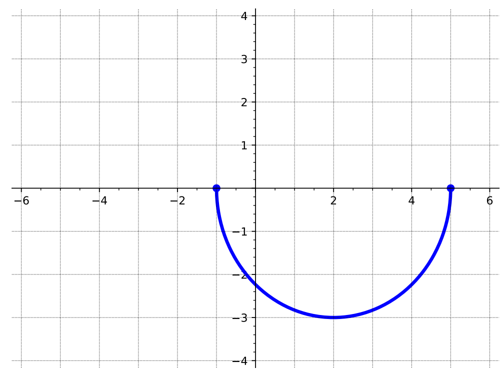
A.



C.



B.

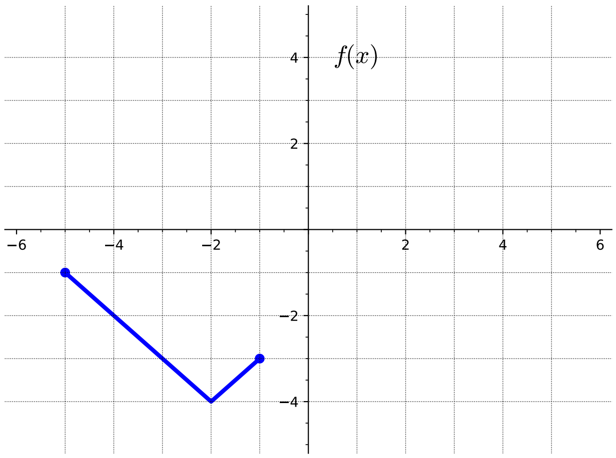


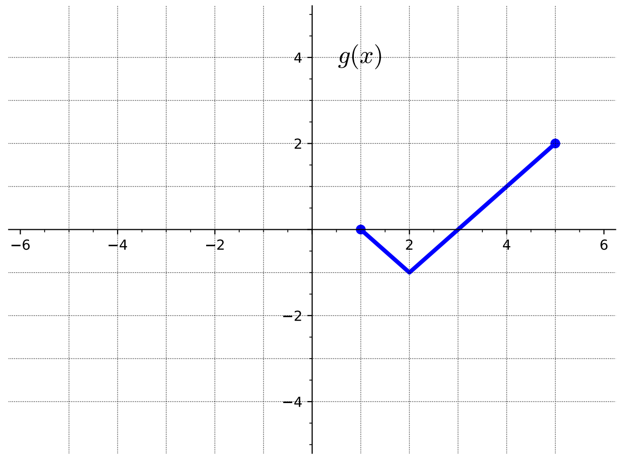
D.

Remark 2.4.18 Notice that in Activity 2.4.17 the resulting graph is different if you perform the reflection first and then the vertical shift, versus the other order. When combining transformations, it is very important to consider the order of the transformations. Be sure to follow the order of operations.

Transformation of Functions (FN4)

Activity 2.4.19 Consider the following two graphs.





(a) How is the graph of $g(x)$ related to that of $f(x)$?

A. Shifted up 3 units

B. Shifted up 1 unit

C. Reflected over the x-axis

D. Reflected over the y-axis

E. Shifted left 1 unit

F. Shifted right 4 units

(b) List the order the transformations must be applied.

(c) Write an equation for the graphed function $g(x)$ using transformations of the graph $f(x)$.

A. $g(x) = -f(x) + 3$

B. $g(x) = f(-x) + 3$

C. $g(x) = f(-x + 3)$

D. $g(x) = -f(x + 3)$

Transformation of Functions (FN4)

Activity 2.4.20 For each of the following, use the information given to find another point on the graph.

(a) If $(1, 6)$ is a point on the graph of $f(x)$, what point must be on the graph of $-f(x-2)$?

A. $(-3, 6)$

B. $(-1, 6)$

C. $(-1, 4)$

D. $(3, -6)$

(b) If $(-2, -4)$ is a point on the graph of $g(x)$, what point must be on the graph of $g(-x) + 3$?

A. $(-2, -7)$

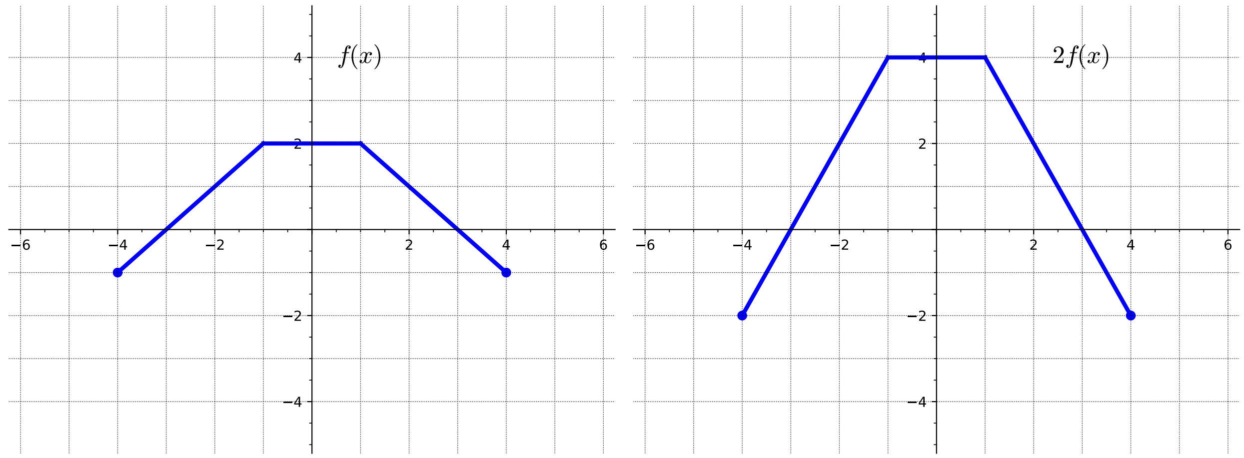
B. $(2, -1)$

C. $(2, -7)$

D. $(-2, 7)$

Transformation of Functions (FN4)

Activity 2.4.21 Consider the following two graphs.



(a) Consider the y-value of the two graphs at $x = 1$. How do they compare?

A. The y-value of $2f(x)$ is twice that of $f(x)$.

B. The y-value of $2f(x)$ is half that of $f(x)$.

C. The y-value of $2f(x)$ and $f(x)$ are the same.

D. The y-value of $2f(x)$ is negative that of $f(x)$.

(b) How is the graph of $2f(x)$ related to that of $f(x)$?

A. Vertically stretched by a factor of 2

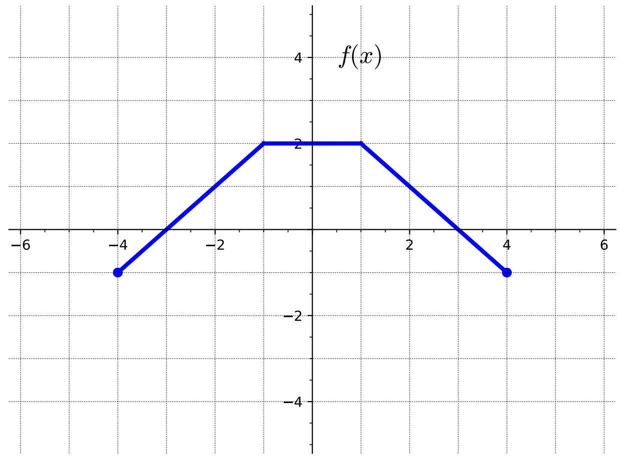
B. Vertically compressed by a factor of 2

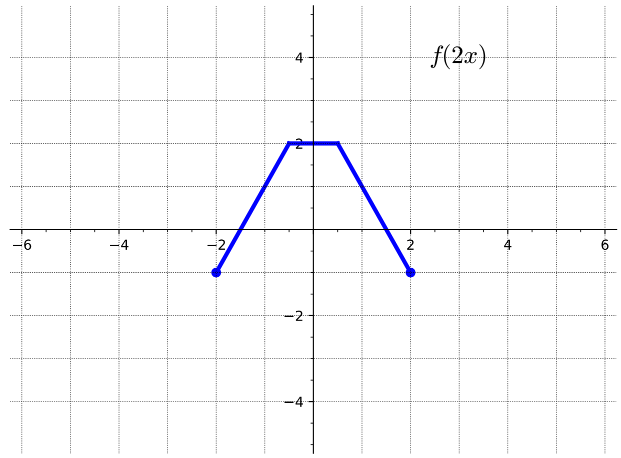
C. Horizontally stretched by a factor of 2

D. Horizontally compressed by a factor of 2

Transformation of Functions (FN4)

Activity 2.4.22 Consider the following two graphs.





(a) Consider an x-value of the two graphs where $y = 1$. How do they compare?

A. The x-value of $f(2x)$ is twice that of $f(x)$.

B. The x-value of $f(2x)$ is half that of $f(x)$.

C. The x-value of $f(2x)$ and $f(x)$ are the same.

D. The x-value of $f(2x)$ is negative that of $f(x)$.

(b) How is the graph of $f(2x)$ related to that of $f(x)$?

A. Vertically stretched by a factor of 2

B. Vertically compressed by a factor of 2

C. Horizontally stretched by a factor of 2

D. Horizontally compressed by a factor of 2

Remark 2.4.23 Notice that in Activity 2.4.21 the y-values are doubled while the x-values remain the same. While, in Activity 2.4.22 the x-values are cut in half while the y-values remain the same.

Transformation of Functions (FN4)

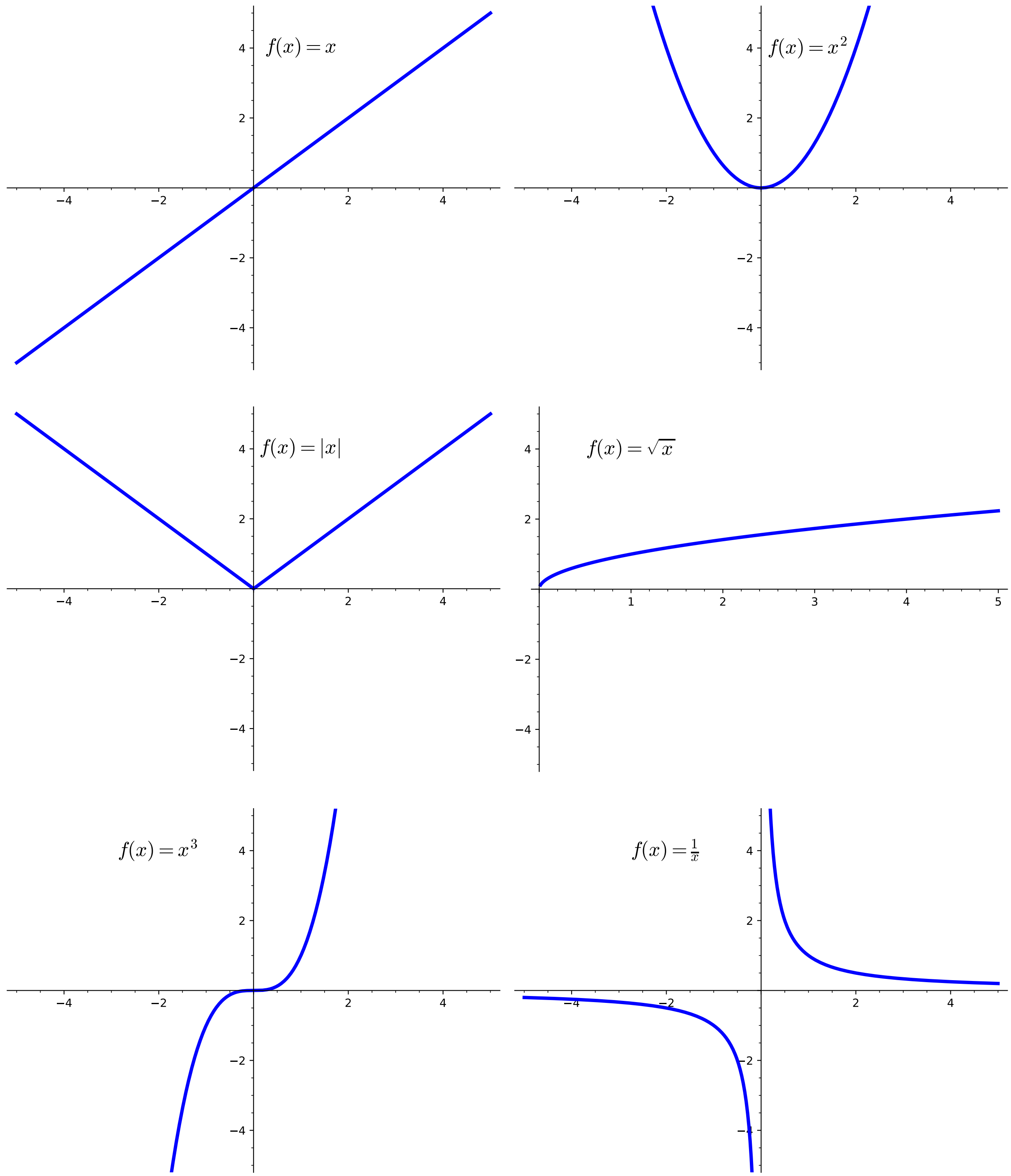
Definition 2.4.24 Given a function $f(x)$, the transformed function $g(x) = af(x)$ is a **vertical stretch** or **vertical compression** of the graph of $f(x)$. That is, all the outputs are multiplied by a. If $a > 1$, the new graph is a vertical stretch of the old graph away from the x-axis. If $0 < a < 1$, the new graph is a vertical compression of the old graph towards the x-axis. Points on the x-axis are unchanged. $\Diamond$

Transformation of Functions (FN4)

Definition 2.4.25 Given a function $f(x)$, the transformed function $g(x) = f(ax)$ is a **horizontal stretch** or **horizontal compression** of the graph of $f(x)$. That is, all the inputs are divided by a. If $a > 1$, the new graph is a horizontal compression of the old graph toward the y-axis. If $0 < a < 1$, the new graph is a horizontal stretch of the old graph away from the y-axis. Points on the y-axis are unchanged. $\Diamond$

Transformation of Functions (FN4)

Remark 2.4.26 We often use a set of basic functions with which to begin transformations. We call these parent functions.



Transformation of Functions (FN4)

Activity 2.4.27 Consider the function $g(x) = 3\sqrt{-x} + 2$

(a) Identify the parent function $f(x)$.

A. $f(x) = x^2$

B. $f(x) = |x|$

C. $f(x) = \sqrt{x}$

D. $f(x) = x$

(b) Graph the parent function $f(x)$.

(c) How is the graph of $g(x)$ related to that of the parent function$f(x)$?

A. Reflected over the x-axis

B. Reflected over the y-axis

C. Shifted down 2 units

D. Shifted up 2 units

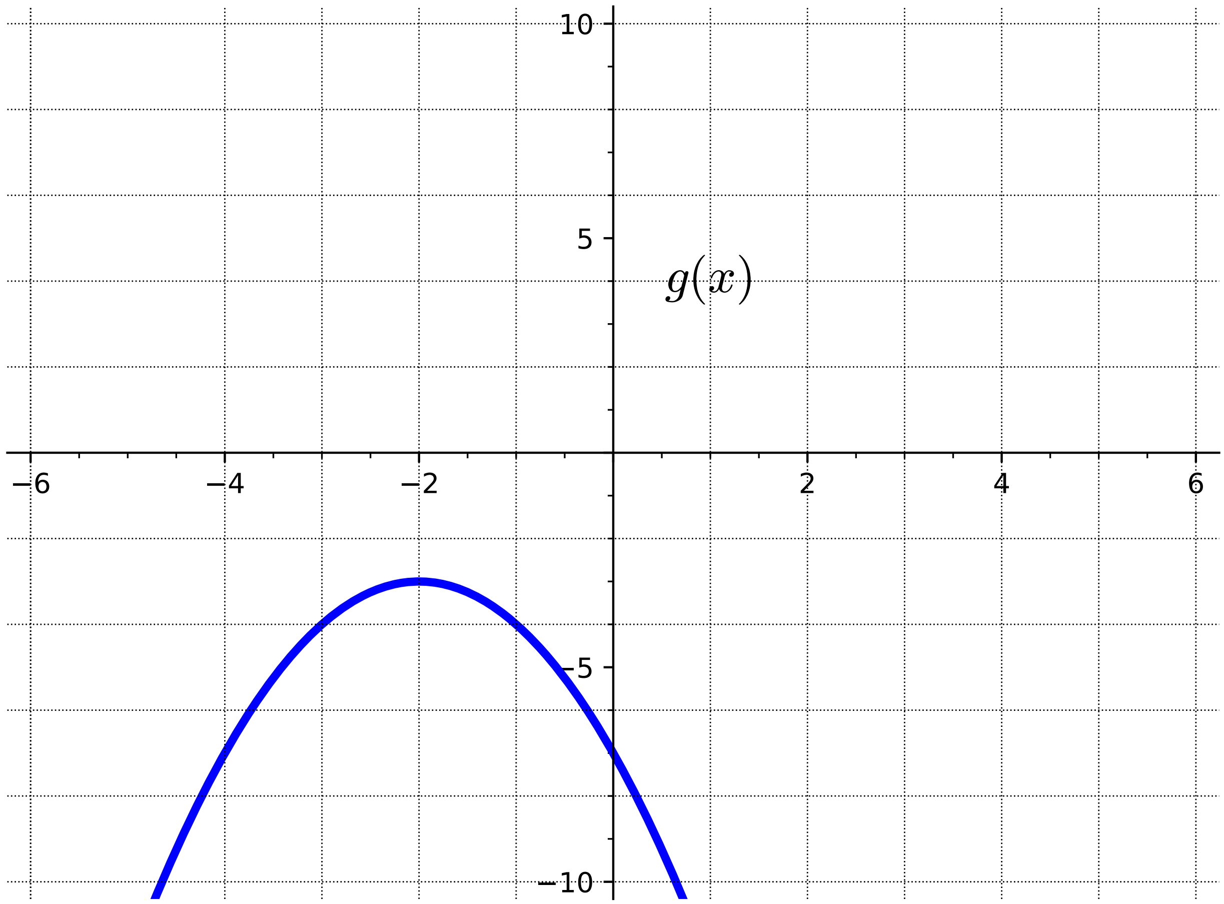
E. Vertically stretched by a factor of 3

F. Horizontally compressed by a factor of 3

(d) Graph the transformed function $g(x)$.

Transformation of Functions (FN4)

Activity 2.4.28 Consider the following graph of the function $g(x)$.



(a) Identify the parent function.

A. $f(x) = x^2$

B. $f(x) = |x|$

C. $f(x) = \sqrt{x}$

D. $f(x) = x$

(b) How is the graph of $g(x)$ related to that of the parent function $f(x)$?

A. Reflected over the x-axis

B. Reflected over the y-axis

C. Shifted down 3 units

D. Shifted up 3 units

E. Shifted left 2 units

F. Shifted right 2 units

(c) Write an equation to represent the transformed function $g(x)$.

A. $g(x) = -(x-2)^2 - 3$

B. $g(x) = -(x+2)^2 + 3$

C. $g(x) = (-x+2)^2 - 3$

D. $g(x) = -(x+2)^2 - 3$

Transformation of Functions (FN4)

Activity 2.4.29 For each of the following, write a formula for the new function $g(x)$ when the graph of $f(x)$ is transformed as described.

(a) $f(x) = |x|$ is shifted 3 units down

A. $g(x) = |x - 3|$

B. $g(x) = |x| + 3$

C. $g(x) = |x| - 3$

D. $g(x) = |x + 3|$

(b) $f(x) = x^2$ is shifted 2 units left and reflected over the x-axis

A. $h(x) = -\sqrt{x - 2}$

B. $h(x) = -(x - 2)^2$

C. $h(x) = \sqrt{-x + 2}$

D. $h(x) = -(x + 2)^2$

(c) $f(x) = x^3$ is reflected over the y-axis, vertically stretched by a factor of 2, and shifted 1 unit up

A. $g(x) = 2(-x)^3 + 1$

B. $g(x) = (-2x)^3 - 1$

C. $g(x) = -(2x)^3 + 1$

D. $g(x) - 2x^3 + 1$

2.5 Combining and Composing Functions (FN5)

Objectives

- Find the sum, difference, product, quotient, and composition of two or more functions and evaluate them.

Combining and Composing Functions (FN5)

Activity 2.5.1 Let $f(x) = x^2 - 3x$ and $g(x) = x^3 - 4x^2 + 7$.

(a) Which of the following seems likely to be the most simplified form of $f(x) + g(x)$?

A. $x^2 - 3x + x^3 - 4x^2 + 7$

B. $x^5 - 7x^3 + 7$

C. $-x^3 + 5x^2 - 3x - 7$

D. $x^3 - 3x^2 - 3x + 7$

(b) Which of the following seems likely to be the most simplified form of $f(x) - g(x)$?

A. $x^3 - 3x^2 - 3x + 7$

B. $-x^3 + 5x^2 - 3x - 7$

C. $-x^3 - 3x^2 - 3x + 7$

D. $x^2 - 3x - x^3 + 4x^2 - 7$

Activity 2.5.2 Let $f(x) = \sqrt{x+1}$ and $g(x) = 5x$.

(a) Which of the following seems likely to be the most simplified form of $f(x) \cdot g(x)$?

A. $\sqrt{5x+1}$

B. $5\sqrt{x+1}$

C. $\sqrt{5x^2+5x}$

D. $5x\sqrt{x+1}$

(b) Which of the following seems likely to be the most simplified form of $\dfrac{f(x)}{g(x)}$?

A. $\dfrac{5x}{\sqrt{x+1}}$

B. $\dfrac{\sqrt{x+1}}{5x}$

C. $\sqrt{\dfrac{x}{5x}+\dfrac{1}{5x}}$

D. $\sqrt{\dfrac{5x}{x}+\dfrac{5x}{1}}$

Remark 2.5.3 In Activity 2.5.1 and Activity 2.5.2, we have found the sum, difference, product, and quotient of two functions. We can use the following notation for these newly created functions:

$$(f+g)(x) = f(x) + g(x)$$
$$(f-g)(x) = f(x) - g(x)$$
$$(f \cdot g)(x) = f(x) \cdot g(x)$$
$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$$

With $\left(\frac{f}{g}\right)(x)$, we note that the quotient is only defined when $g(x) \neq 0$.

Activity 2.5.4 Let $f(x) = \dfrac{1}{3x-5}$.

(a) Find $f(4)$.

A. $\dfrac{4}{3x-5}$

B. $\dfrac{1}{4(3x-5)}$

C. $\dfrac{1}{7}$

D. 7

Hint. See Remark 2.2.5 for a reminder of what this notation means!

(b) If you were asked to find $f(x^3-2)$, how do you think you would proceed?

A. Multiply the original function $\dfrac{1}{3x-5}$ by x^3-2.

B. Plug the expression x^3-2 in for all the x-values in $\dfrac{1}{3x-5}$.

C. Plug the original function $\dfrac{1}{3x-5}$ in for all the x-values in x^3-2.

D. Multiply $3x-5$ by x^3-2.

(c) Find $f(x^3-2)$.

A. $\dfrac{1}{3x-5} \cdot (x^3-2)$

B. $\dfrac{1}{3(x^3-2)-5}$

C. $\left(\dfrac{1}{3x-5}\right)^3 - 2$

D. $(3x-5)(x^3-2)$

(d) What if we gave the expression x^3-2 a name? Let's define $g(x) = x^3-2$. What's another way we could denote $f(x^3-2)$?

A. $f(x) \cdot g(x)$

B. $g(f(x))$

C. $f(g(x))$

D. $\dfrac{f(x)}{g(x)}$

Combining and Composing Functions (FN5)

Definition 2.5.5 Given the functions $f(x)$ and $g(x)$, we define the **composition of** f **and** g to be the new function $h(x)$ given by

$$h(x) = f(g(x)).$$

We also sometimes use the notation

$$f \circ g$$

or

$$(f \circ g)(x)$$

to refer to $f(g(x))$. $\Diamond$

Remark 2.5.6 When discussing the composite function $f(g(x))$, also written as $(f \circ g)(x)$, we often call $g(x)$ the ”inner function” and $f(x)$ the ”outer function”. It is important to note that the inner function is actually the first function that gets applied to a given input, and then the outer function is applied to the output of the inner function.

Combining and Composing Functions (FN5)

Activity 2.5.7 Let $f(x) = \dfrac{1}{3x-5}$ and $g(x) = x^3 - 2$.

(a) Find $f(g(x))$.

A. $\dfrac{x^3-2}{3x-5}$

B. $\dfrac{1}{(3x-5)(x^3-2)}$

C. $\dfrac{1}{3(x^3-2)-5}$

D. $\left(\dfrac{1}{3x-5}\right)^3 - 2$

(b) Find $g(f(x))$.

A. $\dfrac{x^3-2}{3x-5}$

B. $\dfrac{1}{(3x-5)(x^3-2)}$

C. $\dfrac{1}{3(x^3-2)-5}$

D. $\left(\dfrac{1}{3x-5}\right)^3 - 2$

Remark 2.5.8 We can also evaluate the composition of two functions at a particular value just as we did with one function. For example, we may be asked to find something like $f(g(2))$ or $(g \circ f)(-3)$.

Combining and Composing Functions (FN5)

Activity 2.5.9 Let $f(x) = 2x^3$ and $g(x) = \sqrt{6-x}$.

(a) Find $f(g(2))$.

A. 14 B. 16 C. 18 D. 20 E. undefined

(b) Find $(g \circ f)(-3)$.

A. 50 B. 54 C. $\sqrt{60}$ D. $\sqrt{-48}$ E. undefined

(c) Find $(f \circ g)(10)$.

A. $2(\sqrt{-4})^3$

B. 16

C. $\sqrt{-1994}$

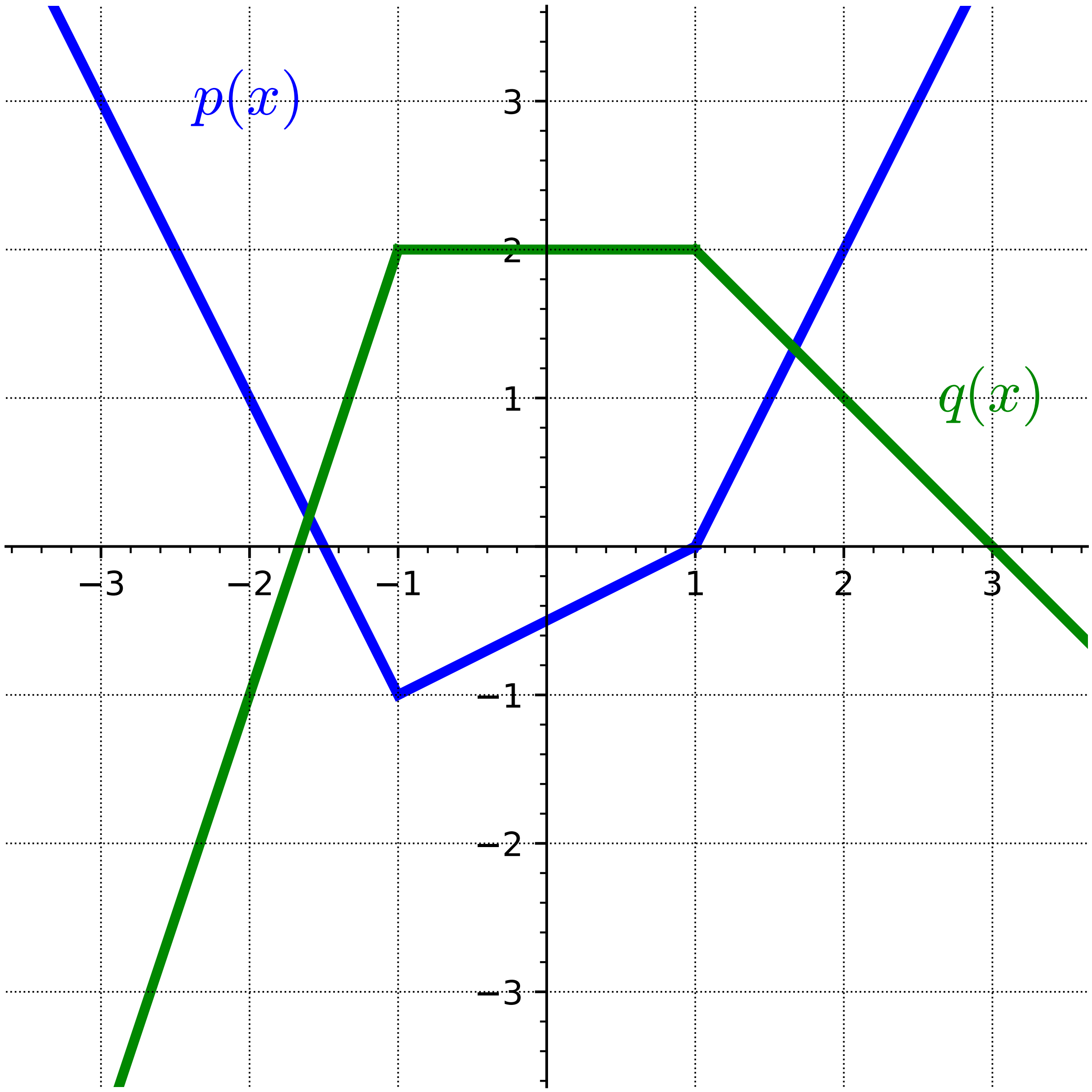
D. -16

E. undefined

Remark 2.5.10 As we saw in Activity 2.5.9, in order for a composite function to make sense, we need to ensure that the range of the inner function lies within the domain of the outer function so that the resulting composite function is defined at every possible input.

Remark 2.5.11 In addition to the possibility that functions are given by formulas, functions can be given by tables or graphs. We can think about composite functions in these settings as well, and the following activities prompt us to consider functions given in this way.

Activity 2.5.12 Let functions p and q be given by the graphs below.



Find each of the following. If something is not defined, explain why.

(a) $(p \circ q)(0)$

(b) $q(p(0))$

(c) $p(p(1))$

(d) $(q \circ p)(-3)$

(e) Find two values of x such that $q(p(x)) = 2$.

Activity 2.5.13 Let functions f and g be given by the tables below.

x	$f(x)$
0	6
1	4
2	3
3	4
4	7

Table 2.5.14

x	$g(x)$
0	1
1	3
2	0
3	5
4	2

Table 2.5.15

Find each of the following. If something is not defined, explain why.

(a) $(f \circ g)(2)$

(b) $(g \circ f)(3)$

(c) $g(f(4))$

(d) For what value(s) of x is $f(g(x)) = 4$?

(e) What are the domain and range of $(f \circ g)(x)$?

Combining and Composing Functions (FN5)

Remark 2.5.16 Now that we've seen how to compose functions, let's take a look at how composition of functions can be applicable in our lives.

Activity 2.5.17 Suppose you have a \$50-off coupon and there is a 15%-off sale on a TV. If you are allowed to apply both the coupon and sale price of the TV, which one should you apply first? Let's investigate to determine the better deal.

(a) Write a cost function, $C(p)$, for the price of the TV, p, if you applied the \$50-off coupon.

(b) Write a cost function, $S(p)$, for the price of the TV, p, if you applied the 15%-off sale.

(c) Suppose you apply the coupon first and then the 15%-off sale. Write a composite function to represent this situation.

(d) Suppose you apply the 15%-off sale first and then the coupon. Write a composite function to represent this situation.

(e) Suppose the original cost of the TV is \$500. Determine the cost of the TV if you applied the coupon first.

(f) Suppose the original cost of the TV is \$500. Determine the cost of the TV if you applied the 15%-off sale first.

(g) Which is the better deal and how much would you save?

2.6 Finding the Inverse Function (FN6)

Objectives

- Find the inverse of a one-to-one function.

Finding the Inverse Function (FN6)

Remark 2.6.1 A function is a process that converts a collection of inputs to a corresponding collection of outputs. One question we can ask is: for a particular function, can we reverse the process and think of the original function's outputs as the inputs?

Finding the Inverse Function (FN6)

Activity 2.6.2 Temperature can be measured using many different units such as Fahrenheit, Celsius, and Kelvin. Fahrenheit is what is usually reported on the news each night in the United States, while Celsius is commonly used for scientific work. We will begin by converting between these two units. To convert from degrees Fahrenheit to Celsius use the following formula.

$$C = \frac{5}{9}(F - 32)$$

(a) Room temperature is around 68 degrees Fahrenheit. Use the above equation to convert this temperature to Celsius.

A. 5.8

B. 20

C. 155.4

D. 293

(b) Solve the equation $C = \frac{5}{9}(F - 32)$ for F in terms of C.

A. $F = \frac{5}{9}C + 32$

B. $F = \frac{5}{9}C - 32$

C. $F = \frac{9}{5}(C + 32)$

D. $F = \frac{9}{5}C + 32$

(c) Alternatively, 20 degrees Celsius is a fairly comfortable temperature. Use your solution for F in terms of C to convert this temperature to Fahrenheit.

A. 43.1

B. -20.9

C. 93.6

D. 68

Finding the Inverse Function (FN6)

Remark 2.6.3 Notice that when you converted 68 degrees Fahrenheit, you got a value of 20 degrees Celsius. Alternatively, when you converted 20 degrees Celsius, you got 68 degrees Fahrenheit. This indicates that the equation you were given for C and the equation you found for F are inverses.

Finding the Inverse Function (FN6)

Definition 2.6.4 Let f be a function. If there exists a function g such that

$$f(g(x)) = x \quad \text{and} \quad g(f(x)) = x$$

for all x, then we say f has an **inverse function**, or that g is the **inverse of** f. When a given function f has an inverse function, we usually denote it as f^{-1}, which is read as "f inverse". $\Diamond$

Finding the Inverse Function (FN6)

Remark 2.6.5 An inverse is a function that "undoes" another function. For any input in the domain, the function g will reverse the process of f.

Finding the Inverse Function (FN6)

Activity 2.6.6 Let $f(x) = 2x + 7$ and $g(x) = \dfrac{x-7}{2}$.

(a) Compute $f(g(x))$.

(b) Compute $g(f(x))$.

(c) What can you conclude about $f(x)$ and $g(x)$?

Finding the Inverse Function (FN6)

Activity 2.6.7 It is important to note that in Definition 2.6.4 we say "if there exists a function," but we don't guarantee that this is always the case. How can we determine whether a function has a corresponding inverse or not? Consider the following two functions f and g represented by the tables.

Table 2.6.8

x	$f(x)$
0	6
1	4
2	3
3	4
4	6

Table 2.6.9

x	$g(x)$
0	3
1	1
2	4
3	2
4	0

(a) Use the definition of $g(x)$ in Table 2.6.9 to find an x such that $g(x) = 4$.

A. $x = 0$

B. $x = 1$

C. $x = 2$

D. $x = 3$

E. $x = 4$

(b) Is it possible to reverse the input and output rows of the function $g(x)$ and have the new table result in a function?

(c) Use the definition of $f(x)$ in Table 2.6.8 to find an x such that $f(x) = 4$.

A. $x = 0$

B. $x = 1$

C. $x = 2$

D. $x = 3$

E. $x = 4$

(d) Is it possible to reverse the input and output rows of the function $f(x)$ and have the new table result in a function?

Finding the Inverse Function (FN6)

Remark 2.6.10 Some functions, like $f(x)$ in Table 2.6.8, have a given output value that corresponds to two or more input values: $f(0) = 6$ and $f(4) = 6$. If we attempt to reverse the process of this function, we have a situation where the new input 6 would correspond to two potential outputs.

Finding the Inverse Function (FN6)

Definition 2.6.11 A **one-to-one function** is a function in which each output value corresponds to exactly one input. $\Diamond$

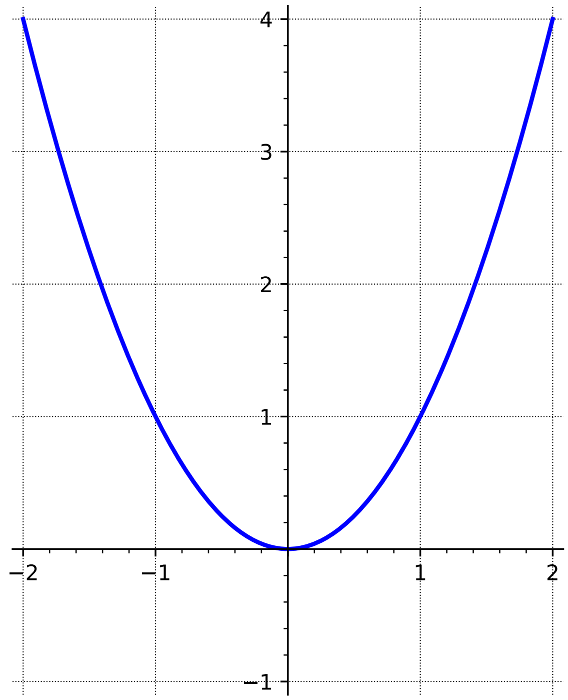
Finding the Inverse Function (FN6)

Remark 2.6.12 A function must be one-to-one in order to have an inverse.

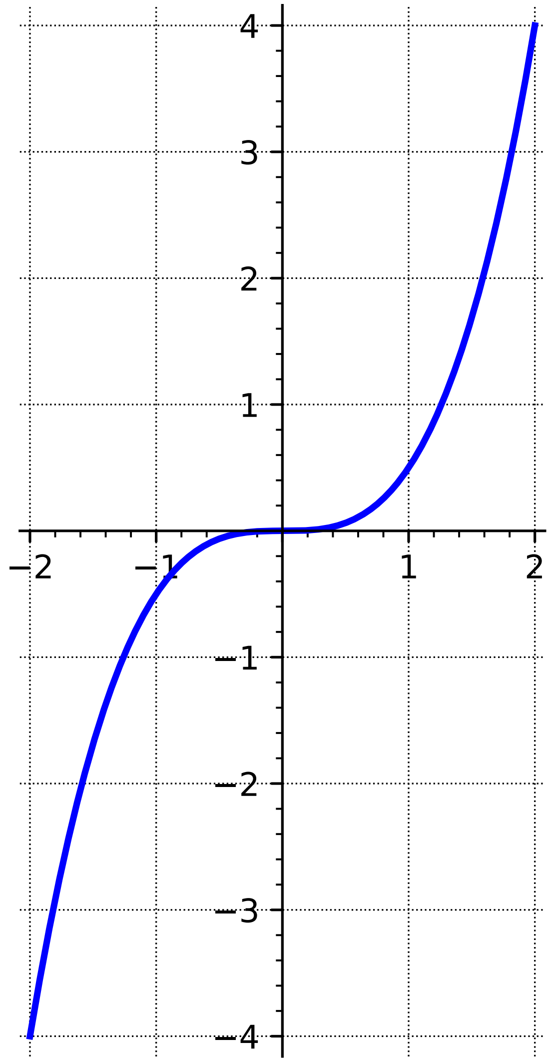
Finding the Inverse Function (FN6)

Activity 2.6.13 For each of the following graphs, determine if they represent a function that is one-to-one or not. If they are not one-to-one, what outputs have the same input?

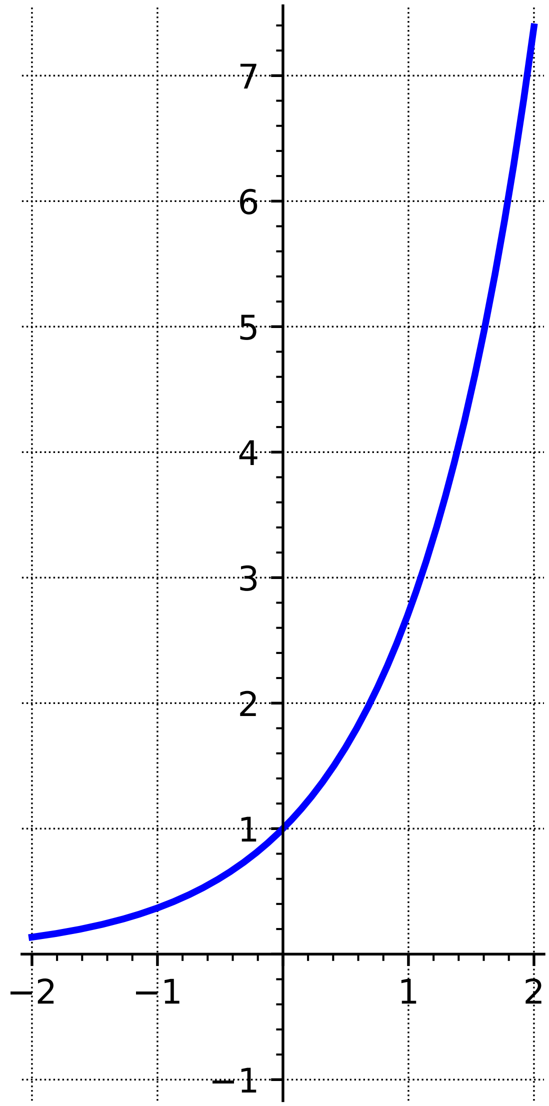
(a)



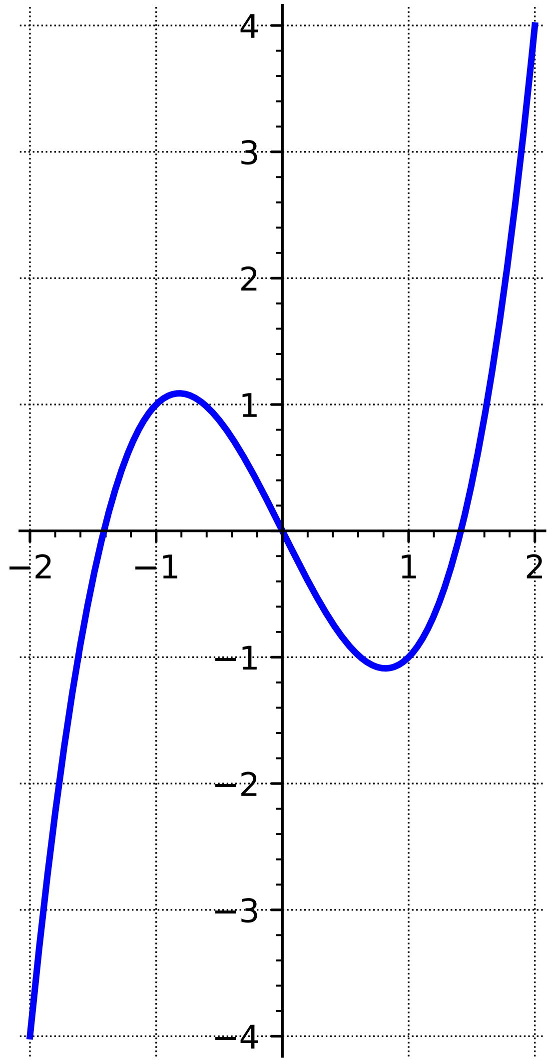
(b)



(c)



(d)



(e) For each graph that was not one-to-one, draw a line connecting the points where two inputs had the same output. What do you notice about the lines?

Finding the Inverse Function (FN6)

Observation 2.6.14 When two outputs have the same input, this means that a horizontal line intersects the graph in two places. This leads us to the **horizontal line test** for one-to-one functions.

Finding the Inverse Function (FN6)

Activity 2.6.15 Consider the function $f(x) = \dfrac{x-5}{3}$.

(a) When you evaluate this expression for a given input value of x, what operations do you perform and in what order?

A. divide by 3, subtract 5

B. subtract 5, divide by 3

C. add 5, multiply by 3

D. multiply by 3, add 5

(b) When you "undo" this expression to solve for a given output value of y, what operations do you perform and in what order?

A. divide by 3, subtract 5

B. subtract 5, divide by 3

C. add 5, multiply by 3

D. multiply by 3, add 5

(c) This set of operations reverses the process for the original function, so can be considered the inverse function. Write an equation to express the inverse function f^{-1}.

A. $f^{-1}(x) = \dfrac{x}{3} - 5$

B. $f^{-1}(x) = \dfrac{x-5}{3}$

C. $f^{-1}(x) = 5(x+3)$

D. $f^{-1}(x) = 3x+5$

(d) Check your answer to the previous question by finding $f(f^{-1}(x))$ and $f^{-1}(f(x))$.

Finding the Inverse Function (FN6)

Observation 2.6.16 To find the inverse of a one-to-one function, perform the reverse operations in the opposite order.

Finding the Inverse Function (FN6)

Activity 2.6.17 Let's look at an alternate method for finding an inverse by solving the function for x and then interchanging the x and y.

$$h(x) = \frac{x}{x+1}$$

(a) Interchange the variables x and y.

A. $y = \frac{x}{x+1}$

B. $x = \frac{y}{x+1}$

C. $x = \frac{y}{y+1}$

D. $x = \frac{x}{y+1}$

(b) Eliminate the denominator.

A. $y(x+1) = x$

B. $x(x+1) = y$

C. $x(y+1) = x$

D. $x(y+1) = y$

(c) Distribute and gather the y terms together.

A. $yx + y = x$

B. $x^2 + x = y$

C. $xy - y = -x$

D. $xy = 0$

(d) Write the inverse function, by factoring and solving for y.

A. $h^{-1}(x) = \frac{x}{x-1}$

B. $h^{-1}(x) = \frac{x}{1-x}$

C. $h^{-1}(x) = \frac{-x}{1-x}$

D. $h^{-1}(x) = \frac{x+1}{x}$

Finding the Inverse Function (FN6)

Activity 2.6.18 Find the inverse of each function, using either method. Check your answer using function composition.

(a) $g(x) = \dfrac{4x-1}{7}$

A. $g^{-1}(x) = \dfrac{7x+1}{4}$

B. $g^{-1}(x) = \dfrac{7x}{4} + 1$

C. $g^{-1}(x) = \dfrac{4x+1}{7}$

D. $g^{-1}(x) = \dfrac{7}{4x-1}$

(b) $f(x) = 2x^3 - 3$

A. $f^{-1}(x) = \dfrac{1}{2}x^{\frac{1}{3}} + 3$

B. $f^{-1}(x) = \left(\dfrac{1}{2}x\right)^{\frac{1}{3}} + 3$

C. $f^{-1}(x) = \left(\dfrac{1}{2}x + 3\right)^{\frac{1}{3}}$

D. $f^{-1}(x) = \dfrac{1}{2}(x+3)^{\frac{1}{3}}$

Finding the Inverse Function (FN6)

Activity 2.6.19 Consider the functions $f(x) = x^2$ and $g(x) = \sqrt{x}$.

(a) Compute each of the following

(i) $\left(\sqrt{9}\right)^2$ (ii) $\left(\sqrt{25}\right)^2$ (iii) $\left(\sqrt{17}\right)^2$

(b) Compute each of the following

(i) $\sqrt{3^2}$

(ii) $\sqrt{11^2}$

(iii) $\sqrt{37^2}$

(iv) $\sqrt{(-4)^2}$

(c) Are the functions $f(x) = x^2$ and $g(x) = \sqrt{x}$ one-to-one?

Hint. Use the horizontal line test.

(d) For which values of x is it true that $f(g(x)) = x$?

(e) For which values of x is it true that $g(f(x)) = x$?

Finding the Inverse Function (FN6)

Observation 2.6.20 While $f(x) = x^2$ is not a one-to-one function and thus cannot have its inverse, we can **restrict the domain** to find an invertible function. In this case, considering $f_0(x) = x^2$ defined only on the interval $[0, \infty)$, $f_0(x)$ is a one-to-one function with inverse $f_0^{-1}(x) = \sqrt{x}$.

Finding the Inverse Function (FN6)

Remark 2.6.21 When finding inverses algebraically, it is tempting to write $\sqrt{x^2} = x$, but this only true for non-negative x-values. In general, $\sqrt{x^2} = |x|$.

Chapter 3

Linear Functions (LF)

Objectives

How do we model scenarios that have a constant rate of change?
By the end of this chapter, you should be able to...

1. Determine the average rate of change of a given function over a given interval. Find the slope of a line.

2. Determine an equation for a line when given two points on the line and when given the slope and one point on the line. Express these equations in slope-intercept or point-slope form and determine the slope and y-intercept of a line given an equation.

3. Graph a line given its equation or some combination of characteristics, such as points on the graph, a table of values, the slope, or the intercepts.

4. Use slope relationships to determine whether two lines are parallel or perpendicular, and find the equation of lines parallel or perpendicular to a given line through a given point.

5. Build linear models from verbal descriptions, and use the models to establish conclusions, including by contextualizing the meaning of slope and intercept parameters.

6. Solve a system of two linear equations in two variables.

7. Solve questions involving applications of systems of equations.

3.1 Slope and Average Rate of Change (LF1)

Objectives

- Determine the average rate of change of a given function over a given interval. Find the slope of a line.

Slope and Average Rate of Change (LF1)

Remark 3.1.1 This section will explore ideas around average rate of change and slope. To help us get started, let's take a look at a context in which these ideas can be helpful.

Slope and Average Rate of Change (LF1)

Activity 3.1.2 Robert came home one day after school to a very hot house! When he got home, the temperature on the thermostat indicated that it was 85 degrees! Robert decided that was too hot for him, so he turned on the air conditioner. The table of values below indicate the temperature of his house after turning on the air conditioner.

Table 3.1.3

Time (minutes)	Temperature (degrees Fahrenheit)
0	85
1	84.3
2	83.6
3	82.9
4	82.2
5	81.5
6	80.8

(a) How much did the temperature change from 0 to 2 minutes?

A. The temperature decreased by 0.7 degrees

B. The temperature decreased by 1.4 degrees

C. The temperature increased by 0.7 degrees

D. The temperature increased by 1.4 degrees

(b) How much did the temperature change from 4 to 6 minutes?

A. The temperature decreased by 0.7 degrees

B. The temperature decreased by 1.4 degrees

C. The temperature increased by 0.7 degrees

D. The temperature increased by 1.4 degrees

(c) If Robert wanted to know how much the temperature was decreasing each minute, how could he figure that out?

(d) How would you describe the overall behavior of the temperature of Robert's house?

Remark 3.1.4 Notice in Activity 3.1.2 that the temperature appears to be decreasing at a constant rate (i.e., the temperature decreased 1.4 degrees for every 2-minute interval). Upon further investigation, you might have also noticed that the temperature decreased by 0.7 degrees every minute.

Activity 3.1.5 Refer back to the data Robert collected of the temperature of his house after turning on the air conditioner (Table 3.1.3).

(a) If this pattern continues, what will the temperature be after 8 minutes?

A. 80.1

B. 78.7

C. 80.8

D. 79.4

(b) If this pattern continues, how long will it take for Robert's house to reach 78 degrees?

A. 12 minutes

B. 9 minutes

C. 10 minutes

D. 11 minutes

Remark 3.1.6 An average rate of change helps us to see and understand how a function is generally behaving. For example, in Activity 3.1.2 and Activity 3.1.5, we began to see how the temperature of Robert's house was decreasing every minute the air conditioner was on. In other words, when looking at average rate of change, we are comparing how one quantity is changing with respect to something else changing.

Slope and Average Rate of Change (LF1)

Definition 3.1.7 The **average rate of change** of a function on a given interval measures how much the function's value changes per unit on that interval. For a function $f(x)$ on the interval $[a, b]$, it is calculated by the following expression:

Recall that we can use function notation to describe x- and y-values. $f(b)$, for instance represents the y-value when plugging in a value for x (or b).

$$\frac{f(b) - f(a)}{b - a}.$$

◊

Remark 3.1.8 Notice that to calculate the average rate of change over an interval $[a, b]$, we are using the two endpoints of the interval, namely $(a, f(a))$ and $(b, f(b))$.

Slope and Average Rate of Change (LF1)

Activity 3.1.9 Use the table below to answer the questions.

Table 3.1.10

x	$f(x)$
-5	28
-4	19
-3	12
-2	7
-1	4

(a) Applying Definition 3.1.7, what is the average rate of change when $x = -5$ to $x = -2$?

A. $\frac{1}{7}$

B. -3

C. -7

D. 7

(b) What is the average rate of change on the interval $[-4, -1]$?

A. -5

B. -3

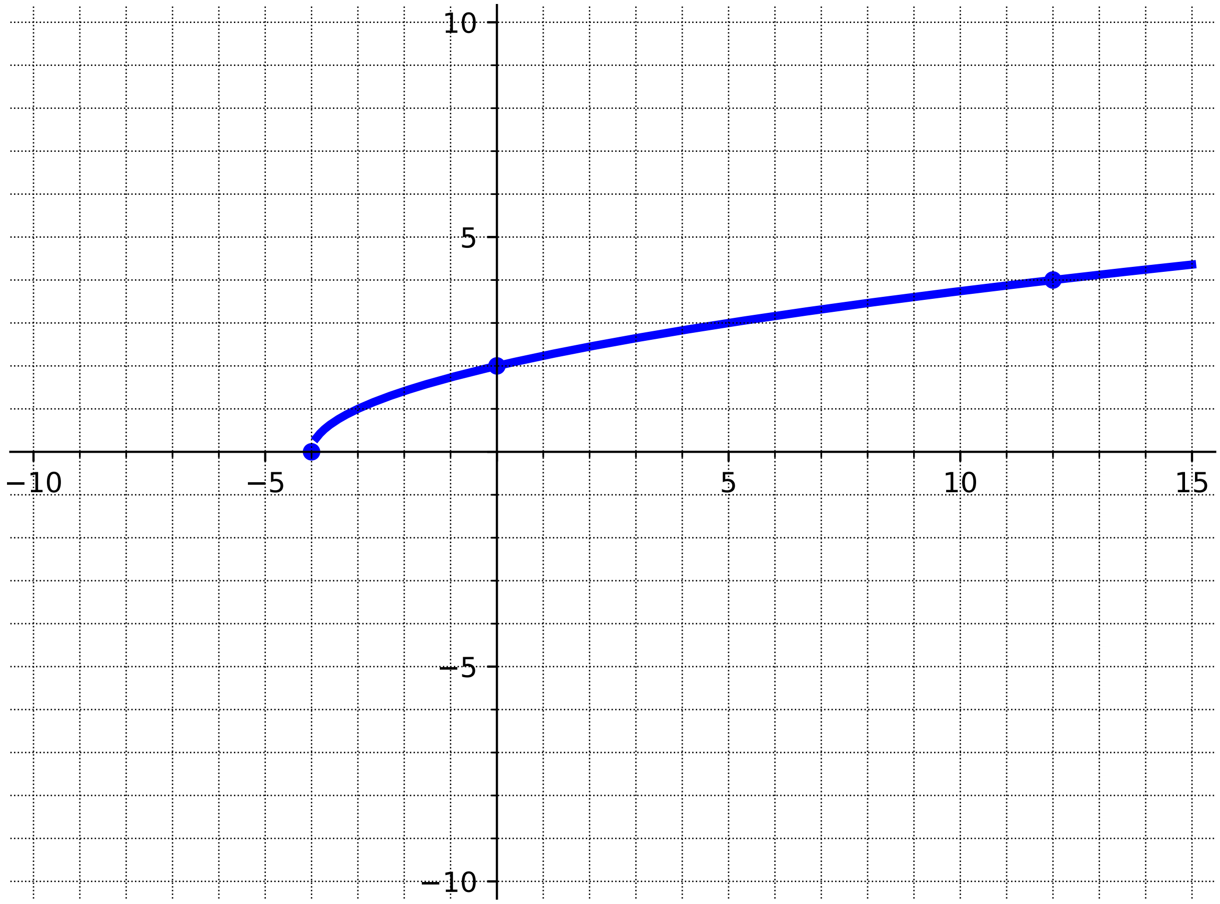
C. 5

D. 3

(c) Does this function have a constant average rate of change?

Slope and Average Rate of Change (LF1)

Activity 3.1.11 Use the graph to calculate the average rate of change on the given intervals.



(a) Applying Definition 3.1.7, what is the average rate of change on the interval $[-4, 0]$?

A. $-\frac{1}{2}$

B. $\frac{1}{2}$

C. -2

D. 2

(b) What is the average rate of change on the interval $[0, 12]$?

A. $\frac{1}{6}$

B. -6

C. $-\frac{1}{6}$

D. 6

Slope and Average Rate of Change (LF1)

Activity 3.1.12 Just like with tables and graphs, you should be able to find the average rate of change when given a function. For this activity, use the function

$$f(x) = -3x^2 - 1$$

to answer the following questions.

(a) Applying Definition 3.1.7, what is the average rate of change on the interval $[-2, 3]$?

A. $\frac{41}{5}$

B. -3

C. $-\frac{1}{3}$

D. $\frac{5}{41}$

(b) What is the average rate of change on the interval $[0, 4]$?

A. $\frac{2}{25}$

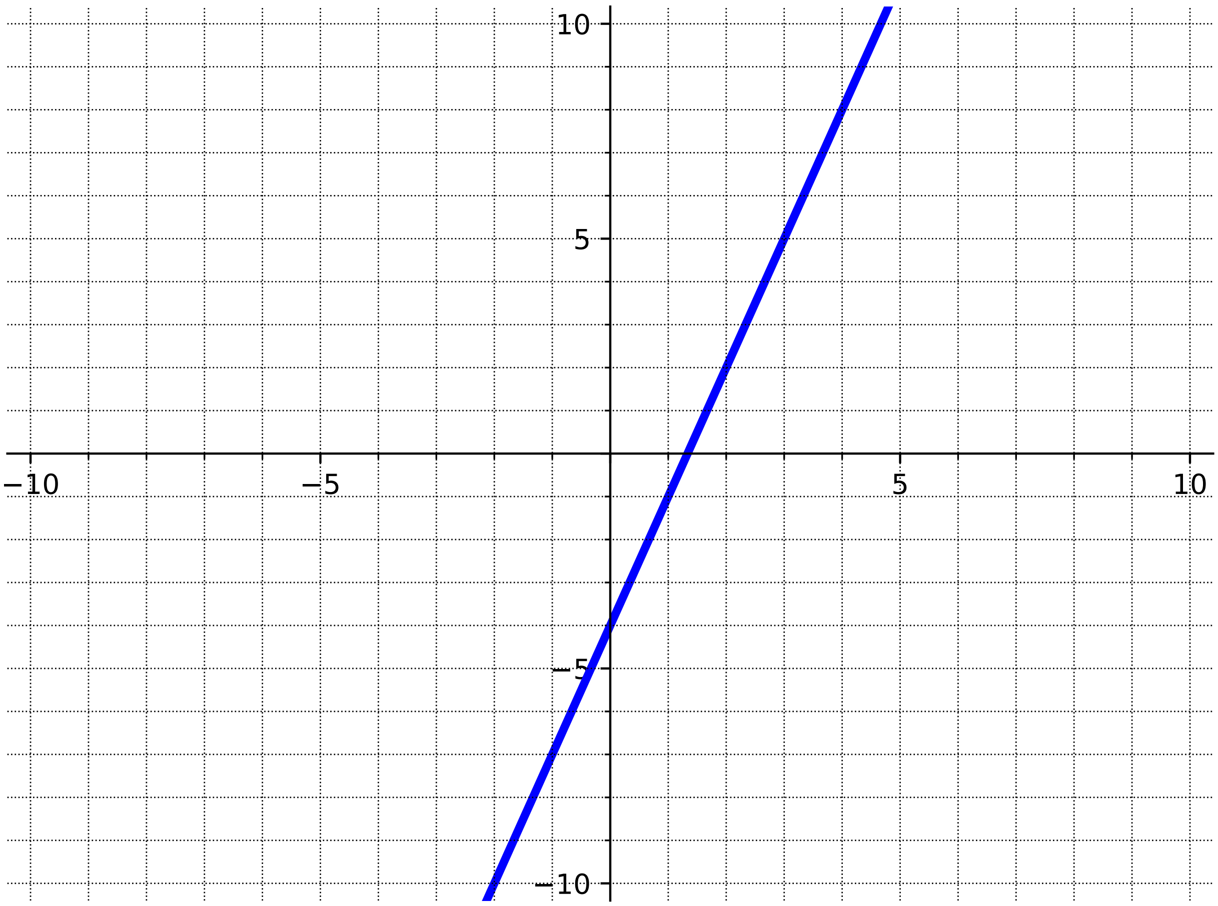
B. -50

C. $-\frac{1}{12}$

D. -12

Slope and Average Rate of Change (LF1)

Activity 3.1.13 Use the given graph of the function, $f(x) = 3x - 4$, to investigate the average rate of change of a linear function.



(a) What is the average rate of change on the interval $[-2, 0]$?

A. $\frac{1}{3}$

B. 3

C. $-\frac{1}{7}$

D. -7

(b) What is the average rate of change on the interval $[-1, 5]$? Notice that you cannot see the point at $x = 5$. How could you use the equation of the line to determine the y-value when $x = 5$?

A. 3

B. $\frac{1}{3}$

C. $\frac{2}{3}$

D. -3

(c) Based on your observations in parts (a) and (b), what do you think will be the average rate of change on the interval $[5, 25]$?

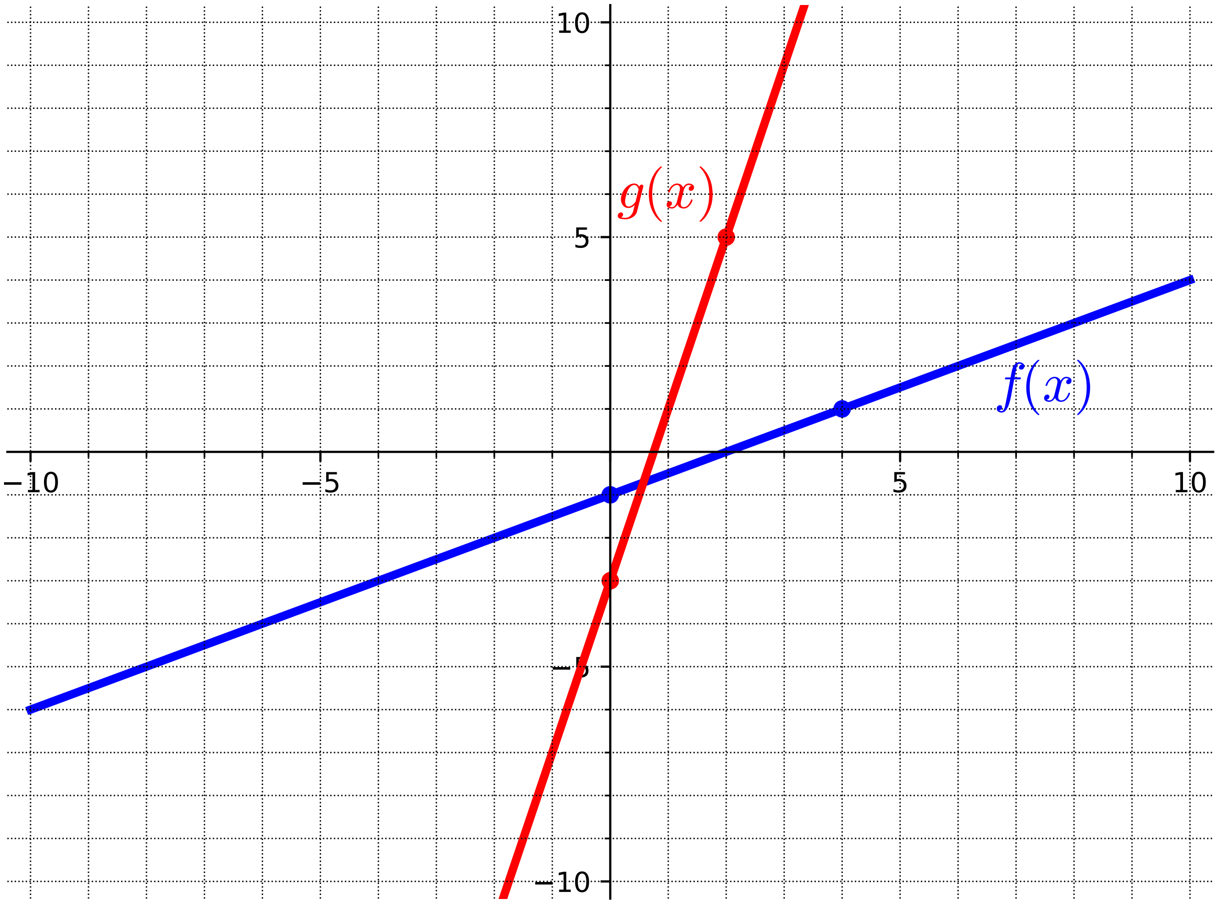
Remark 3.1.14 Notice in Activity 3.1.13, the average rate of change was the same regardless of which interval you were given. But in Activity 3.1.11, the average rate of change was not the same across different intervals.

Slope and Average Rate of Change (LF1)

Definition 3.1.15 The **slope** of a line is a constant that represents the direction and steepness of the line. For a linear function, the slope never changes - meaning it has a constant average rate of change. $\Diamond$

Slope and Average Rate of Change (LF1)

Activity 3.1.16 The steepness of a line depends on the vertical and horizontal distances between two points on the line. Use the graph below to compare the steepness, or slope, of the two lines.



(a) What is the vertical distance between the two points on the line $y = g(x)$ (the red line)?

A. 2

B. 4

C. 8

D. $\frac{1}{2}$

(b) What is the horizontal distance between the two points on the line $y = g(x)$ (the red line)?

A. 2

B. 4

C. 8

D. $\frac{1}{2}$

(c) Using information from parts (a) and (b), what value could we use to describe the steepness of the line $y = g(x)$ (the red line)?

A. 2

B. 4

C. 8

D. $\frac{1}{2}$

(d) What is the vertical distance between the two points on the line $y = f(x)$ (the blue line)?

A. 2

B. 4

C. 8

D. $\frac{1}{2}$

(e) What is the horizontal distance between the two points on the line $y = f(x)$ (the blue line)?

A. 2

B. 4

C. 8

D. $\frac{1}{2}$

(f) Using information from parts (d) and (e), what value could we use to describe the steepness of the line $y = f(x)$ (the blue line)?

A. 2

B. 4

C. 8

D. $\frac{1}{2}$

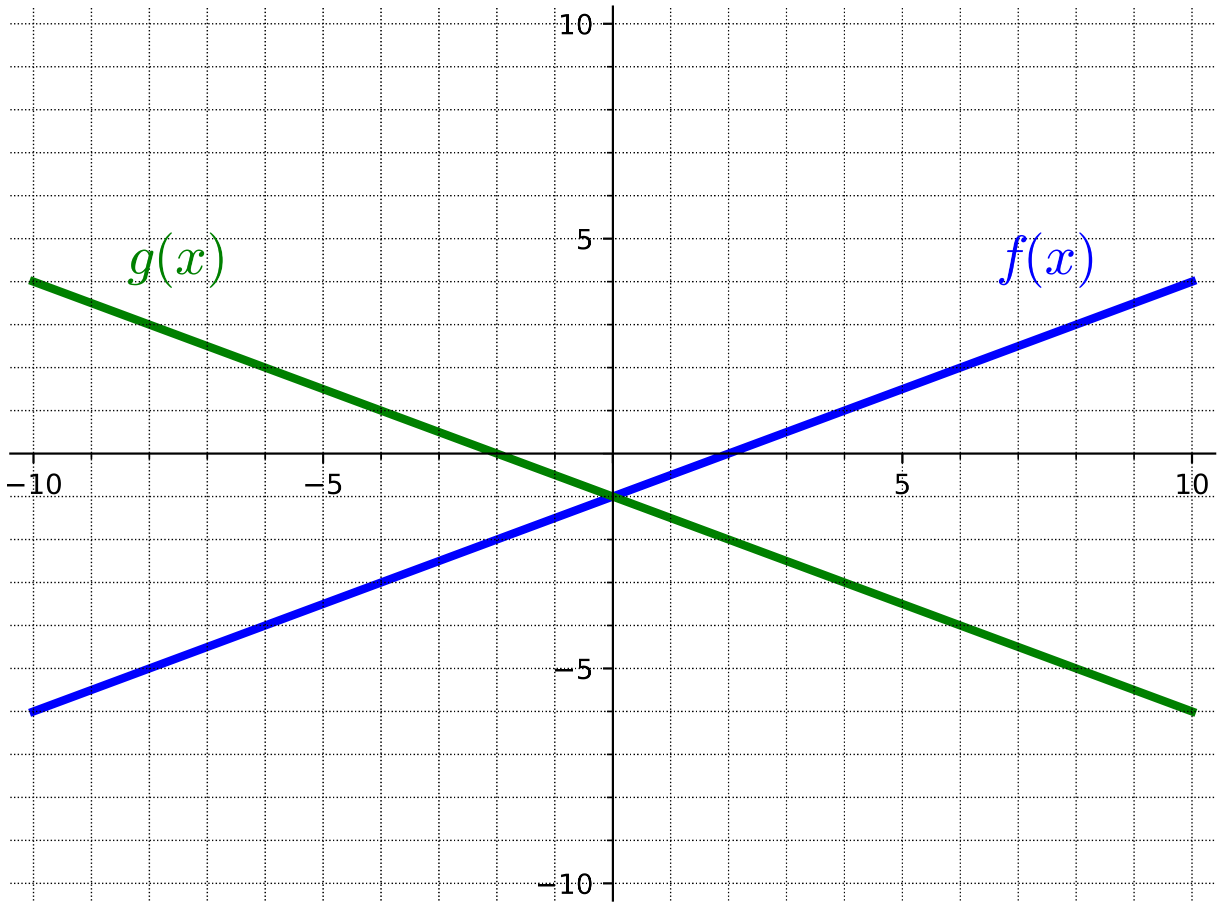
(g) Which line is the steepest?

Remark 3.1.17 The steepness, or slope, of a line can be found by the change in y (the vertical distance between two points on the line) divided by the change in x (the horizontal distance between two points on the line). Slope can be calculated as "rise over run."

Slope is a way to describe the steepness of a line. The red line in Activity 3.1.16 has a larger value for it's slope than the blue line. Thus, the red line is steeper than the blue line.

Slope and Average Rate of Change (LF1)

Activity 3.1.18 Now that we know how to find the slope (or steepness) of a line, let's look at other properties of slope. Use the graph below to answer the following questions.



(a) What is the slope of the line $y = f(x)$ (the blue line)?

A. $\frac{1}{2}$

B. 2

C. $-\frac{1}{2}$

D. -2

(b) What is the slope of $y = g(x)$ (the green line)?

A. $\frac{1}{2}$

B. 2

C. $-\frac{1}{2}$

D. -2

(c) How are the slopes of the lines similar?

(d) How are the slopes of the lines different?

Remark 3.1.19 Notice in Activity 3.1.18 that the slope does not just indicate how steep a line is, but also its direction. A negative slope indicates that the line is decreasing (from left to right) and a positive slope indicates that the line is increasing (from left to right).

Slope and Average Rate of Change (LF1)

Activity 3.1.20 Suppose $(-3, 7)$ and $(7, 2)$ are two points on a line.

(a) Plot these points on a graph and find the slope by using "rise over run."

A. $\frac{1}{2}$

B. 2

C. $-\frac{1}{2}$

D. -2

(b) Now calculate the slope by using the change in y over the change in x.

A. $\frac{1}{2}$

B. 2

C. $-\frac{1}{2}$

D. -2

(c) What do you notice about the slopes you got in parts (a) and (b)?

Slope and Average Rate of Change (LF1)

Remark 3.1.21 We can calculate slope (m) by finding the change in y and dividing by the change in x. Mathematically, this means that when given (x_1, y_1) and (x_2, y_2),

$$m = \frac{y_2 - y_1}{x_2 - x_1}.$$

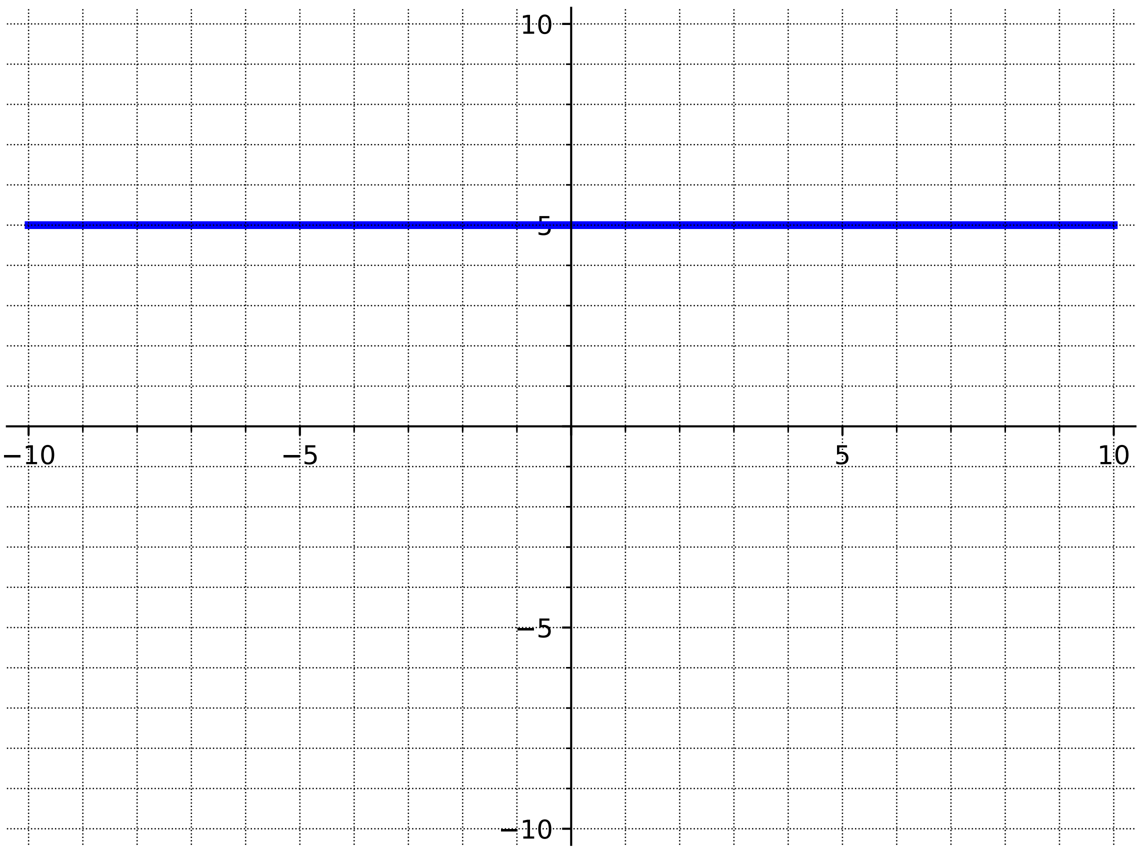
Slope and Average Rate of Change (LF1)

Activity 3.1.22 Calculate the slope of each representation of a line using the slope formula.

(a)

x	$f(x)$
-2	-7
-1	-4
0	-1
1	2
2	5

(b)



(c) $(-4, 7)$ and $(-4, 1)$

Remark 3.1.23 In Activity 3.1.22, there were slopes that were 0 and undefined. When a line is vertical, the slope is undefined. This means that there is only a vertical distance between two points and there is no horizontal distance. When a line is horizontal, the slope is 0. This means that the line never rises vertically, giving a vertical distance of zero.

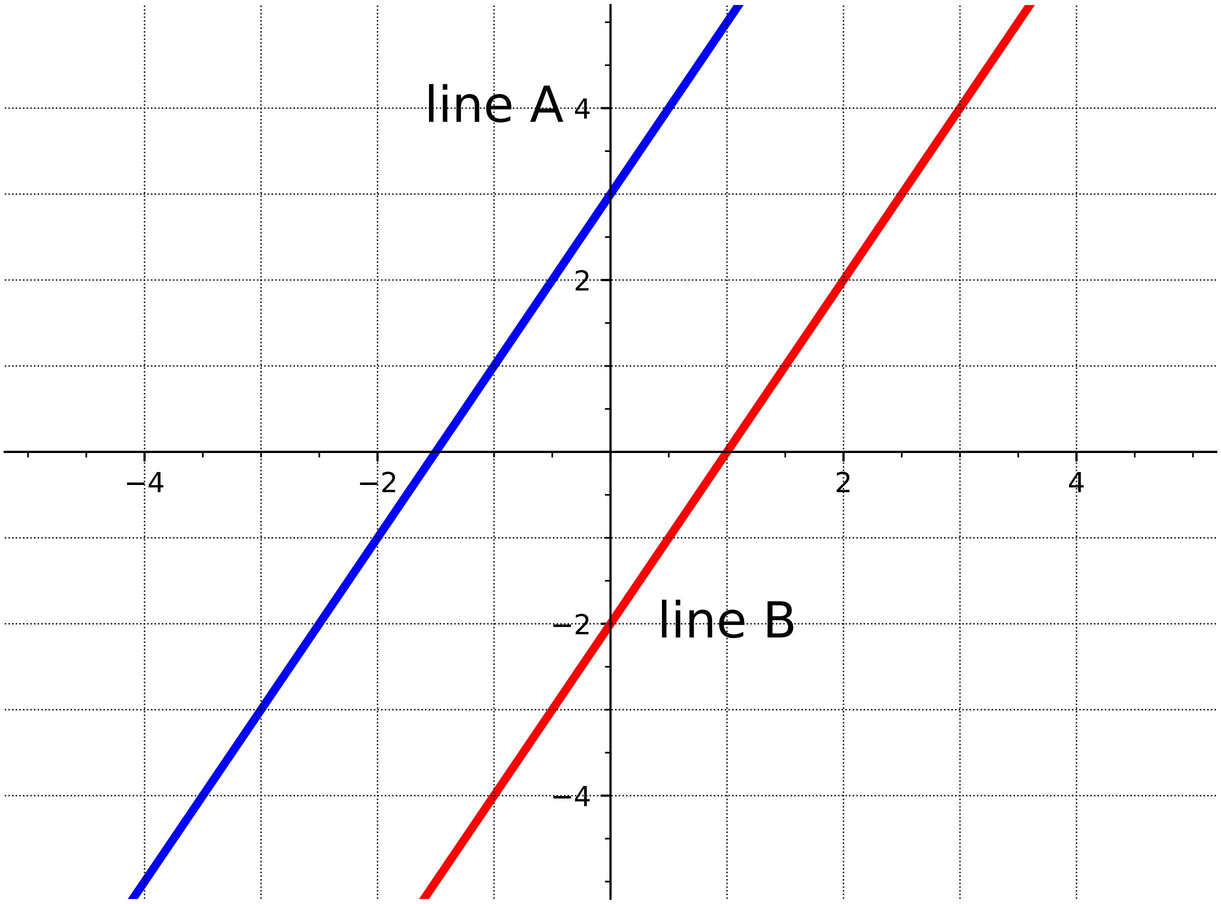
3.2 Equations of Lines (LF2)

Objectives

- Determine an equation for a line when given two points on the line and when given the slope and one point on the line. Express these equations in slope-intercept or point-slope form and determine the slope and y-intercept of a line given an equation.

Equations of Lines (LF2)

Activity 3.2.1 Consider the graph of two lines.



(a) Find the slope of line A.

A. 1

B. 2

C. $\frac{1}{2}$

D. -2

(b) Find the slope of line B.

A. 1

B. 2

C. $\frac{1}{2}$

D. -2

(c) Find the y-intercept of line A.

A. -2

B. -1.5

C. 1

D. 3

(d) Find the y-intercept of line B.

A. -2
B. -1.5
C. 1
D. 3

(e) What is the same about the two lines?

(f) What is different about the two lines?

Remark 3.2.2 Notice that in Activity 3.2.1 the lines have the same slope but different y-intercepts. It is not enough to just know one piece of information to determine a line, you need both a slope and a point.

Equations of Lines (LF2)

Definition 3.2.3 Linear functions can be written in **slope-intercept form**

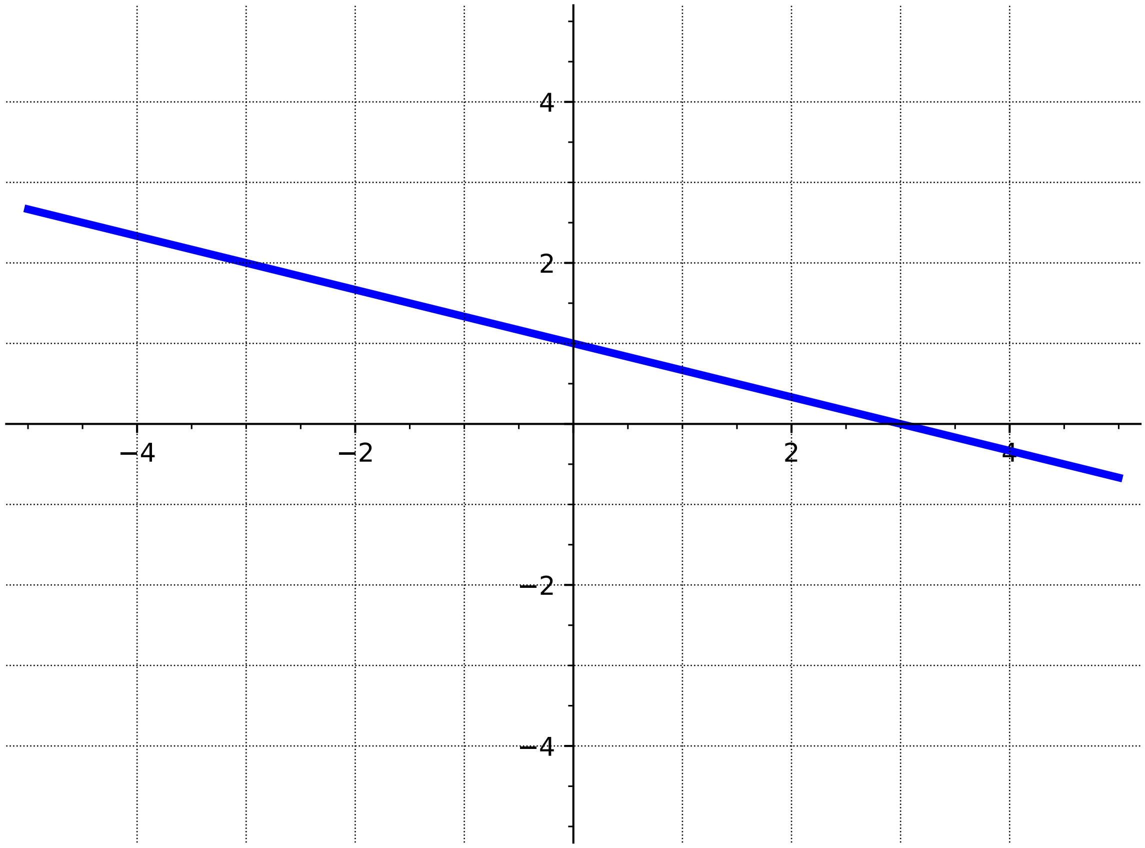
$$f(x) = mx + b$$

where b is the y-intercept (or starting value) and m is the slope (or constant rate of change).

$\Diamond$

Equations of Lines (LF2)

Activity 3.2.4 Write the equation of each line in slope-intercept form.



(a)

A. $y = -3x + 1$

B. $y = -x + 3$

C. $y = -\frac{1}{3}x + 1$

D. $y = -\frac{1}{3}x + 3$

(b) The slope is 4 and the y-intercept is $(0, -3)$.

A. $f(x) = 4x - 3$

B. $f(x) = 3x - 4$

C. $f(x) = -4x + 3$

D. $f(x) = 4x + 3$

(c) Two points on the line are $(0, 1)$ and $(2, 4)$.

A. $y = 2x + 1$

B. $y = -\frac{3}{2}x + 4$

C. $y = \dfrac{3}{2}x + 1$

D. $y = \dfrac{3}{2}x + 4$

(d)

x	$f(x)$
-2	-8
0	-2
1	1
4	10

A. $f(x) = -3x - 2$

B. $f(x) = -\dfrac{1}{3}x - 2$

C. $f(x) = 3x + 1$

D. $f(x) = 3x - 2$

Equations of Lines (LF2)

Activity 3.2.5 Let's try to write the equation of a line given two points that don't include the y-intercept.

(a) Plot the points $(2, 1)$ and $(-3, 4)$.

(b) Find the slope of the line joining the points.

A. $-\frac{5}{3}$

B. $-\frac{3}{5}$

C. $\frac{3}{5}$

D. -3

(c) When you draw a line connecting the two points, it's often hard to draw an accurate enough graph to determine the y-intercept of the line exactly. Let's use the slope-intercept form and one of the given points to solve for the y-intercept. Try using the slope and one of the points on the line to solve the equation $y = mx + b$ for b.

A. 2

B. $\frac{11}{5}$

C. $\frac{5}{2}$

D. 3

(d) Write the equation of the line in slope-intercept form.

Equations of Lines (LF2)

Remark 3.2.6 It would be nice if there was another form of the equation of a line that works for any points and does not require the y-intercept.

Equations of Lines (LF2)

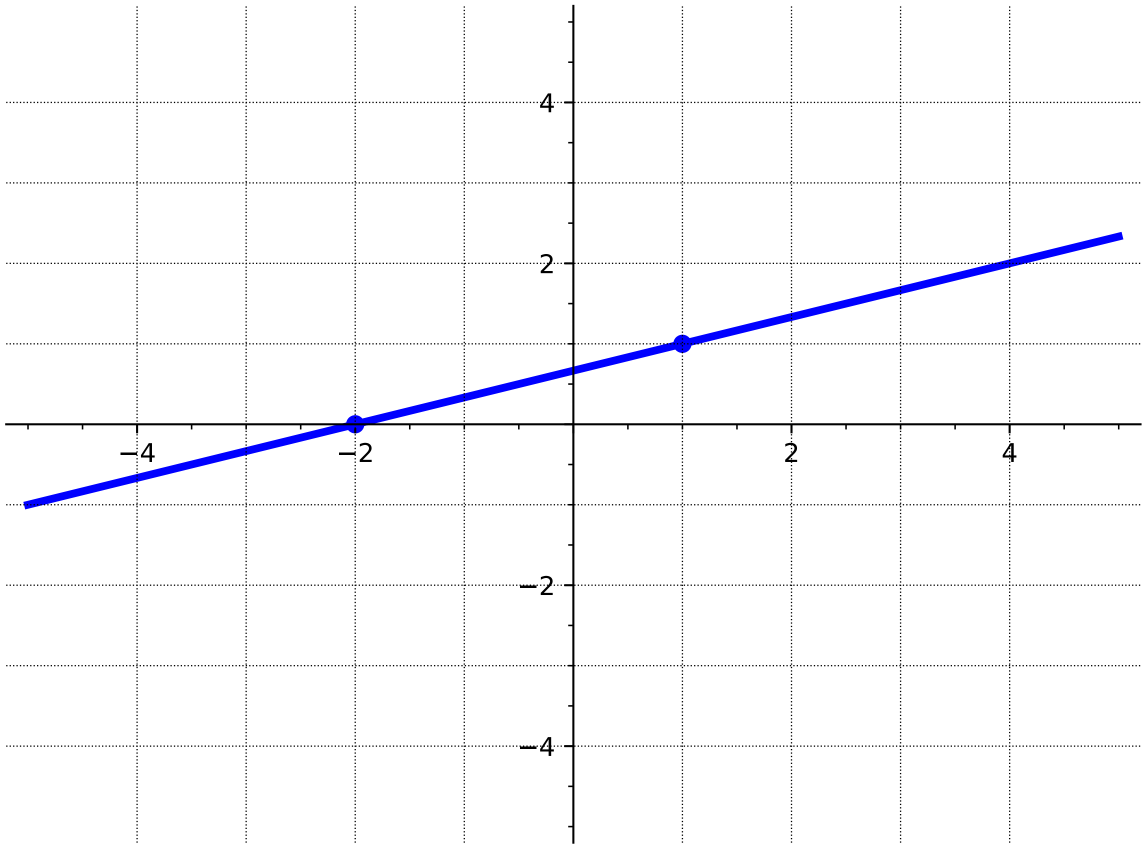
Definition 3.2.7 Linear functions can be written in **point-slope form**

$$y - y_0 = m(x - x_0)$$

where (x_0, y_0) is any point on the line and m is the slope. ♢

Equations of Lines (LF2)

Activity 3.2.8 Write an equation of each line in point-slope form.



(a)

A. $y = \frac{1}{3}x + \frac{2}{3}$

B. $y - 1 = 3(x - 1)$

C. $y - 1 = \frac{1}{3}(x - 1)$

D. $y + 2 = \frac{1}{3}(x + 2)$

E. $y = \frac{1}{3}(x + 2)$

(b) The slope is 4 and $(-1, -7)$ is a point on the line.

A. $y + 7 = 4(x + 1)$

B. $y - 7 = 4(x - 1)$

C. $y + 1 = 4(x + 7)$

D. $y - 4 = 7(x - 1)$

(c) Two points on the line are $(1, 0)$ and $(2, -4)$.

Equations of Lines (LF2)

A. $y = -4x + 1$

B. $y - 0 = -2(x - 1)$

C. $y + 4 = -4(x - 2)$

D. $y + 4 = -3(x - 2)$

(d)

x	$f(x)$
-2	-8
1	1
4	10

A. $y + 8 = 3(x - 2)$

B. $y - 1 = -\frac{1}{3}(x - 1)$

C. $y + 8 = -\frac{1}{3}(x + 2)$

D. $y - 10 = 3(x - 4)$

Activity 3.2.9 Consider again the two points from Activity 3.2.5, $(2, 1)$ and $(-3, 4)$.

(a) Use point-slope form to find an equation of the line.

A. $y = -\frac{3}{5}x + \frac{11}{5}$

B. $y - 1 = -\frac{3}{5}(x - 2)$

C. $y - 4 = -\frac{3}{5}(x + 3)$

D. $y - 2 = -\frac{3}{5}(x - 1)$

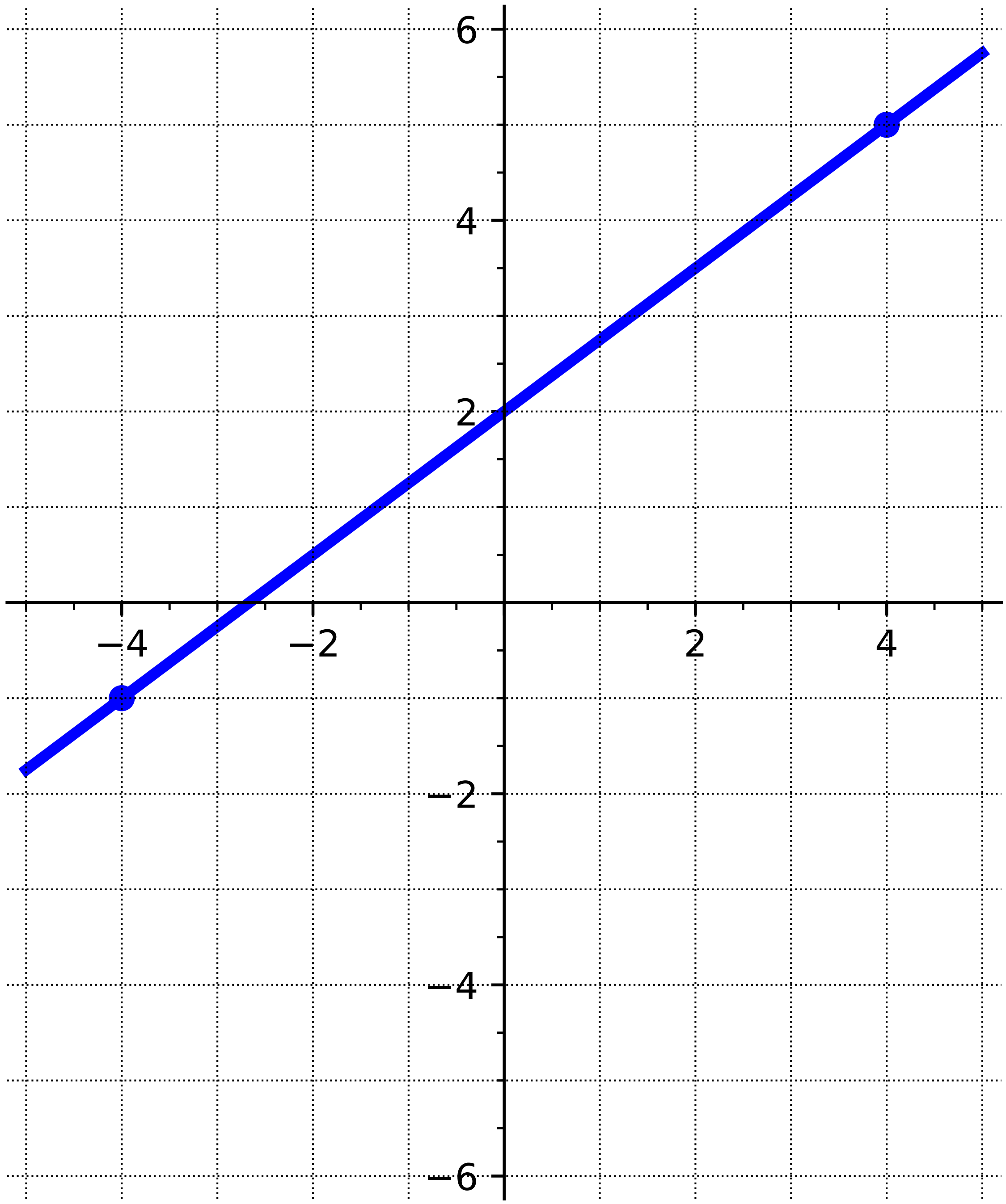
(b) Solve the point-slope form of the equation for y to rewrite the equation in slope-intercept form. Identify the slope and intercept of the line.

Remark 3.2.10 Notice that it was possible to use either point to find an equation of the line in point-slope form. But, when rewritten in slope-intercept form the equation is unique.

Equations of Lines (LF2)

Activity 3.2.11 For each of the following lines, determine which form (point-slope or slope-intercept) would be "easier" and why. Then, write the equation of each line.

(a) The line whose graph is given below.



(b) The line whose slope is $-\frac{1}{2}$ and passes through the point $(1, -3)$.

(c) The line that passes through the points $(0, 3)$ and $(2, 0)$.

Remark 3.2.12 It is always possible to use both forms to write the equation of a line and they are both valid. Although, sometimes the given information lends itself to make one form easier.

Equations of Lines (LF2)

Activity 3.2.13 Write the equation of each line.

(a) The slope is 0 and $(-1, -7)$ is a point on the line.

A. $y = -7$

B. $y = 7x$

C. $y = -x$

D. $x = -1$

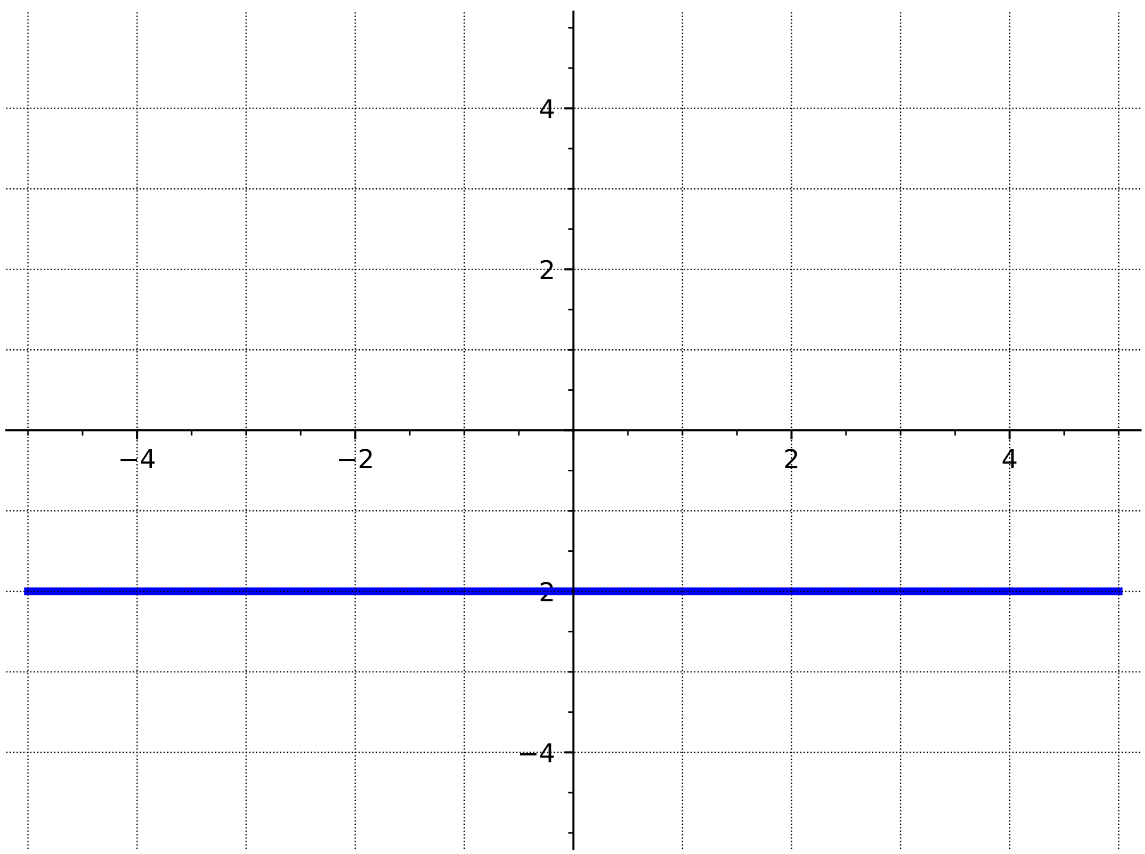
(b) Two points on the line are $(3, 0)$ and $(3, 5)$.

A. $y = 3x + 3$

B. $y = 3x + 5$

C. $x = 3$

D. $y = 3$



(c) A. $x = -2$

B. $y - 2 = x$

C. $y = -2x - 2$

D. $y = -2$

Equations of Lines (LF2)

Definition 3.2.14 A **horizontal line** has a slope of zero and has the form $y = k$ where k is a constant. A **vertical line** has an undefined slope and has the form $x = h$ where h is a constant. $\diamondsuit$

Equations of Lines (LF2)

Definition 3.2.15 The equation of a line can also be written in **standard form**. Standard form looks like $Ax + By = C$. ◊

Remark 3.2.16 It is possible to rearrange a line written in standard form to slope-intercept form by solving for y.

Equations of Lines (LF2)

Activity 3.2.17 Given a line in standard form

$$5x + 4y = 2.$$

Find the slope and y-intercept of the line.

3.3 Graphs of Linear Equations (LF3)

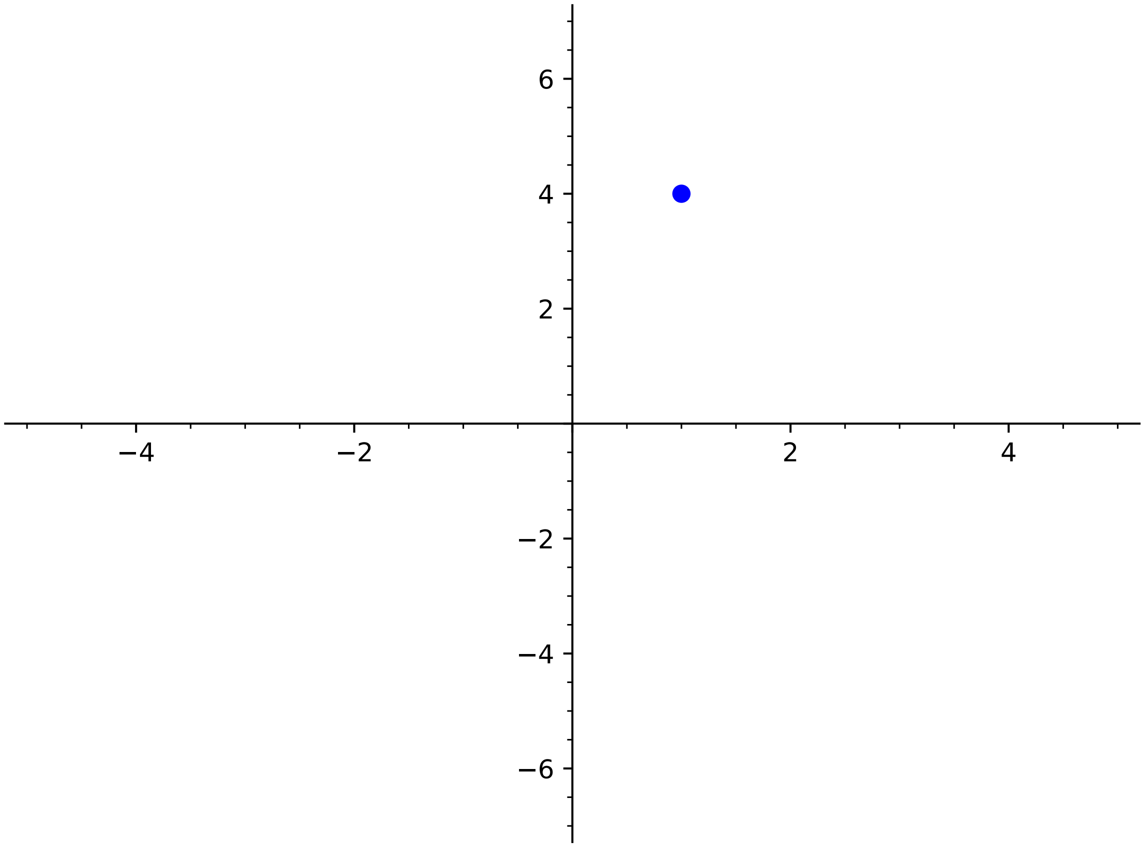
Objectives

- Graph a line given its equation or some combination of characteristics, such as points on the graph, a table of values, the slope, or the intercepts.

Graphs of Linear Equations (LF3)

Activity 3.3.1

(a) Draw a line that goes through the point $(1, 4)$.



(b) Was this the only possible line that goes through the point $(1, 4)$?

A. Yes. The line is unique.

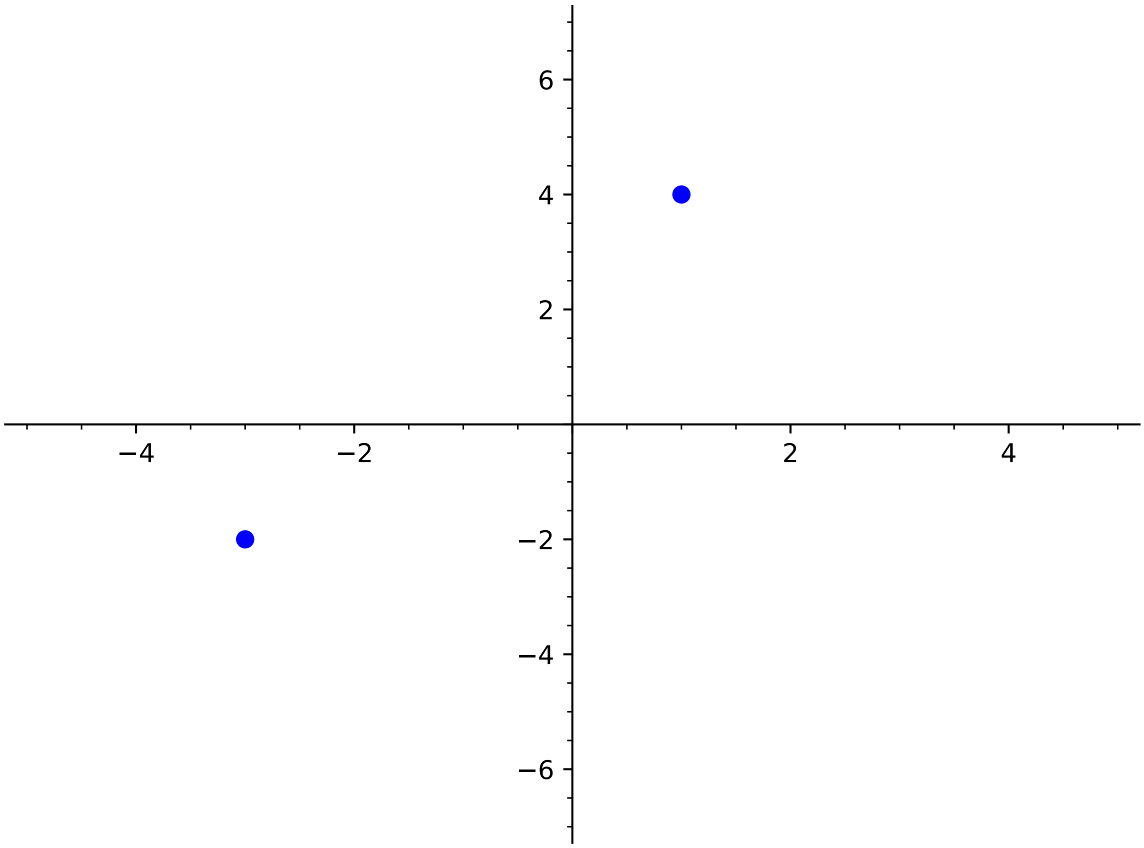
B. No. There is exactly one more line possible.

C. No. There are a lot of lines that go through $(1, 4)$.

D. No. There are an infinite number of lines that go through $(1, 4)$.

(c) Now draw a line that goes through the points $(1, 4)$ and $(-3, -2)$.

Graphs of Linear Equations (LF3)



(d) Was this the only possible line that goes through the points $(1, 4)$ and $(-3, -2)$?

A. Yes. The line is unique.

B. No. There is exactly one more line possible.

C. No. There are a lot of lines that go through $(1, 4)$ and $(-3, -2)$.

D. No. There are an infinite number of lines that go through $(1, 4)$ and $(-3, -2)$.

Graphs of Linear Equations (LF3)

Observation 3.3.2 If you are given two points, then you can always graph the line containing them by plotting them and connecting them with a line.

Graphs of Linear Equations (LF3)

Activity 3.3.3

(a) Graph the line containing the points $(-7, 1)$ and $(6, -2)$.

(b) Graph the line containing the points $(-3, 0)$ and $(0, 8)$.

(c) Graph the line given by the table below.

x	y
-3	-12
-2	-9
-1	-6
0	-3
1	0
2	3

(d) Let's say you are given a table that listed six points that are on the same line. How many of those points are necessary to use to graph the line?

A. One point is enough.

B. Two points are enough.

C. Three points are enough.

D. You need to plot all six points.

E. You can use however many you want.

Remark 3.3.4 In Activity 3.3.3, we were given at least two points in each question. However, sometimes we are not directly given two points to graph a line. Instead we are given some combination of characteristics about the line that will help us *find* two points. These characteristics could include a point, the intercepts, the slope, or an equation.

Graphs of Linear Equations (LF3)

Activity 3.3.5 A line has a slope of $-\frac{1}{3}$ and its y-intercept is 4.

(a) We were given the y-intercept. What point does that correspond to?

A. $(4, 0)$

B. $(0, 4)$

C. $\left(4, -\frac{1}{3}\right)$

D. $\left(-\frac{1}{3}, 4\right)$

(b) After we plot the y-intercept, how can we use the slope to find another point?

A. Start at the y-intercept, then move up one space and to the left three spaces to find another point.

B. Start at the y-intercept, then move up one space and to the right three spaces to find another point.

C. Start at the y-intercept, then move down one space and to the left three spaces to find another point.

D. Start at the y-intercept, then move down one space and to the right three spaces to find another point.

(c) Graph the line that has a slope of $-\frac{1}{3}$ and its y-intercept is 4.

Graphs of Linear Equations (LF3)

Activity 3.3.6 A line is given by the equation $y = -2x + 5$.

(a) What form is the equation given in?

A. Standard form

B. Point-slope form

C. Slope-intercept form

D. The form it is in doesn't have a name.

(b) The form gives us one point right away: the y-intercept. Which of the following is the y-intercept?

A. $(-2, 0)$

B. $(0, -2)$

C. $(5, 0)$

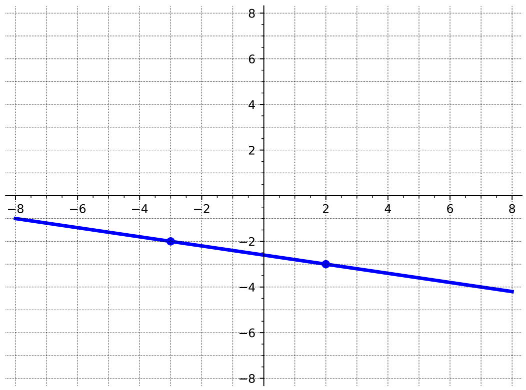
D. $(0, 5)$

(c) y

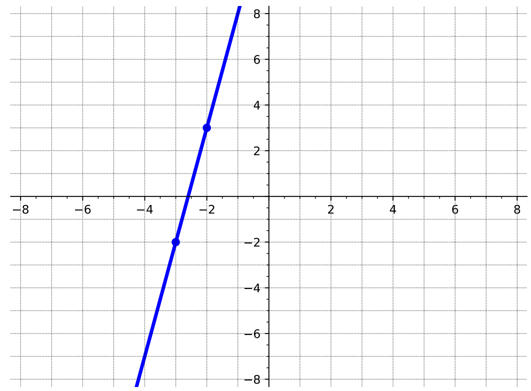
Graphs of Linear Equations (LF3)

Activity 3.3.7 A line contains the point $(-3,-2)$ and has slope $\frac{1}{5}$. Which of the following is the graph of that line?

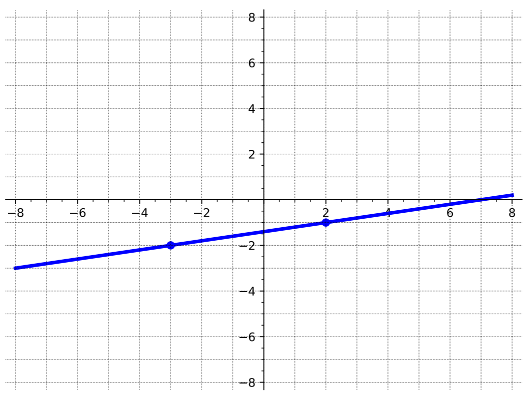
A.



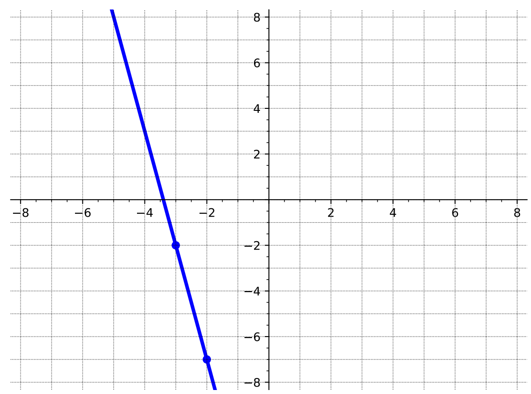
C.



B.



D.



Activity 3.3.8 A line is given by the equation $y - 6 = -4(x + 2)$.

(a) What form is the equation given in?

A. Standard form

B. Point-slope form

C. Slope-intercept form

D. The form it is in doesn't have a name.

(b) The form gives us one point right away. Which of the following is a point on the line?

A. $(-2, -6)$

B. $(-2, 6)$

C. $(2, -6)$

D. $(2, 6)$

(c)

Activity 3.3.9 Recall from Definition 3.2.14 that the equation of a horizontal line has the form $y = k$ where k is a constant and a vertical line has the form $x = h$ where h is a constant.

(a) Which type of line has a slope of zero?

A. Horizontal

B. Vertical

(b) Which type of line has an undefined slope?

A. Horizontal

B. Vertical

(c) Graph the vertical line that goes through the point $(4, -2)$.

(d) What is the equation of the vertical line through the point $(4, -2)$?

A. $x = 4$

B. $y = 4$

C. $x = -2$

D. $y = -2$

(e) Graph the horizontal line that goes through the point $(4, -2)$.

(f) What is the equation of the horizontal line through the point $(4, -2)$?

A. $x = 4$

B. $y = 4$

C. $x = -2$

D. $y = -2$

Graphs of Linear Equations (LF3)

Activity 3.3.10 Graph each line described below.

(a) The line containing the points $(-3, 4)$ and $(5, -2)$.

(b) The line whose x-intercept is -2 and whose y-intercept is 7.

(c) The line whose slope is $\frac{2}{5}$ that goes through the point $(4, 6)$.

(d) The line whose slope is $-\frac{1}{3}$ and whose y-intercept is -4.

(e) The vertical line through the point $(-2, -7)$.

(f) The horizontal line through the point $(-6, 3)$.

(g) The line with equation $y = -\frac{5}{3}x - 6$.

(h) The line with equation $y - 5 = \frac{7}{2}(x - 2)$.

(i) The line with equation $3x - 6y = 8$.

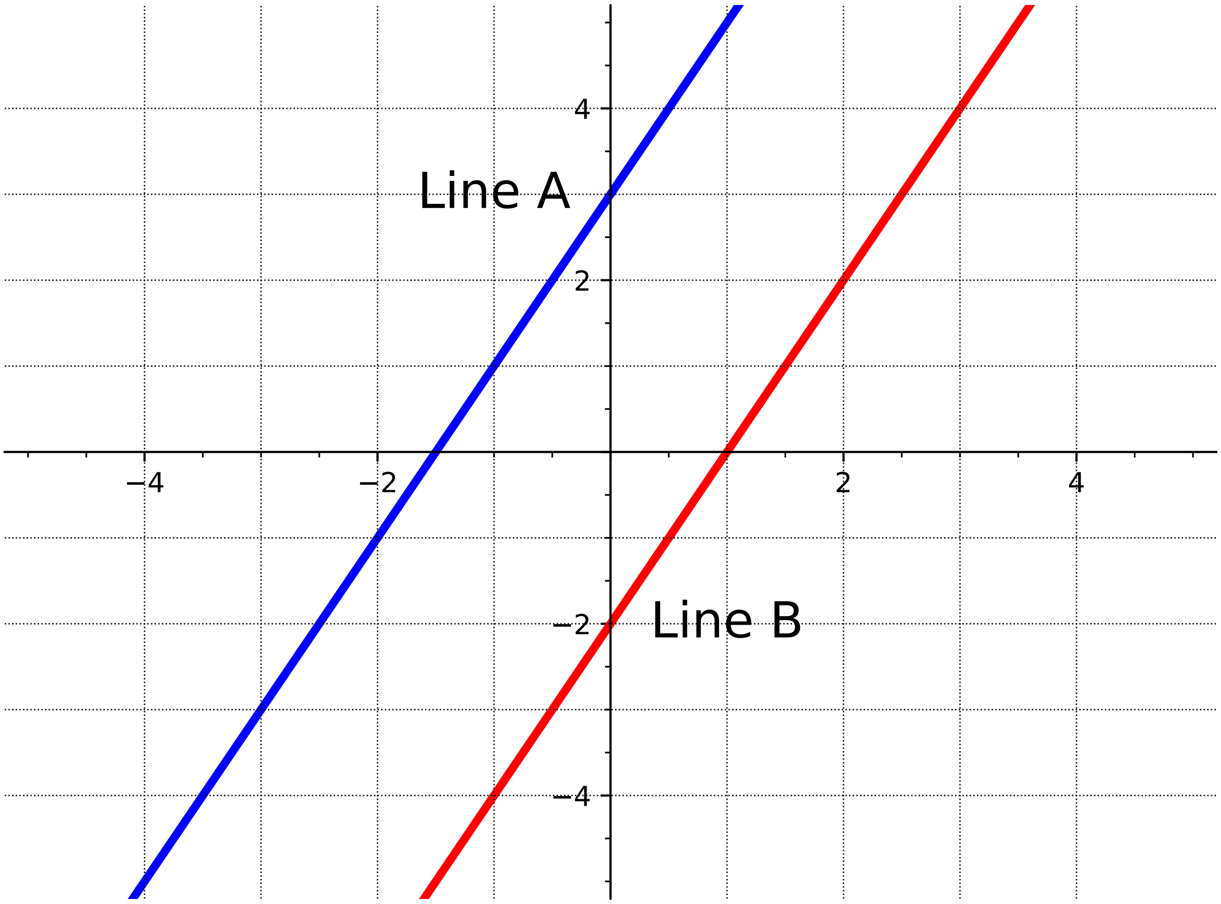
3.4 Parallel and Perpendicular Lines (LF4)

Objectives

- Use slope relationships to determine whether two lines are parallel or perpendicular, and find the equation of lines parallel or perpendicular to a given line through a given point.

Parallel and Perpendicular Lines (LF4)

Activity 3.4.1 Let's revisit Activity 3.2.1 to investigate special types of lines.



(a) What is the slope of line A?

A. 1

B. 2

C. $\frac{1}{2}$

D. -2

(b) What is the slope of line B?

A. 1

B. 2

C. $\frac{1}{2}$

D. -2

(c) What is the y-intercept of line A?

A. -2

B. -1.5

C. 1

D. 3

(d) What is the y-intercept of line B?

A. -2

B. -1.5

C. 1

D. 3

(e) What is the same about the two lines?

(f) What is different about the two lines?

Remark 3.4.2 Notice that in Activity 3.4.1 the two lines never touch.

Parallel and Perpendicular Lines (LF4)

Definition 3.4.3 Parallel lines are lines that always have the same distance apart (equidistant) and will never meet. Parallel lines have the same slope, but different y-intercepts. $\Diamond$

Parallel and Perpendicular Lines (LF4)

Activity 3.4.4 Suppose you have the function,

$$f(x) = -\frac{1}{2}x - 1$$

(a) What is the slope of $f(x)$?

A. -1

B. 2

C. 1

D. $-\frac{1}{2}$

(b) Applying Definition 3.4.3, what would the slope of a line parallel to $f(x)$ be?

A. -1

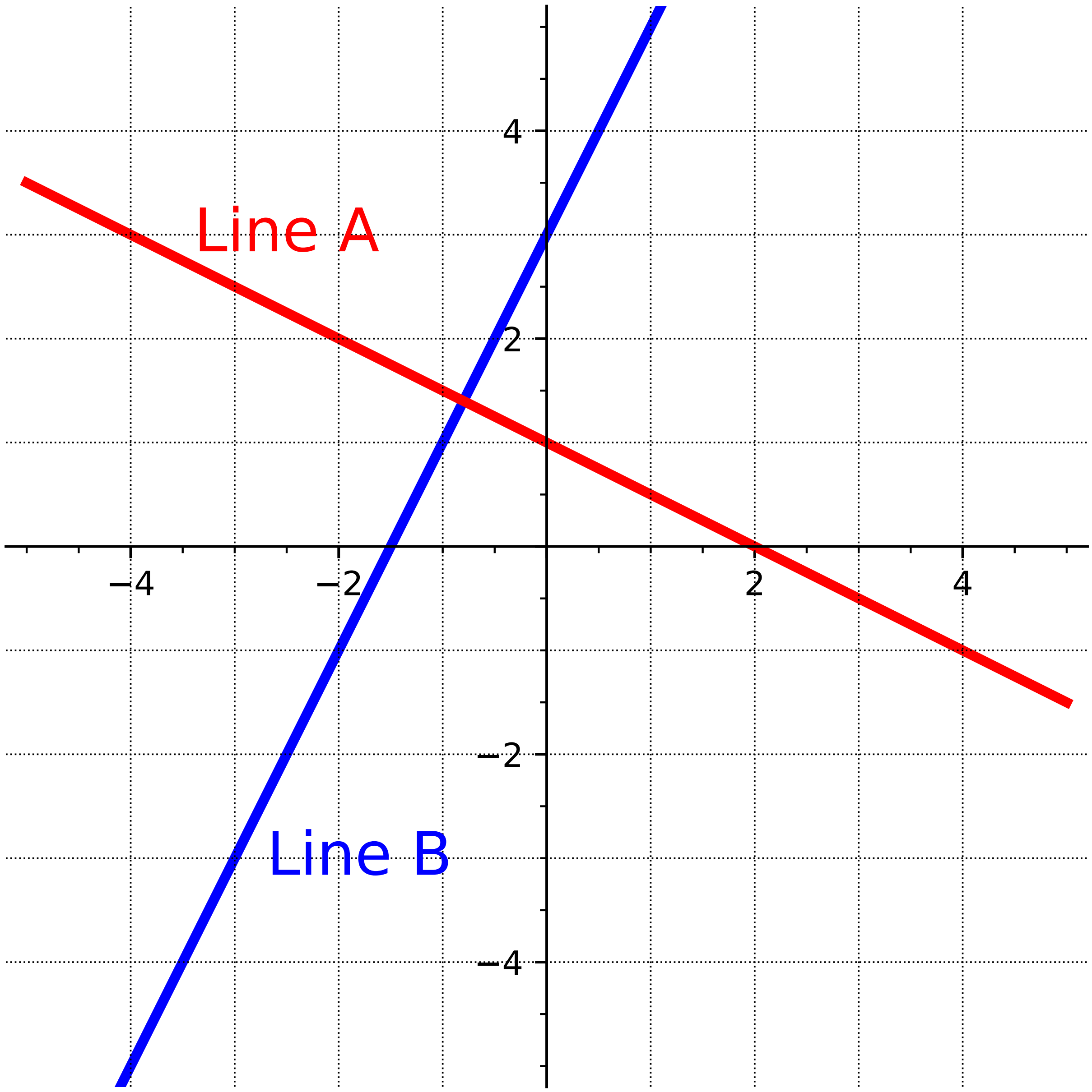
B. 2

C. 1

D. $-\frac{1}{2}$

(c) Find the equation of a line parallel to $f(x)$ that passes through the point $(-4, 2)$.

Activity 3.4.5 Consider the graph of the two lines below.



(a) What is the slope of line A?

A. 3

B. 2

C. $-\frac{1}{2}$

D. -2

(b) What is the slope of line B?

A. 3

B. 2

C. $-\frac{1}{2}$

D. -2

(c) What is the y-intercept of line A?

A. -2

B. $-\frac{1}{2}$

C. 2

D. 3

(d) What is the y-intercept of line B?

A. -2

B. $-\frac{1}{2}$

C. 1

D. 3

(e) If you were to think of slope as ”rise over run,” how would you write the slope of each line?

(f) How would you compare the slopes of the two lines?

Remark 3.4.6 Notice in Activity 3.4.5, that even though the two lines have different slopes, the slopes are somewhat similar. For example, if you take the slope of Line A $\left(-\frac{1}{2}\right)$ and flip and negate it, you will get the slope of Line B $\left(\frac{2}{1}\right)$.

Parallel and Perpendicular Lines (LF4)

Definition 3.4.7 Perpendicular lines are two lines that meet or intersect each other at a right angle. The slopes of two perpendicular lines are *negative reciprocals* of each other (given that the slope exists!). $\Diamond$

Parallel and Perpendicular Lines (LF4)

Activity 3.4.8 Suppose you have the function,

$$f(x) = 3x + 5$$

(a) What is the slope of $f(x)$?

A. $-\frac{1}{3}$

B. 3

C. 5

D. $-\frac{1}{5}$

(b) Applying Definition 3.4.7, what would the slope of a line perpendicular to $f(x)$ be?

A. $-\frac{1}{3}$

B. 3

C. 5

D. $-\frac{1}{5}$

(c) Find an equation of the line perpendicular to $f(x)$ that passes through the point $(3, 6)$.

Parallel and Perpendicular Lines (LF4)

Activity 3.4.9 For each pair of lines, determine if they are parallel, perpendicular, or neither.

(a)

$$f(x) = -3x + 4$$
$$g(x) = 5 - 3x$$

(b)

$$f(x) = 2x - 5$$
$$g(x) = 6x - 5$$

(c)

$$f(x) = 6x - 5$$
$$g(x) = \frac{1}{6}x + 8$$

(d)

$$f(x) = \frac{4}{5}x + 3$$
$$g(x) = -\frac{5}{4}x - 1$$

Parallel and Perpendicular Lines (LF4)

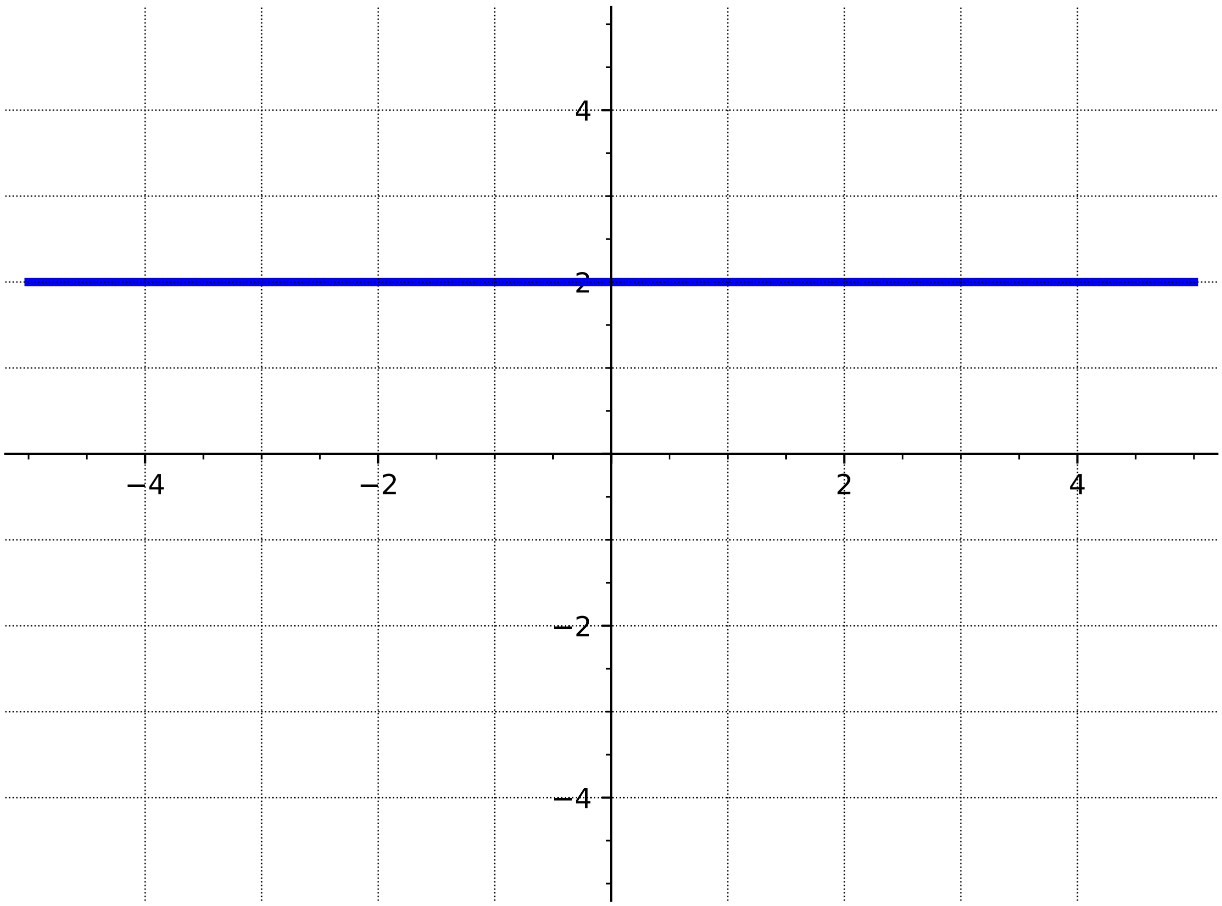
Activity 3.4.10 Consider the linear equation, $f(x) = -\frac{2}{3}x - 4$ and the point A: $(-6, 4)$.

(a) Find an equation of the line that is parallel to $f(x)$ and passes through the point A.

(b) Find an equation of the line that is perpendicular to $f(x)$ and passes through the point A.

Parallel and Perpendicular Lines (LF4)

Activity 3.4.11 Consider the line, $y = 2$, as shown in the graph below.



(a) What is the slope of the line $y = 2$?

A. undefined

B. 0

C. 1

D. $-\frac{1}{2}$

(b) What is the slope of a line that is parallel to $y = 2$?

A. undefined

B. 0

C. 1

D. $-\frac{1}{2}$

(c) Find an equation of the line that is parallel to $y = 2$ and passes through the point $(-1, -4)$.

(d) What is the slope of a line that is perpendicular to $y = 2$?

A. undefined

B. 0

C. 1

D. $-\frac{1}{2}$

Parallel and Perpendicular Lines (LF4)

(e) Find an equation of the line that is perpendicular to $y = 2$ and passes through the point $(-1, 2)$.

3.5 Linear Models and Meanings (LF5)

Objectives

- Build linear models from verbal descriptions, and use the models to establish conclusions, including by contextualizing the meaning of slope and intercept parameters.

Remark 3.5.1 We begin by revisiting Activity 2.2.15 from Section 2.2.

Linear Models and Meanings (LF5)

Activity 3.5.2 Ellie has \$13 in her piggy bank, and she gets an additional \$1.50 each week for her allowance. Assuming she does not spend any money, the total amount of allowance, $A(w)$, she has after w weeks can be modeled by the function

$$A(w) = 13 + 1.50w.$$

(a) How much money will be in her piggy bank after 5 weeks?

(b) After how many weeks will she have \$40 in her piggy bank?

Linear Models and Meanings (LF5)

Remark 3.5.3 The function in the previous activity is an example of a **linear model**. A linear model is a linear function that describes, or models, a real-life application. In this section we will build and use linear models.

Linear Models and Meanings (LF5)

Activity 3.5.4 Jack bought a package containing 40 cookies. Each day he takes two in his lunch to work.

(a) How many cookies are left in the package after 3 days?

A. 46

B. 42

C. 38

D. 36

E. 34

(b) Fill out the following table to represent the number of cookies left in the package after the given number of days.

number of days	number of cookies left
0	
2	
5	
10	
20	

(c) What is the y-intercept? Explain what it represents in the context of the problem.

(d) What is the rate of change? Explain what it represents in the context of the problem.

(e) Write a linear function to model the situation. Let d represent the number of days elapsed and $C(d)$ represent the number of cookies in the package. (Hint: Use the previous two questions to help!)

(f) Find $C(6)$. Explain what that means in the context of the problem.

(g) How many days will it take to empty the package? What does this correspond to on the graph?

Linear Models and Meanings (LF5)

Activity 3.5.5 Daisy's Doughnut Shop sells delicious doughnuts. Each month, they incur a fixed cost of \$2000 for rent, insurance, and other expenses. Then, for each doughnut they produce, it costs them an additional \$0.25.

(a) In January, Daisy's Doughnut Shop produced 1000 doughnuts. What was their total monthly cost to run the shop?

A. \$2000.25

B. \$2002.50

C. \$2025.00

D. \$2250.00

E. \$4500.00

(b) Fill out the following table to represent the cost for producing various amounts of doughnuts.

number of doughnuts	**cost**
0	
500	
1000	
1500	
2000	

(c) What is the y-intercept? Explain what it represents in the context of the problem.

(d) What is the rate of change? Explain what it represents in the context of the problem.

(e) Write a linear function to model the situation. Let x represent the number of doughnuts produced and $C(x)$ represent the total cost. (Hint: Use the previous two questions to help!)

(f) Find $C(1300)$. Explain what that means in the context of the problem.

(g) Find the x-intercept. Explain what it means in the context of the problem.

Linear Models and Meanings (LF5)

Activity 3.5.6 A taxi costs \$5.00 up front, and then charges \$0.73 per mile traveled.

(a) Write a linear function to model this situation.

(b) How much will it cost for a 13 mile taxi ride?

(c) If the taxi ride cost \$11.06, how many miles did it travel?

Linear Models and Meanings (LF5)

Activity 3.5.7 When reporting the weather, temperature is given in degrees Fahrenheit (F) or degrees Celsius (C). The two scales are related linearly, which means we can find a linear model to describe their relationship. This model lets us convert between the two scales.

(a) Water freezes at $0°C$ and $32°F$. Water boils at $100°C$ and $212°F$. Use this information to write two ordered pairs.

Hint. Choose Celsius to be your input value, and Fahrenheit to be the output value.

(b) Use the two points to write a linear model for this situation. Use C and F as your variables.

(c) If the temperature outside is $25°C$, what is the temperature in Fahrenheit?

(d) If the temperature outside is $50°F$, what is the temperature in Celsius?

(e) What temperature value is the same in Fahrenheit as it is in Celsius?

Linear Models and Meanings (LF5)

Activity 3.5.8 Erin needs to print t-shirts for her company retreat. She has found two businesses that produce quality shirts, but they have different pricing structures. Shirts-R-Us requires a \$60 set up fee, then charges \$7 for each shirt produced. Graphix! has no set up fee and charges \$8.50 per shirt produced.

(a) Write a linear function $S(x)$ that models the pricing structure for producing x shirts at Shirts-R-Us.

(b) Write a linear function $G(x)$ that models the pricing structure for producing x shirts at Graphix!.

(c) At which business would it be less expensive to buy 1 shirt? 10 shirts? 100 shirts? Explain your reasoning.

(d) It depended on how many shirts were needed to determine which business was less expensive. For what range of number of shirts should Erin choose Shirts-R-Us and for what range should she choose Graphix!?

Hint. Try finding where the cost is the same at both businesses. And looking at a graph may help as well!

3.6 Systems of Linear Equations (LF6)

Objectives

- Solve a system of two linear equations in two variables.

Systems of Linear Equations (LF6)

Observation 3.6.1 Often times when solving a real-world application, more than one equation is necessary to describe the information. We'll investigate some of those in this section.

Systems of Linear Equations (LF6)

Activity 3.6.2 Admission into a carnival for 4 children and 2 adults is \$128.50. For 6 children and 4 adults, the admission is \$208. Assuming a different price for children and adults, what is the price of the child's admission and the price of the adult admission?

(a) Let c represent the cost of a child's admission and a the cost of an adult admission. Write an equation to represent the total cost for 4 children and 2 adults.

A. $2c + 4a = \$128.50$

B. $c + a = \$128.50$

C. $4c + 2a = \$128.50$

D. $c + a = \$336.50$

(b) Now write an equation to represent the total cost for 6 children and 4 adults.

A. $6c + 4a = \$208$

B. $c + a = \$208$

C. $4c + 6a = \$208$

D. $c + a = \$336.50$

(c) Using the above equations, check by substitution which admission prices would satisfy both equations?

A. $c = \$20$ and $a = \$24.25$

B. $c = \$24.50$ and $a = \$15.25$

C. $c = \$20$ and $a = \$22$

D. $c = \$14.50$ and $a = \$30.25$

Systems of Linear Equations (LF6)

Definition 3.6.3 A **system of linear equations** consists of two or more linear equations made up of two or more variables. A **solution to a system of equations** is a value for each of the variables that satisfies all the equations at the same time. $\diamond$

Activity 3.6.4 Consider the following system of equations.

$$\begin{cases} y = 2x + 4 \\ 3x + 2y = 1 \end{cases}$$

Which of the ordered pairs is a solution to the system?

A. $(3, 10)$

B. $(0, 4)$

C. $(1, -1)$

D. $(-1, 2)$

Remark 3.6.5 While we can test points to determine if they are solutions, it is not feasible to test every possible point to find a solution. We need a method to solve a system.

Systems of Linear Equations (LF6)

Activity 3.6.6 Consider the following system of equations.

$$\begin{cases} 3x - y = 2 \\ x + 4y = 5 \end{cases}$$

(a) Rewrite the first equation in terms of y.

(b) Rewrite the second equation in terms of y.

(c) Graph the two equations on the same set of axes. Where do the lines intersect?

(d) Check that the point of intersection of the two lines is a solution to the system of equations.

Remark 3.6.7 Sometimes it is difficult to determine the exact intersection point of two lines using a graph. Let's explore another possible method for solving a system of equations.

Systems of Linear Equations (LF6)

Activity 3.6.8 Consider the following system of equations.

$$\begin{cases} 3x + y = 4 \\ x + 3y = 10 \end{cases}$$

(a) Graph the two equations on the same set of axes. Is it possible to determine exactly where the lines intersect?

(b) Solve the first equation for y and substitute into the second equation. What is the resulting equation?

A. $x + 4 - 3x = 10$

B. $x + 3(4 - 3x) = 10$

C. $4 - 3x + 3y = 10$

D. $3x + (4 - 3x) = 4$

(c) Solve the resulting equation from part (b) for x.

A. $x = -3$

B. $x = \frac{1}{4}$

C. $x = \frac{7}{3}$

D. $x = 0$

(d) Substitute the value of x into the first equation to find the value of y.

A. $y = -5$

B. $y = \frac{13}{4}$

C. $y = -3$

D. $y = \frac{4}{3}$

(e) Write the solution to the system of equations (the found values of x and y) as an ordered pair.

Remark 3.6.9 This method of solving a system of equations is referred to as the **Substitution Method**.

1. Solve one of the equations for one variable.
2. Substitute the expression into the other equation to solve for the remaining variable.
3. Substitute that value into either equation to find the value of the first variable.

Systems of Linear Equations (LF6)

Activity 3.6.10 Solve the following system of equations using the substitution method.

$$\begin{cases} x + 2y = -1 \\ -x + y = 3 \end{cases}$$

Remark 3.6.11 While the substitution method will always work, sometimes the resulting equations will be difficult to solve. Let's explore a third method for solving a system of two linear equations with two variables.

Systems of Linear Equations (LF6)

Activity 3.6.12 Consider the following system of equations.

$$\begin{cases} 5x + 7y = 12 \\ 3x - 7y = 37 \end{cases}$$

(a) Add the two equations together. What is the resulting equation?

A. $2x = -15$

B. $14y = 49$

C. $8x + 14y = 49$

D. $8x = 49$

(b) Use the resulting equation after addition, to solve for the variable.

A. $x = -\frac{15}{2}$

B. $y = \frac{49}{14}$

C. $x = \frac{49}{22}$

D. $x = \frac{49}{8}$

(c) Use the value to find the solution to the system of equations.

Remark 3.6.13 This method of solving a system of equations is referred to as the **Elimination Method**.

1. Combine the two equations using addition or subtraction to eliminate one of the variables.

2. Solve the resulting equation.

3. Substitute that value into either equation to find the value of the other variable.

Activity 3.6.14 Solve the following system of equations using the elimination method.

$$\begin{cases} 7x - 4y = 3 \\ 3y - 7x = 8 \end{cases}$$

Systems of Linear Equations (LF6)

Activity 3.6.15 Consider the following system of equations.

$$\begin{cases} 5x - 9y = 6 \\ -10x + 4y = 2 \end{cases}$$

Notice that if you simply add the two equations together, it will not eliminate a variable. Substitution will also be difficult since it involves fractions.

(a) What value can you multiply the first equation by so that when you add the result to the second equation one variable cancels?

A. -1 and the x will cancel

B. 2 and the x will cancel

C. 3 and the y will cancel

D. -2 and the y will cancel

(b) Perform the multiplication and add the two equations. What is the resulting equation?

A. $-5y = 8$

B. $-14y = 14$

C. $-14y = 8$

D. $-5x = 4$

(c) What is the solution to the system of equations?

A. $(-1, -1.6)$

B. $(-1, -0.6)$

C. $(-0.6, -1)$

D. $(-1.6, -1)$

Systems of Linear Equations (LF6)

Activity 3.6.16 For each system of equations, determine which method (graphical, substitution, or elimination) might be best for solving.

(a)

$$\begin{cases} 5x + 9y = 16 \\ x + 2y = 4 \end{cases}$$

A. Graphical

B. Substitution

C. Elimination

(b)

$$\begin{cases} y = 4x - 6 \\ y = -5x + 21 \end{cases}$$

A. Graphical

B. Substitution

C. Elimination

(c)

$$\begin{cases} x + y = 10 \\ x - y = 12 \end{cases}$$

A. Graphical

B. Substitution

C. Elimination

Activity 3.6.17 Solve each of the systems of equations from Activity 3.6.16 using the method you chose.

3.7 Applications of Systems of Linear Equations (LF7)

Objectives

- Solve questions involving applications of systems of equations.

Applications of Systems of Linear Equations (LF7)

Remark 3.7.1 Now that we have explored multiple methods for solving systems of linear equations, let's put those in to practice using some real-world application problems.

Activity 3.7.2 Let's begin by revisiting the carnival admission problem from Section 3.6.

Admission into a carnival for 4 children and 2 adults is \$128.50. For 6 children and 4 adults, the admission is \$208. Assuming a different price for children and adults, what is the price of the child's admission and the price of the adult admission?

(a) First, set up a system of equations representing the given information. Use x to represent the child admission price and y for the adult admission price.

A. $\begin{cases} x + y = 128.50 \\ x + y = 208 \end{cases}$

B. $\begin{cases} 2x + 4y = 128.50 \\ 4x + 6y = 208 \end{cases}$

C. $\begin{cases} 4x + 2y = 128.50 \\ 6x + 4y = 208 \end{cases}$

D. $\begin{cases} 6x + 4y = 128.50 \\ 4x + 2y = 208 \end{cases}$

(b) Solve the system of equations.

A. $(27, 10.25)$

B. $(15.25, 24.5)$

C. $(24.5, 15.25)$

D. $(10, 37)$

(c) Write your solution in terms of the price of admission for children and adults.

Activity 3.7.3 Let's revisit another application we've encountered before in Section 1.2, Activity 1.2.7.

Ammie's favorite snack to share with friends is candy salad, which is a mixture of different types of candy. Today she chooses to mix Nerds Gummy Clusters, which cost \$8.38 per pound, and Starburst Jelly Beans, which cost \$7.16 per pound. If she makes seven pounds of candy salad and spends a total of \$55.61, how many pounds of each candy did she buy?

(a) Set up a system of equations to represent the mixture problem. Let N represent the pounds of Nerds Gummy Clusters and S represent the pounds of Starburst Jelly Beans in the mixture.

A. $\begin{cases} N + S = 7 \\ 7.16N + 8.38S = 55.61 \end{cases}$

B. $\begin{cases} N + S = 7 \\ 8.38N + 7.16S = 55.61 \end{cases}$

C. $\begin{cases} N + S = 55.61 \\ 8.38N + 7.16S = 389.27 \end{cases}$

D. $\begin{cases} N + S = 7 \\ 7N + 7S = 55.61 \end{cases}$

(b) Now solve the system of equations and put your answer in the context of the problem.

A. Ammie bought 2.5 lbs of Nerds Gummy Clusters and 4.5 lbs of Starburst Jelly Beans.

B. Ammie bought 3.5 lbs of Nerds Gummy Clusters and 3.5 lbs of Starburst Jelly Beans.

C. Ammie bought 4.5 lbs of Nerds Gummy Clusters and 2.5 lbs of Starburst Jelly Beans.

D. Ammie bought 5.5 lbs of Nerds Gummy Clusters and 1.5 lbs of Starburst Jelly Beans.

Applications of Systems of Linear Equations (LF7)

Activity 3.7.4 A couple has a total household income of \$104,000. The wife earns \$16,000 less than twice what the husband earns. How much does the wife earn?

(a) Set up a system of equations to represent the situation. Let w represent the wife's income and h represent the husband's income.

A. $\begin{cases} w + h = 104000 \\ h = 2w - 16000 \end{cases}$

B. $\begin{cases} w + h = 104000 \\ 2h = w - 16000 \end{cases}$

C. $\begin{cases} w + 2h = 104000 \\ w = h - 16000 \end{cases}$

D. $\begin{cases} w + h = 104000 \\ w = 2h - 16000 \end{cases}$

(b) Solve the system of equations. How much does the wife earn?

A. The wife earns \$29,300.

B. The wife earns \$40,000.

C. The wife earns \$64,000.

D. The wife earns \$74,600.

Applications of Systems of Linear Equations (LF7)

Activity 3.7.5 Kenneth currently sells suits for Company A at a salary of \$22,000 plus a \$10 commission for each suit sold. Company B offers him a position with a salary of \$28,000 plus a \$4 commission for each suit sold. How many suits would Kenneth need to sell for the options to be equal?

Set-up and solve a system of equations to represent the situation.

Chapter 4

Polynomial and Rational Functions (PR)

Objectives

How do we model polynomial or rational change?
By the end of this chapter, you should be able to...

1. Graph quadratic functions and identify their axis of symmetry, and maximum or minimum point.
2. Use quadratic models to solve an application problem and establish conclusions.
3. Determine the zeros and their multiplicities of a polynomial in factored form. Describe and graph the behavior of a polynomial function at the intercepts and the ends.
4. Rewrite a rational function as a polynomial plus a proper rational function.
5. Determine the zeros of a polynomial function with real coefficients.
6. Solve rational equations.
7. Find the domain and range, vertical and horizontal asymptotes, and intercepts of a rational function and use this information to sketch the graph.
8. Solve quadratic inequalities and express the solution graphically and with interval notation.
9. Solve rational inequalities and express the solution graphically and using interval notation.

4.1 Graphing Quadratic Functions (PR1)

Objectives

- Graph quadratic functions and identify their axis of symmetry, and maximum or minimum point.

Graphing Quadratic Functions (PR1)

Observation 4.1.1 Quadratic functions have many different applications in the real world. For example, say we want to identify a point at which the maximum profit or minimum cost occurs. Before we can interpret some of these situations, however, we will first need to understand how to read the graphs of quadratic functions to locate these least and greatest values.

Activity 4.1.2 Use the graph of the quadratic function $f(x) = 3(x-2)^2 - 4$ to answer the questions below.

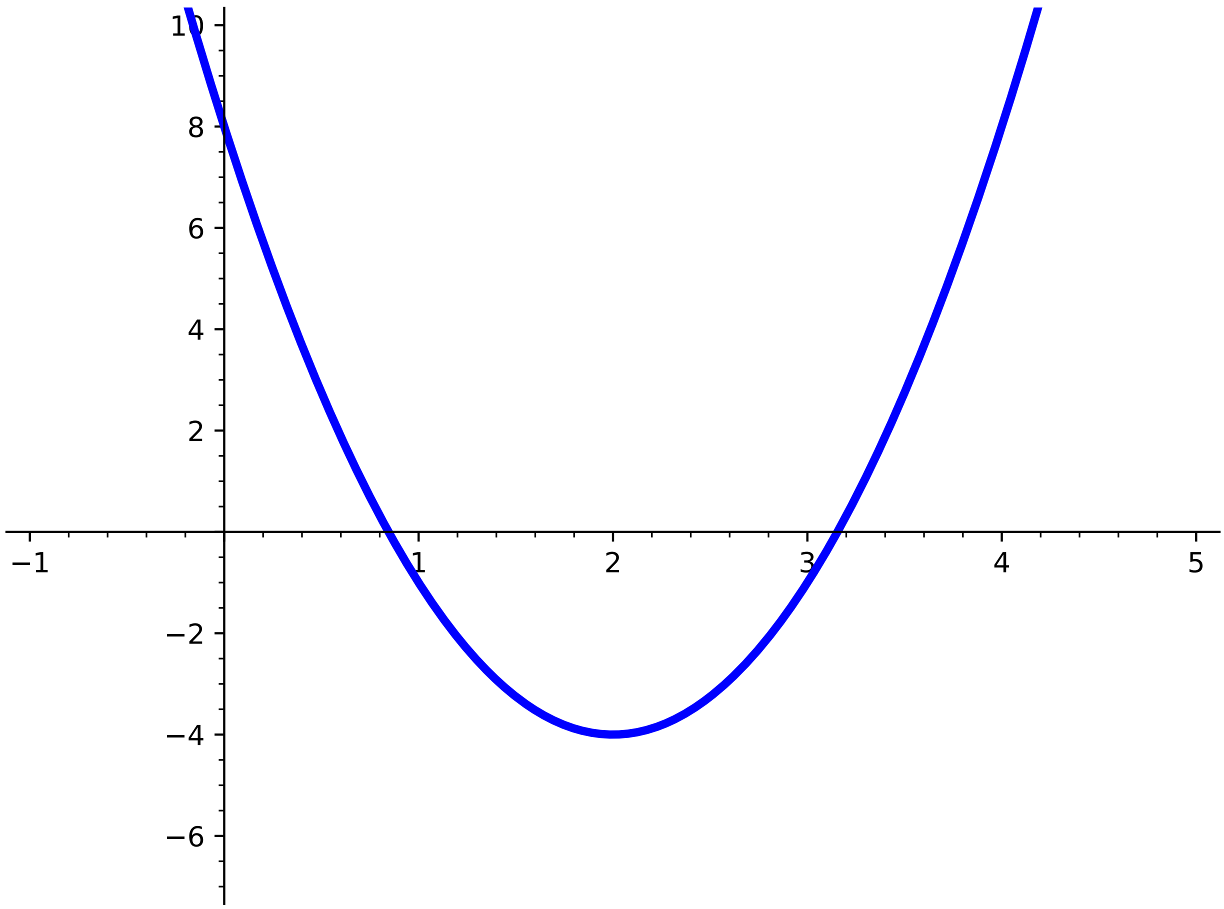


Figure 4.1.3

(a) Make a table for values of $f(x)$ corresponding to the given x-values. What is happening to the y-values as the x-values increase? Do you notice any other patterns of the y-values of the table?

Table 4.1.4

x	$f(x)$
-2	
-1	
0	
1	
2	
3	
4	
5	

(b) At which point (x, y) does $f(x)$ have a minimum value? That is, is there a point on the graph that is lower than all other points?

A. The minimum value appears to occur near $(0, 8)$.

B. The minimum value appears to occur near $\left(-\frac{1}{5}, 10\right)$.

C. The minimum value appears to occur near $(2, -4)$.

D. There is no minimum value of this function.

(c) At which point (x, y) does $f(x)$ have a maximum value? That is, is there a point on the graph that is higher than all other points?

A. The maximum value appears to occur near $(-2, 44)$.

B. The maximum value appears to occur near $\left(-\frac{1}{5}, 10\right)$.

C. The maximum value appears to occur near $(2, -4)$.

D. There is no maximum value of this function.

Graphing Quadratic Functions (PR1)

Definition 4.1.5 The point at which a quadratic function has a maximum or minimum value is called the **vertex**. The **vertex form** of a quadratic function is given by

$$f(x) = a(x - h)^2 + k,$$

where (h, k) is the **vertex** of the parabola. $\Diamond$

Definition 4.1.6 The **axis of symmetry**, also known as the line of symmetry, is the line that makes the shape of an object symmetrical. For a quadratic function, the axis of symmetry always passes through the vertex (h, k) and so is the vertical line $x = h$. $\Diamond$

Graphing Quadratic Functions (PR1)

Activity 4.1.7 Use the given quadratic function, $f(x) = 3(x-2)^2 - 4$, to answer the following:

(a) Applying Definition 4.1.6, what is the vertex and axis of symmetry of $f(x)$?

A. vertex: $(2, -4)$; axis of symmetry: $x = 2$

B. vertex: $(-2, 4)$; axis of symmetry: $x = -2$

C. vertex: $(-2, -4)$; axis of symmetry: $x = -2$

D. vertex: $(2, 4)$; axis of symmetry: $x = 2$

(b) Compare what you got in part a with the values you found in Activity 4.1.2. What do you notice?

Graphing Quadratic Functions (PR1)

Definition 4.1.8 The **standard form** of a quadratic function is given by

$$f(x) = ax^2 + bx + c,$$

where a, b, and c are real coefficients. $\Diamond$

Activity 4.1.9 Completing the square (see Section 1.5) is a very useful tool or method to convert the quadratic equation in standard form into vertex form. Let's start with the standard form of $y = 2x^2 - 4x + 7$ and convert it into vertex form.

(a) Before we begin, isolate the terms with x on one side of the equation. What equation do you now have?

(b) Notice that the a-value is not 1. Before we complete the square, we will want to factor out the coefficient of the x^2 term. What does your equation look like when factoring out 2?

(c) Now apply the "completing the square" steps to determine the constant term that is added to both sides of the equation.

Hint. Refer back to Definition 1.5.13 to help you determine how to find the constant term. Be careful when the coefficient is not 1!

(d) Factor the perfect square trinomial.

(e) Now isolate y and then determine the vertex and axis of symmetry.

Remark 4.1.10 In Activity 4.1.10, we were able to convert from standard form to vertex form. If you were to do this for the general form of $y = ax^2 + bx + c$, you will see that the x-value of the vertex will always be of the form $-\frac{b}{2a}$. For instance, suppose we start with

$$y = ax^2 + bx + c.$$

Following similar steps as we did in Activity 4.1.10, we would get:

$$y - c = ax^2 + bx$$

$$y - c = a\left(x^2 + \frac{b}{a}x\right)$$

$$y - c + a\left(\frac{b}{2a}\right)^2 = a\left(x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2\right)$$

$$y - c + a\left(\frac{b}{2a}\right)^2 = a\left(x + \frac{b}{2a}\right)^2$$

$$y = a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a}$$

In the vertex form, $y = a(x - h)^2 + k$, the vertex (h, k) can be identified as:

$$h = -\frac{b}{2a}$$

$$k = c - \frac{b^2}{4a}$$

Observation 4.1.11 Just as with the vertex form of a quadratic, we can use the standard form of a quadratic ($f(x) = ax^2 + bx + c$) to find the **axis of symmetry** and the **vertex** by using the values of a, b, and c. Given the standard form of a quadratic, the axis of symmetry is the vertical line $x = -\dfrac{b}{2a}$ and the vertex is at the point $\left(-\dfrac{b}{2a}, f\left(-\dfrac{b}{2a}\right)\right)$.

Activity 4.1.12 Use the graph of the quadratic function to answer the questions below.

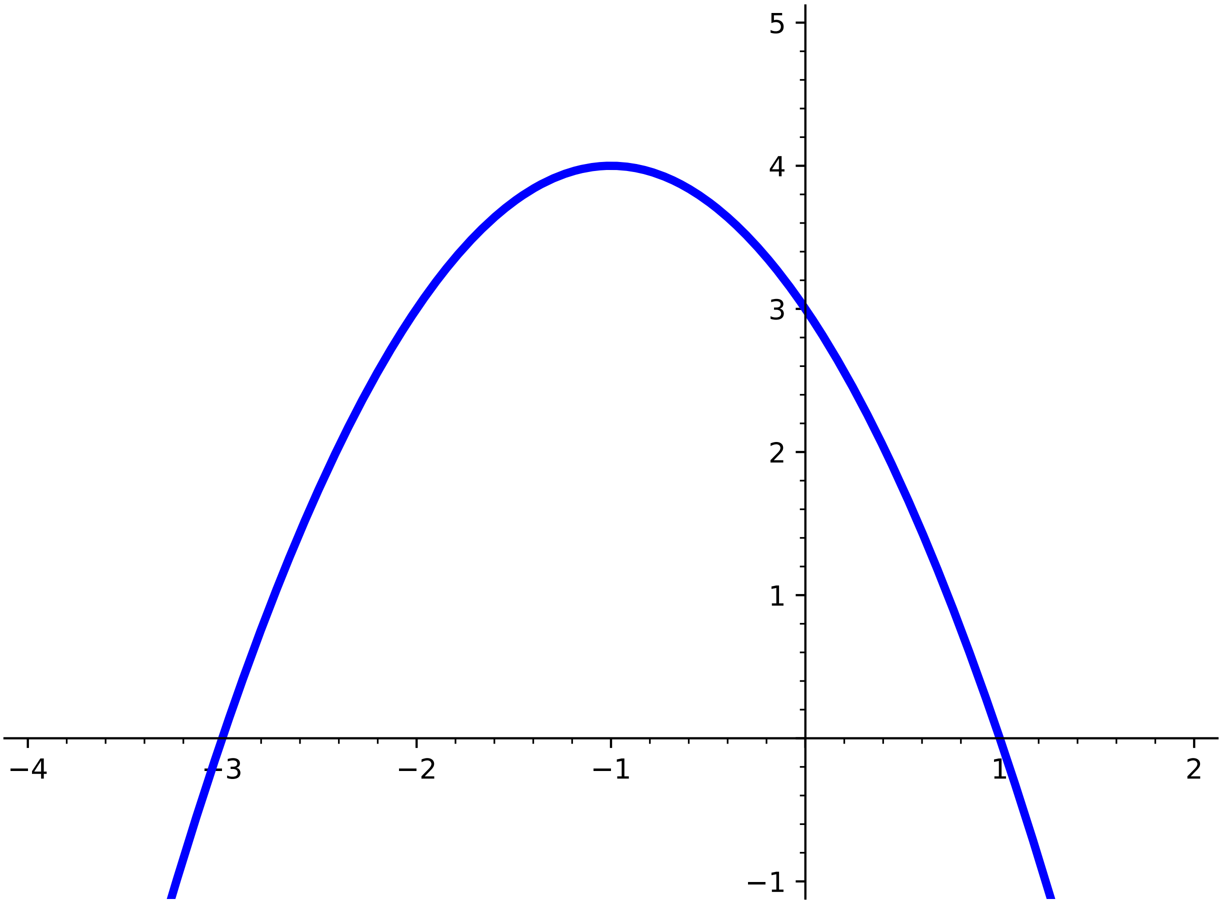


Figure 4.1.13

(a) Which of the following quadratic functions could be the graph shown in the figure?

A. $f(x) = x^2 + 2x + 3$

B. $f(x) = -(x+1)^2 + 4$

C. $f(x) = -x^2 - 2x + 3$

D. $f(x) = (x+1)^2 + 4$

(b) What is the maximum or minimum value?

A. -1

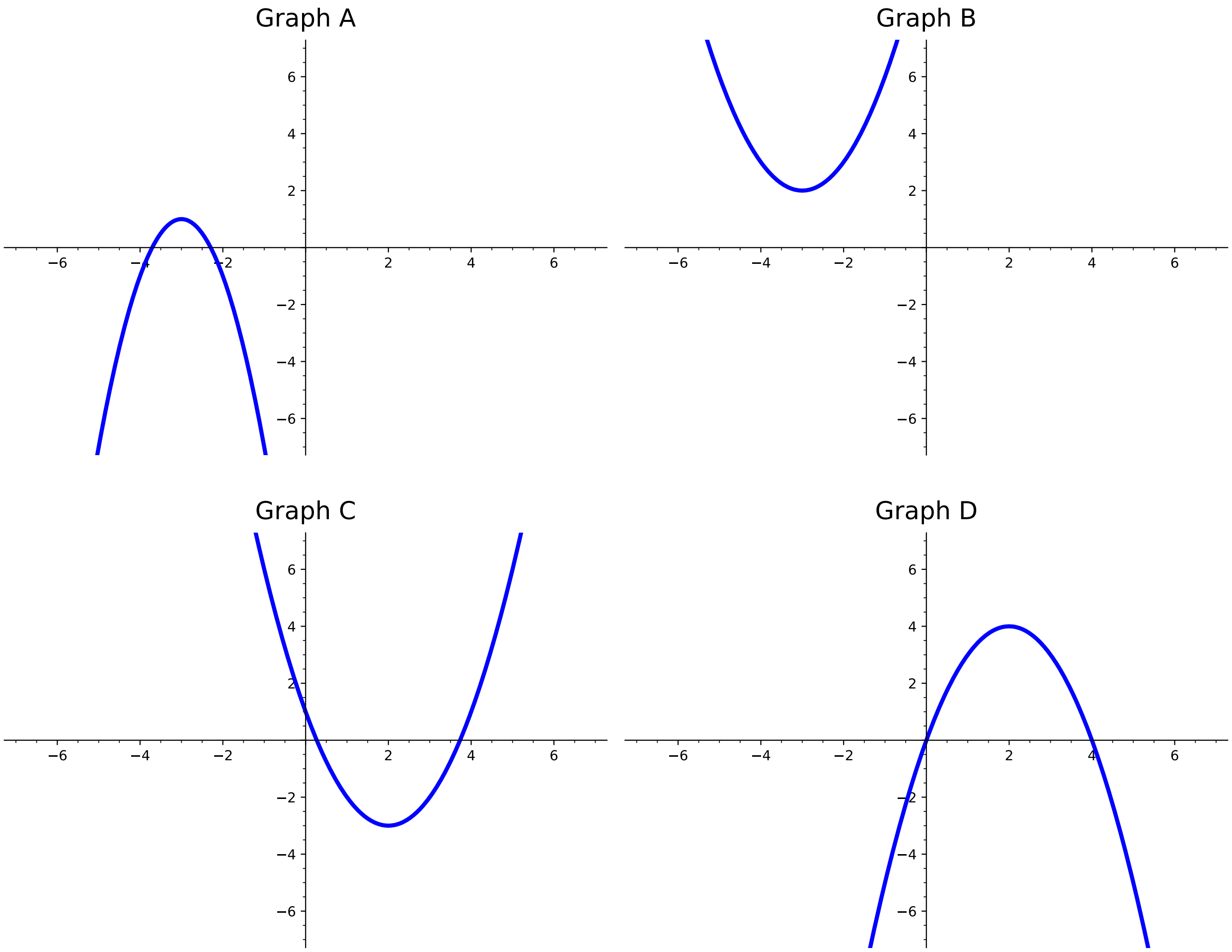
B. 4

C. -3

D. 1

Graphing Quadratic Functions (PR1)

Activity 4.1.14 Consider the following four graphs of quadratic functions:



(a) Which of the graphs above have a maximum?

A. Graph A
B. Graph B
C. Graph C
D. Graph D

(b) Which of the graphs above have a minimum?

A. Graph A
B. Graph B
C. Graph C
D. Graph D

(c) Which of the graphs above have an axis of symmetry at $x = 2$?

A. Graph A
B. Graph B
C. Graph C
D. Graph D

(d) Which of the graphs above represents the function $f(x) = -(x-2)^2 + 4$?

A. Graph A
B. Graph B
C. Graph C
D. Graph D

(e) Which of the graphs above represents the function $f(x) = x^2 - 4x + 1$?

A. Graph A
B. Graph B
C. Graph C
D. Graph D

Remark 4.1.15 Notice that the maximum or minimum value of the quadratic function is the y-value of the vertex.

Graphing Quadratic Functions (PR1)

Activity 4.1.16 A quadratic function $f(x)$ has a maximum value of 7 and axis of symmetry at $x = -2$.

(a) Sketch a graph of a function that meets the criteria for $f(x)$.

(b) Was your graph the only possible answer? Try to sketch another graph that meets this criteria.

Remark 4.1.17 Other points, such as x- and y-intercepts, may be helpful in sketching a more accurate graph of a quadratic function.

Graphing Quadratic Functions (PR1)

Activity 4.1.18 Consider the following two quadratic functions $f(x) = x^2 - 4x + 20$ and $g(x) = 2x^2 - 8x + 24$ and answer the following questions:

(a) Applying Definition 4.1.9, what is the vertex and axis of symmetry of $f(x)$?

A. vertex: $(2, -16)$; axis of symmetry: $x = 2$

B. vertex: $(-2, 16)$; axis of symmetry: $x = -2$

C. vertex: $(-2, -16)$; axis of symmetry: $x = -2$

D. vertex: $(2, 16)$; axis of symmetry: $x = 2$

(b) Applying Definition 4.1.9, what is the vertex and axis of symmetry of $g(x)$?

A. vertex: $(2, -16)$; axis of symmetry: $x = 2$

B. vertex: $(-2, 16)$; axis of symmetry: $x = -2$

C. vertex: $(-2, -16)$; axis of symmetry: $x = -2$

D. vertex: $(2, 16)$; axis of symmetry: $x = 2$

(c) What do you notice about $f(x)$ and $g(x)$?

(d) Now graph both $f(x)$ and $g(x)$ and draw a sketch of each graph on one coordinate plane. How are they similar/different?

Remark 4.1.19 Notice that in Activity 4.1.20, two different functions could have the same vertex and axis of symmetry. When $|a| > 0$, the graph narrows. When $0 < |a| < 1$, the graph widens (refer back to Section 2.4).

Graphing Quadratic Functions (PR1)

Activity 4.1.20 Answer the following for each quadratic function below.

1. Determine if the parabola opens upwards or downwards.
2. Determine the coordinates of the vertex of the parabola. Is this point a maximum or a minimum of the quadratic function?
3. Determine the axis of symmetry for the parabola.
4. Sketch the quadratic function. Be sure to include the vertex along with at least two points on each side of the vertex.

(a) $f(x) = x^2 + 2x - 3$

(b) $f(x) = -5(x-3)^2 + 20$

(c) $f(x) = 5x^2 + 30x + 40$

(d) $f(x) = 2(x+4)^2 - 3$

4.2 Quadratic Models and Meanings (PR2)

Objectives

- Use quadratic models to solve an application problem and establish conclusions.

Quadratic Models and Meanings (PR2)

Observation 4.2.1 Objects launched into the air follow a path that can be described by a quadratic function. We can also use quadratic functions to model area, profit, and population. Knowing the key components of a quadratic function allow us to find maximum profit, the point where an object hits the ground, or how much of an object to make for a minimum cost.

Quadratic Models and Meanings (PR2)

Activity 4.2.2 A water balloon is tossed vertically from a fifth story window. It's height $h(t)$, in feet, at a time t, in seconds, is modeled by the function

$$h(t) = -16t^2 + 40t + 50$$

(a) Complete the following table. Do all the values have meaning in terms of the model?

Table 4.2.3

t	$h(t)$
0	
1	
2	
3	
4	
5	

(b) Compute the slope of the line joining $t = 0$ and $t = 1$. Then, compute the slope of the line joining $t = 1$ and $t = 2$. What do you notice about the slopes?

(c) What is the meaning of $h(0) = 50$?

A. The initial height of the water balloon is 50 feet.

B. The water balloon reaches a maximum height of 50 feet.

C. The water balloon hits the ground after 50 seconds.

D. The water balloon travels 50 feet before hitting the ground.

(d) Find the vertex of the quadratic function $h(t)$.

A. $(0, 50)$

B. $(1, 74)$

C. $(1.25, 75)$

D. $(3.4, 0)$

(e) What is the meaning of the vertex?

A. The water balloon reaches a maximum height of 50 feet at the start.

B. After 1 second, the water balloon reaches a maximum height of 74 feet.

C. After 1.25 seconds, the water balloon reaches the maximum height.

D. After 3.4 seconds, the water balloon hits the ground.

Quadratic Models and Meanings (PR2)

Activity 4.2.4 The population of a small city is given by the function $P(t) = -50t^2 + 1200t + 32000$, where t is the number of years after 2015.

(a) When will the population of the city reach a maximum?

A. 2020

B. 2022

C. 2025

D. 2027

(b) Determine when the population of the city is increasing and when it is decreasing.

(c) When will the population of the city reach 36,000 people?

A. 2019

B. 2025

C. 2027

D. 2035

Quadratic Models and Meanings (PR2)

Activity 4.2.5 The unit price of an item affects its supply and demand. That is, if the unit price increases, the demand for the item will usually decrease. For example, an online streaming service currently has 84 million subscribers at a monthly charge of \$6. Market research has suggested that if the owners raise the price to \$8, they would lose 4 million subscribers. Assume that subscriptions are linearly related to the price.

(a) Which of the following represents a linear function which relates the price of the streaming service p to the number of subscribers Q (in millions)?

A. $Q(p) = -2p$

B. $Q(p) = -2p + 84$

C. $Q(p) = -2p - 4$

D. $Q(p) = -2p + 96$

(b) Using the fact that revenue R is price times the number of items sold, $R = pQ$, which of the following represents the revenue in terms of the price?

A. $R(p) = -2p^2$

B. $R(p) = -2p^2 + 84p$

C. $R(p) = -2p^2 - 4p$

D. $R(p) = -2p^2 + 96p$

(c) What price should the streaming service charge for a monthly subscription to maximize their revenue?

A. \$10

B. \$19.50

C. \$24

D. \$28.25

(d) How many subscribers would the company have at this price?

A. 39.5 million

B. 48 million

C. 57 million

D. 76 million

(e) What is the maximum revenue?

A. 760 million

B. 1112 million

C. 1152 million

D. 1116 million

Quadratic Models and Meanings (PR2)

Activity 4.2.6 The owner of a ranch decides to enclose a rectangular region with 240 feet of fencing. To help the fencing cover more land, he plans to use one side of his barn as part of the enclosed region. What is the maximum area the rancher can enclose?

(a) Draw a picture to represent the fenced area against the barn. Use w to represent the length of fence parallel to the barn and l to represent the two sides perpendicular to the barn.

(b) Find an equation for the area of the fence in terms of the length l. It may be useful to find an equation for the total amount of fencing in terms of the length l and width w.

A. $A = lw$

B. $A = l^2$

C. $A = l(240 - 2l)$

D. $A = l\left(120 - \frac{l}{2}\right)$

(c) Use the area equation to find the maximum area the rancher can enclose.

4.3 Graphs of Polynomial Functions (PR3)

Objectives

- Determine the zeros and their multiplicities of a polynomial in factored form. Describe and graph the behavior of a polynomial function at the intercepts and the ends.

Graphs of Polynomial Functions (PR3)

Remark 4.3.1 Just like with quadratic functions, we should be able to determine key characteristics that will help guide us in creating a sketch of any polynomial function. We can start by finding both x and y-intercepts and then explore other characteristics polynomial functions can have. Recall that the **zeros** of a function are the x**-intercepts** - i.e., the values of x that cross or touch the x-axis. Just like with quadratic functions, we can find the zeros of a function by setting the function equal to 0 and solving for x.

Graphs of Polynomial Functions (PR3)

Activity 4.3.2 Given the function, $f(x) = (x-2)(x+1)(x-3)^2$, determine the following characteristics.

(a) How many zeros does $f(x)$ have?

A. 1
B. 2
C. 3
D. 4

(b) What are the zeros of $f(x)$?

A. $1, 2, 3$
B. $-1, 2, 3$
C. $1, -2, -3$
D. $-1, 2, -3$

(c) What is the y-intercept of $f(x)$?

A. -1
B. -6
C. 6
D. -18

(d) Now that we have found both the x and y-intercepts of $f(x)$, do we have enough information to draw a possible sketch of $f(x)$? What other characteristics would be useful to help us draw an accurate sketch of $f(x)$?

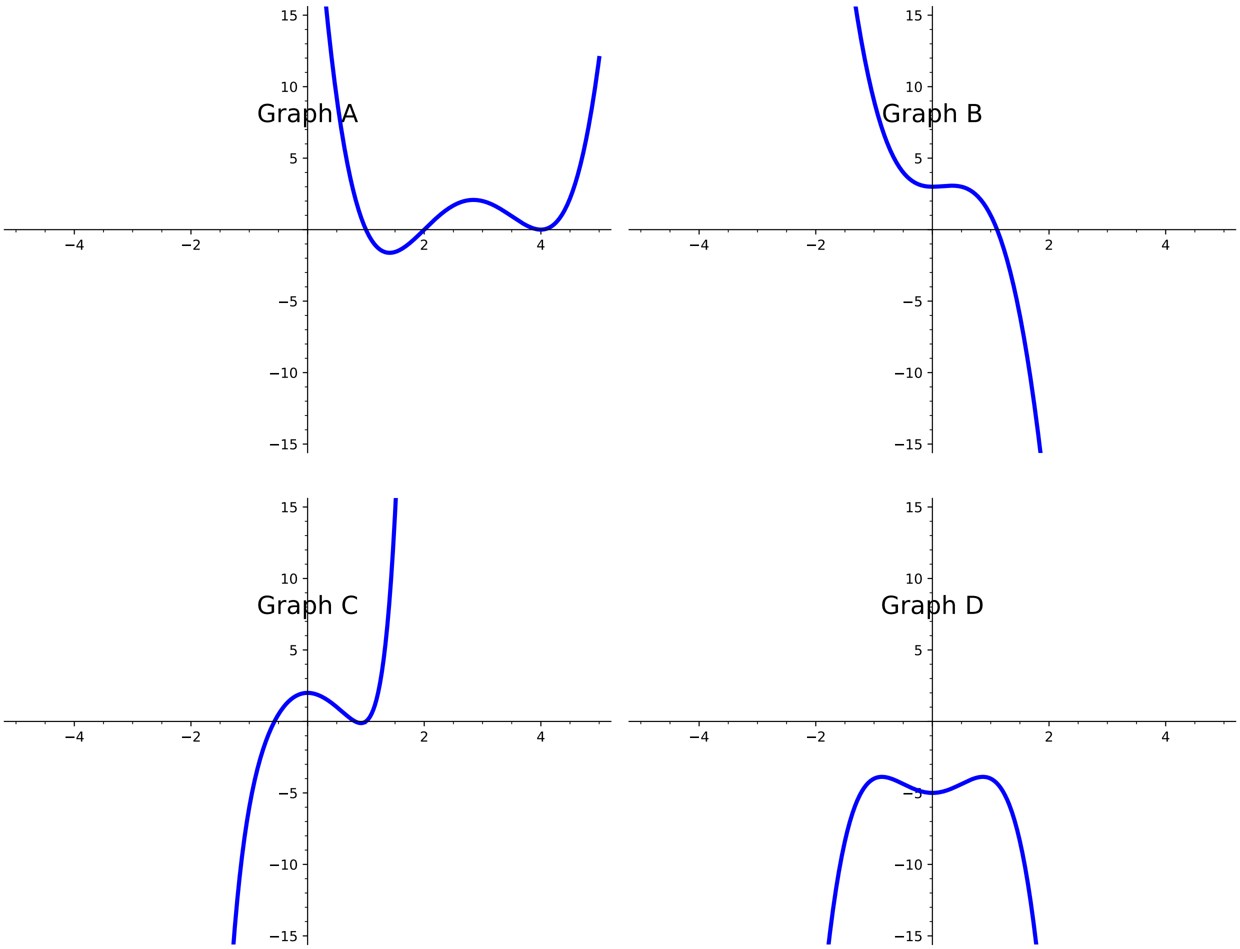
Definition 4.3.3 The **end behavior** of a polynomial function describes the behavior of the graph at the "ends" of the function. In other words, as we move to the right of the graph (as the x values increase), what happens to the y values? Similarly, as we move to the left of the graph (as the x values decrease), what happens to the y values?

Although we are looking at the "ends" to determine the end behavior, note that polynomials do not actually have ends. In other words, polynomial functions have y-values that exist for every x-value.

$\Diamond$

Graphs of Polynomial Functions (PR3)

Activity 4.3.4 Use the graphs of the following polynomial functions to answer the questions below.



(a) How would you describe the behavior of Graph A as you approach the ends?

A. Graph A rises on the left and on the right.

B. Graph A rises on the left, but falls on the right.

C. Graph A rises on the right, but falls on the left.

D. Graph A falls on the left and on the right.

(b) How would you describe the behavior of Graph B as you approach the ends?

A. Graph B rises on the left and on the right.

B. Graph B rises on the left, but falls on the right.

C. Graph B rises on the right, but falls on the left.

D. Graph B falls on the left and on the right.

(c) How would you describe the behavior of Graph C as you approach the ends?

A. Graph C rises on the left and on the right.

B. Graph C rises on the left, but falls on the right.

C. Graph C rises on the right, but falls on the left.

D. Graph C falls on the left and on the right.

(d) How would you describe the behavior of Graph D as you approach the ends?

A. Graph D rises on the left and on the right.

B. Graph D rises on the left, but falls on the right.

C. Graph D rises on the right, but falls on the left.

D. Graph D falls on the left and on the right.

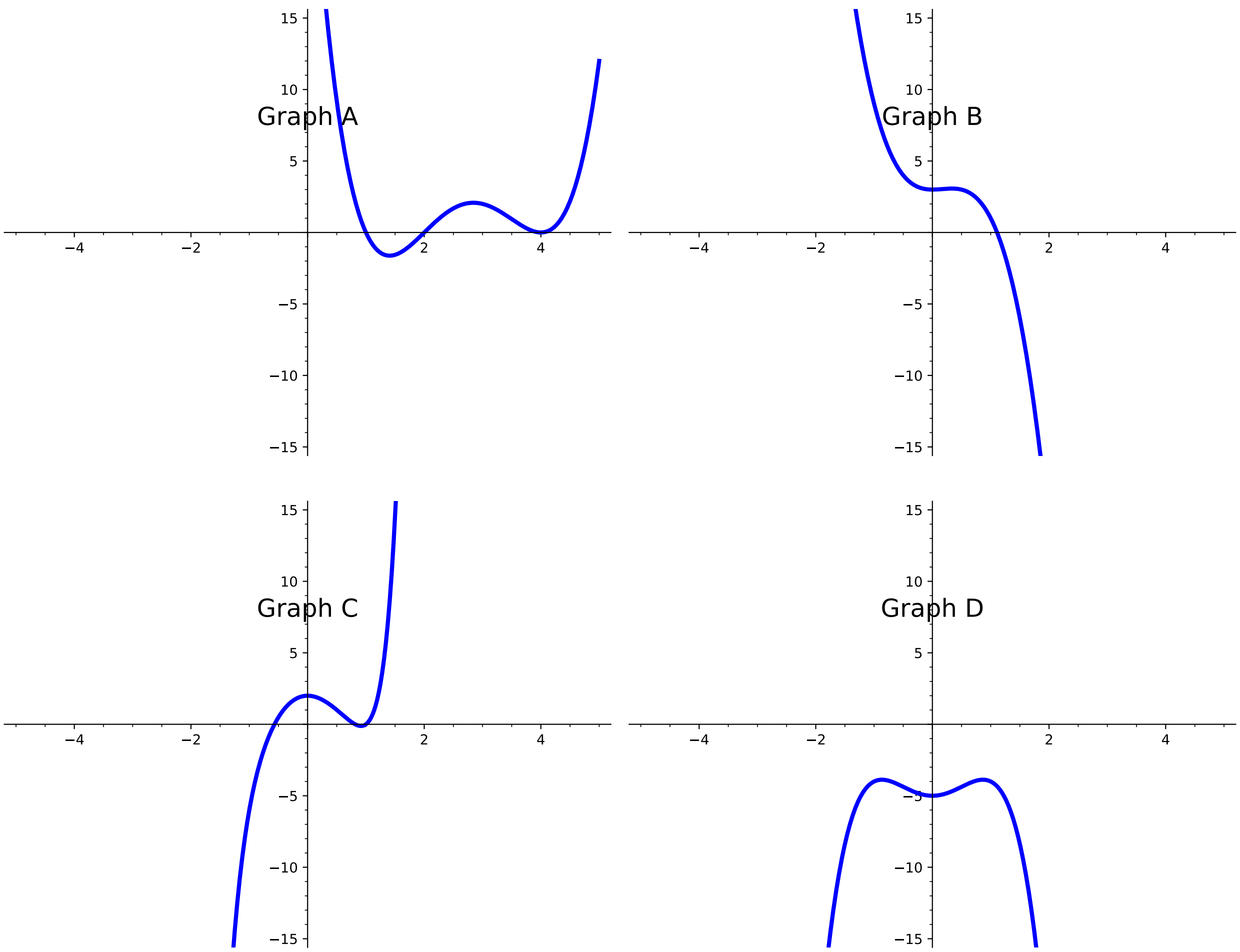
Graphs of Polynomial Functions (PR3)

Definition 4.3.5 Typically, when given an equation of a polynomial function, we look at the **degree** and **leading coefficient** to help us determine the behavior of the ends. The **degree** is the highest exponential power in the polynomial. The **leading coefficient** is the number written in front of the variable with the highest exponential power. ♢

Graphs of Polynomial Functions (PR3)

Activity 4.3.6 Let's refer back to the graphs in Activity 4.3.4 and look at the equations of those polynomial functions. Let's apply Definition 4.3.5 to see if we can determine how the degree and leading coefficients of those graphs affect their end behavior.

- Graph A: $f(x) = -11x^3 + 32 + 42x^2 + x^4 - 64x$
- Graph B: $g(x) = 2x^2 + 3 - 4x^3$
- Graph C: $h(x) = x^7 + 2x^3 - 5x^2 + 2$
- Graph D: $j(x) = 3x^2 - 2x^4 - 5$



(a) What is the degree and leading coefficient of Graph A?

A. Degree: -64; Leading Coefficient: 4

B. Degree: 4; Leading Coefficient: 0

C. Degree: 1; Leading Coefficient: -64

D. Degree: 4; Leading Coefficient: 1

(b) What is the degree and leading coefficient of Graph B?

A. Degree: 3; Leading Coefficient: -4

B. Degree: -4; Leading Coefficient: 3

C. Degree: 2; Leading Coefficient: 3

D. Degree: 3; Leading Coefficient: 4

(c) What is the degree and leading coefficient of Graph C?

A. Degree: -5; Leading Coefficient: 2

B. Degree: 0; Leading Coefficient: 7

C. Degree: -5; Leading Coefficient: 3

D. Degree: 7; Leading Coefficient: 1

(d) What is the degree and leading coefficient of Graph D?

A. Degree: -2; Leading Coefficient: 4

B. Degree: 3; Leading Coefficient: 2

C. Degree: 4; Leading Coefficient: -2

D. Degree: -5; Leading Coefficient: 4

(e) Notice that Graph A and Graph D have their ends going in the same direction. What conjectures can you make about the relationship between their degrees and leading coefficients with the behavior of their graphs?

(f) Notice that Graph B and Graph C have their ends going in opposite directions. What conjectures can you make about the relationship between their degrees and leading coefficients with the behavior of their graphs?

Remark 4.3.7 From Activity 4.3.6, we saw that the degree and leading coefficient of a polynomial function can give us more clues about the behavior of the function. In summary, we know:

- If the degree is even, the ends of the polynomial function will be going in the same direction. If the leading coefficient is positive, both ends will be pointing up. If the leading coefficient is negative, both ends will be pointing down.
- If the degree is odd, the ends of the polynomial function will be going in opposite directions. If the leading coefficient is positive, the left end will fall and the right end will rise. If the leading coefficient is negative, the left end will rise and the right end will fall.

Definition 4.3.8 When describing end behavior, mathematicians typically use **arrow notation**. Just as the name suggests, arrows are used to indicate the behavior of certain values on a graph.

For end behavior, students are often asked to determine the behavior of y-values as x-values either increase or decrease. The statement "As $x \to \infty$, $f(x) \to -\infty$" can be translated to "As x approaches infinity (or as x increases), $f(x)$ (or the y-values) go to negative infinity (i.e., it decreases)." $\Diamond$

Activity 4.3.9 Use the graph of $f(x)$ to answer the questions below.

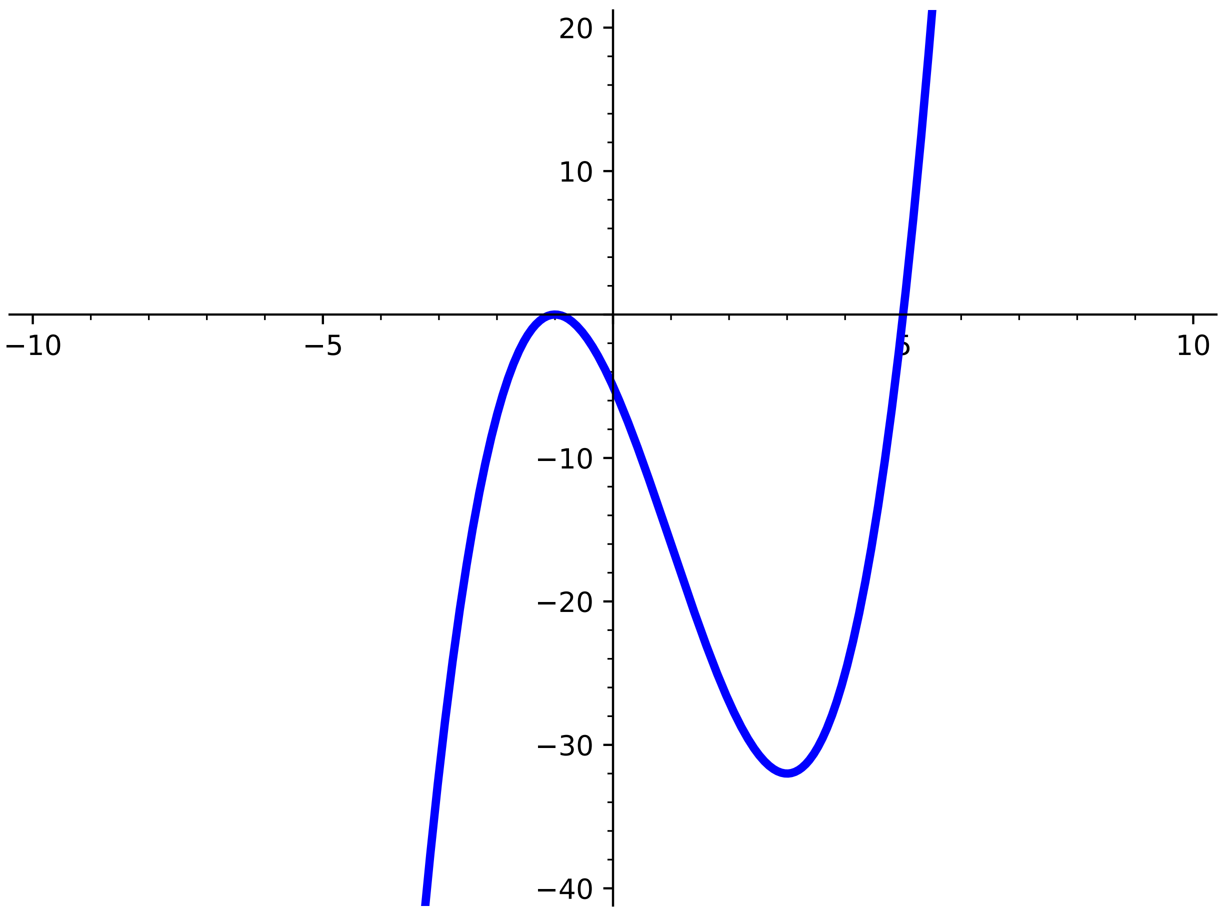


Figure 4.3.10

(a) How would you describe the end behavior of $f(x)$?

A. $f(x)$ rises on the left and on the right.
B. $f(x)$ rises on the left, but falls on the right.
C. $f(x)$ rises on the right, but falls on the left.
D. $f(x)$ falls on the left and on the right.

(b) How would you describe the end behavior of $f(x)$ using arrow notation?

A. As $x \to -\infty$, $f(x) \to -\infty$
As $x \to \infty$, $f(x) \to -\infty$
B. As $x \to -\infty$, $f(x) \to -\infty$
As $x \to \infty$, $f(x) \to \infty$
C. As $x \to -\infty$, $f(x) \to \infty$
As $x \to \infty$, $f(x) \to -\infty$
D. As $x \to -\infty$, $f(x) \to \infty$
As $x \to \infty$, $f(x) \to \infty$

Activity 4.3.11 Let's now look at the graph of $f(x) = (x-2)(x+1)(x-3)^2$ to answer the questions below.

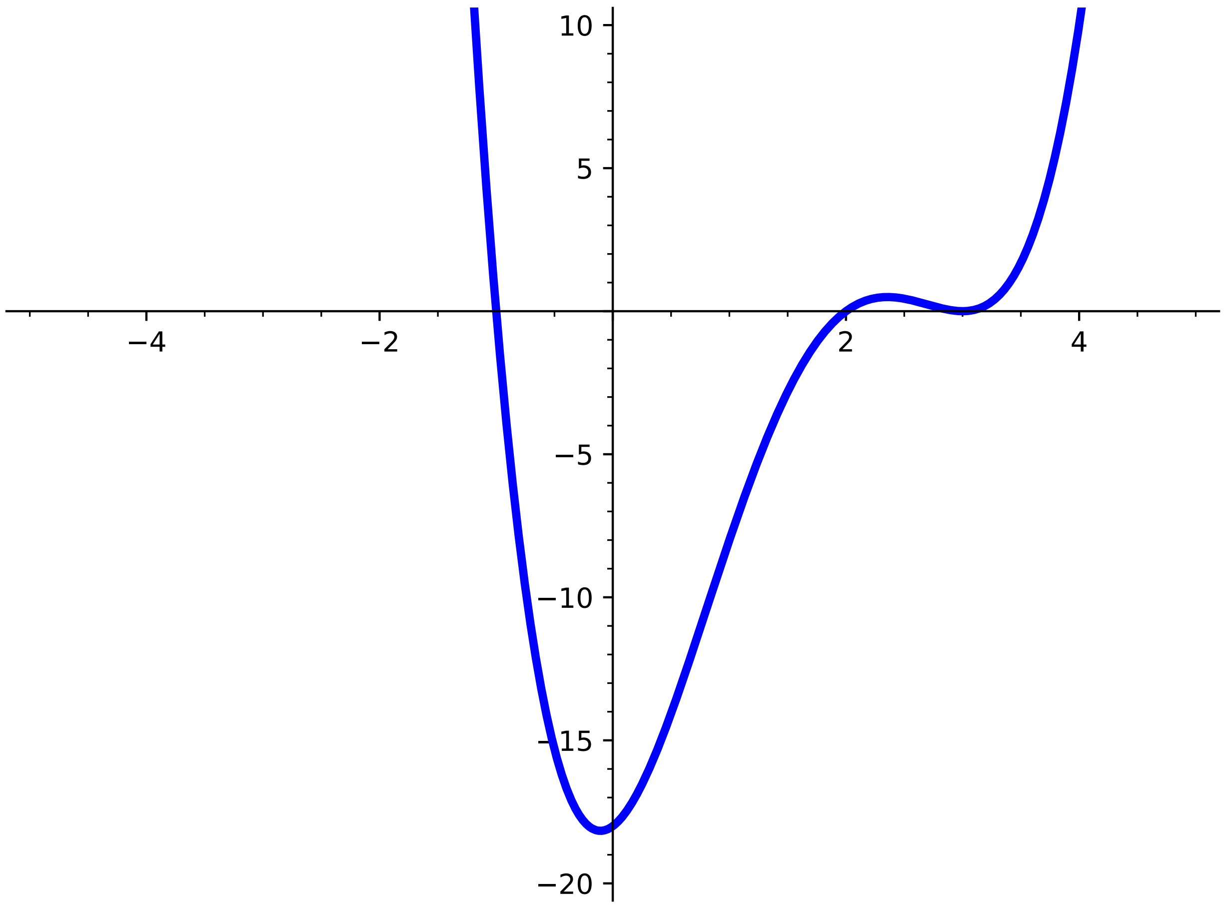


Figure 4.3.12

(a) What are the zeros of $f(x)$?

A. $1, 2, 3$

B. $-1, 2, 3$

C. $1, -2, -3$

D. $-1, 2, -3$

(b) Describe the behavior at each zero. What do you notice?

Definition 4.3.13 The **multiplicity** of a polynomial function is the number of times a given factor appears in the factored form of the equation of a polynomial.

The zero, $x = 3$ in Activity 4.3.11 has multiplicity 2 because the factor $(x-3)$ occurs twice (the exponent is a 2). ♢

Graphs of Polynomial Functions (PR3)

Activity 4.3.14 Use the function, $f(x) = (x-3)^2(x-1)$ to answer the following questions:

(a) What are the zeros of $f(x)$?

A. $-1, -3$

B. $-1, 3$

C. $1, -3$

D. $1, 3$

(b) Using Definition 4.3.13, determine the multiplicities of the zeros you found in part (a).

(c) Now graph $f(x)$ and look at the zeros on the graph. What do you notice about the behavior of the graph at each zero?

Remark 4.3.15 Refer back to Activity 4.3.11 and Activity 4.3.14. Notice that when the graph crosses the x-axis at the zero, the mulitplicity of that zero is odd. When the graph touches the x-axis at the zero, the multiplicity is even. In other words, factors of $f(x)$ with odd exponents will cross the x-axis and factors of $f(x)$ with even exponents will touch the x-axis.

Graphs of Polynomial Functions (PR3)

Definition 4.3.16 When graphing polynomial functions, you may notice that these functions have some ”hills” and ”valleys.” These characteristics of the graph are known as the **local maxima** and **local minima** of the graph - similar to what we've already seen with quadratic functions. Unlike quadratic functions, however, a polynomial graph can have many local maxima/minima (quadratic functions only have one). ♢

Activity 4.3.17 Now that we know all the different characteristics of polynomials, we should also be able to identify them from a graph. Use the graph below to find the given characteristics.

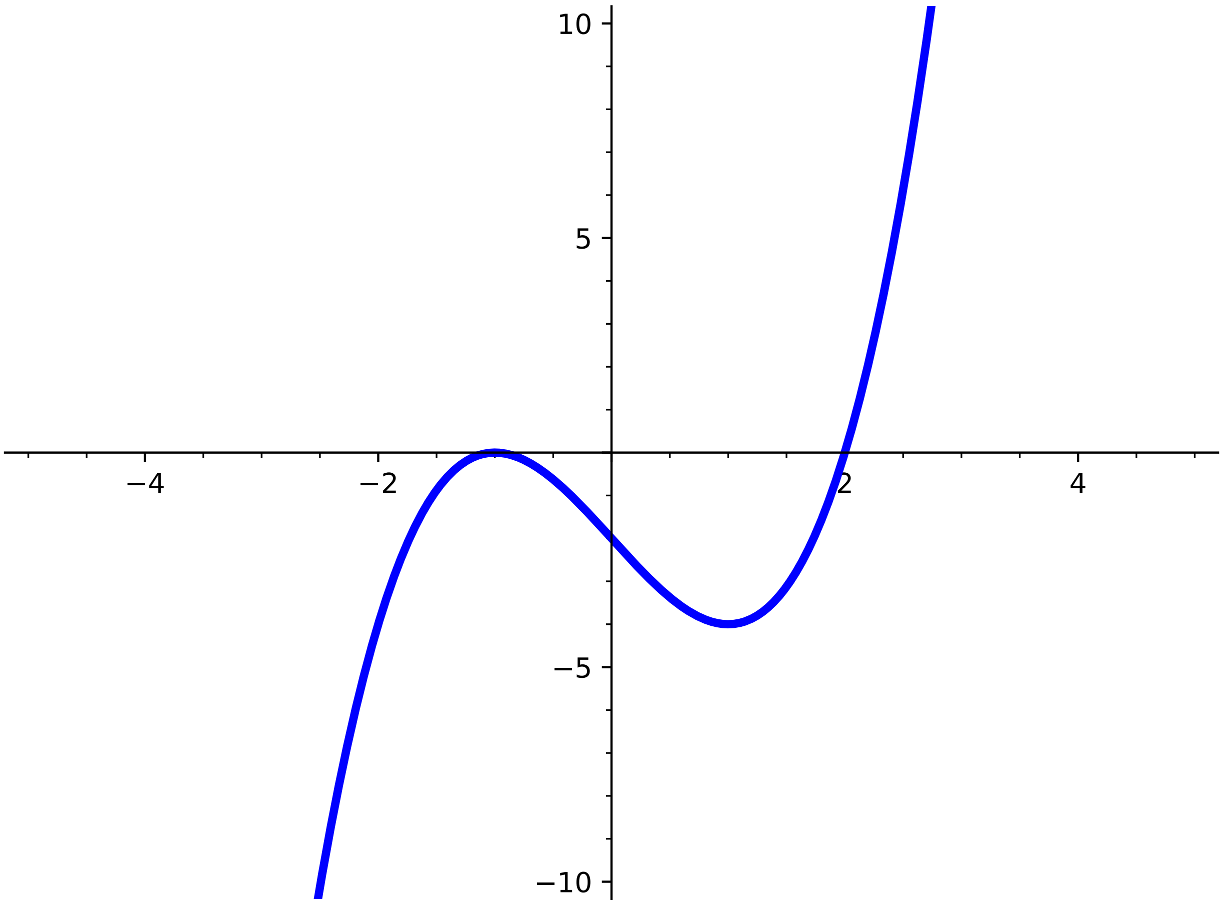


Figure 4.3.18

(a) What are the x-intercept(s) of the polynomial function? Select all that apply.

A. $(1, 0)$
B. $(-1, 0)$
C. $(2, 0)$
D. $(0, -2)$

(b) What are the y-intercept(s) of the polynomial function?

A. $(1, 0)$
B. $(-1, 0)$
C. $(2, 0)$
D. $(0, -2)$

(c) How many zeros does this polynomial function have?

A. 0
B. 1
C. 2
D. 3

(d) At what point is the local minimum located?

A. $(2, -4)$

B. $(-1, 0)$

C. $(-2, 0)$

D. $(1, -4)$

E. $(2, 0)$

(e) At what point is the local maximum located?

A. $(2, -4)$

B. $(-1, 0)$

C. $(-2, 0)$

D. $(1, -4)$

E. $(2, 0)$

(f) How do you describe the behavior of the polynomial function as $x \to \infty$?

A. the y-values go to negative infinity

B. $f(x) \to \infty$

C. the y-values go to positive infinity

D. $f(x) \to -\infty$

(g) How do you describe the behavior of the polynomial function as $x \to -\infty$?

A. the y-values go to negative infinity

B. $f(x) \to \infty$

C. the y-values go to positive infinity

D. $f(x) \to -\infty$

Graphs of Polynomial Functions (PR3)

Activity 4.3.19 Sketch the function, $f(x) = (x - 2)(x + 1)(x - 3)^2$, by first finding the given characteristics.

(a) Find the zeros of $f(x)$.

(b) Find the multiplicities and describe the behavior at each zero.

(c) Find the y-intercept of $f(x)$.

(d) Describe the end behavior of $f(x)$.

(e) Estimate where any local maximums and minimums may occur.

Graphs of Polynomial Functions (PR3)

Activity 4.3.20 Sketch the graph of a function $f(x)$ that meets all of the following criteria. Be sure to scale your axes and label any important features of your graph.

- The x-intercepts of $f(x)$ are $0, 2$, and 5.
- 0 and 2 have even multiplicity and 5 has odd multiplicity.
- The end behavior of $f(x)$ is given as: As $x \to \infty$, $f(x) \to \infty$ and As $x \to -\infty$, $f(x) \to -\infty$

Graphs of Polynomial Functions (PR3)

Activity 4.3.21 Use the given function, $f(x) = (x+1)^2(x-5)$, to answer the following questions.

(a) What are the zeros of $f(x)$?

A. $-1, -5$

B. $-1, 5$

C. $1, -5$

D. $1, 5$

(b) What are the multiplicities at each zero?

A. At $x = -1$, the multiplicity is even.
At $x = 5$, the multiplicity is even.

B. At $x = -1$, the multiplicity is even.
At $x = 5$, the multiplicity is odd.

C. At $x = -1$, the multiplicity is odd.
At $x = 5$, the multiplicity is even.

D. At $x = -1$, the multiplicity is odd.
At $x = 5$, the multiplicity is odd.

(c) What is the end behavior of $f(x)$?

A. $f(x)$ rises on the left and on the right.

B. $f(x)$ rises on the left, but falls on the right.

C. $f(x)$ rises on the right, but falls on the left.

D. $f(x)$ falls on the left and on the right.

(d) Using what you know about the zeros, multiplicities, and end behavior, where on the graph can we estimate the local maxima and minima to be?

(e) Now look at the graph of $f(x)$. At which zero does a local maximum or local minimum occur? Explain how you know.

Remark 4.3.22 We can estimate where these local maxima and minima occur by looking at other characteristics, such as multiplicities and end behavior.

From Activity 4.3.22, we saw that when the function touches the x-axis at a zero, then that zero could be either a local maximum or a local minimum of the graph. When the function crosses the x-axis, however, the local maximum or local minimum occurs between the zeros.

4.4 Polynomial Long Division (PR4)

Objectives

- Rewrite a rational function as a polynomial plus a proper rational function.

Polynomial Long Division (PR4)

Observation 4.4.1 We have seen previously that we can reduce rational functions by factoring, for example

$$\frac{x^2+5x+4}{x^3+3x^2+2x} = \frac{(x+1)(x+4)}{x(x+2)(x+1)} = \frac{x+4}{x(x+2)}.$$

In this section, we will explore the question: what can we do to simplify rational functions if we are not able to reduce by easily factoring?

Polynomial Long Division (PR4)

Definition 4.4.2 Recall that a fraction is called **proper** if its numerator is smaller than its denominator, and **improper** if the numerator is larger than the denominator (so $\frac{3}{5}$ is a proper fraction, but $\frac{32}{7}$ is an improper fraction). Similarly, we define a **proper rational function** to be a rational function where the degree of the numerator is less than the degree of the denominator. $\diamondsuit$

Activity 4.4.3 properimproper

A. $\dfrac{x^3+x}{x^2+4}$

B. $\dfrac{3}{x^2+3x+4}$

C. $\dfrac{7+x^3}{x^2+x+1}$

D. $\dfrac{x^4+x+1}{x^4+4x^2}$

Polynomial Long Division (PR4)

Observation 4.4.4 When dealing with an improper fraction such as $\frac{32}{7}$, it is sometimes useful to rewrite this as an integer plus a proper fraction, e.g. $\frac{32}{7} = 4 + \frac{4}{7}$. Similarly, it will sometimes be useful to rewrite an improper rational function as the sum of a polynomial and a proper rational function, such as $\frac{x^3 + x}{x^2 + 4} = x - \frac{3x}{x^2 + 4}$.

Polynomial Long Division (PR4)

Activity 4.4.5 Consider the improper fraction $\frac{357}{11}$.

(a) Use long division to write $\frac{357}{11}$ as an integer plus a proper fraction.

(b) Now we will carefully redo this process in a way that we can generalize to rational functions. Note that we can rewrite 357 as $357 = 3 \cdot 10^2 + 5 \cdot 10 + 7$, and 11 as $11 = 1 \cdot 10 + 1$. By comparing the leading terms in these expansions, we see that to knock off the leading term of 357, we need to multiply 11 by $3 \cdot 10^1$.

Using the fact that $357 = 11 \cdot 30 + 27$, rewrite $\frac{357}{11}$ as $\frac{357}{11} = 30 + \frac{?}{11}$.

(c) Note now that if we can rewrite $\frac{27}{11}$ as an integer plus a proper fraction, we will be done, since $\frac{357}{11} = 30 + \frac{27}{11}$.

Rewrite $\frac{27}{11} =? + \frac{?}{11}$ as an integer plus a proper fraction.

(d) Combine your work in the previous two parts to rewrite $\frac{357}{11}$ as an integer plus a proper fraction. How does this compare to what you obtained in part (a)?

Polynomial Long Division (PR4)

Activity 4.4.6 Now let's consider the rational function $\frac{3x^2+5x+7}{x+1}$. We want to rewrite this as a polynomial plus a proper rational function.

(a) Looking at the leading terms, what do we need to multiply $x+1$ by so that it would have the same leading term as $3x^2+5x+7$?

A. 3

B. x

C. $3x$

D. $3x+5$

(b) Rewrite $3x^2+5x+7 = 3x(x+1)+?$, and use this to rewrite $\frac{3x^2+5x+7}{x+1} = 3x + \frac{?}{x+1}$.

(c) Now focusing on $\frac{2x+7}{x+1}$, what do we need to multiply $x+1$ by so that it would have the same leading term as $2x+7$?

A. 2

B. x

C. $2x$

D. $2x+7$

(d) Rewrite $\frac{2x+7}{x+1} = 2 + \frac{?}{x+1}$.

(e) Combine this with the previous parts to rewrite $\frac{3x^2+5x+7}{x+1} = 3x+?+\frac{?}{x+1}$.

Polynomial Long Division (PR4)

Activity 4.4.7 Next we will use the notation of long division to rewrite the rational function $\frac{3x^2+5x+7}{x+1}$ as a polynomial plus a proper rational function.

(a) First, let's use long division notation to write the quotient.

$$(x+1)\ \overline{)\,3x^2+5x+7} \quad \text{quotient: } ?$$

What do we need to multiply $x+1$ by so that it would have the same leading term as $3x^2+5x+7$?

(b) Now to rewrite $3x^2+5x+7$ as $3x(x+1)+?$, place the product $3x(x+1)$ below and subtract.

$$\begin{array}{r} 3x \\ (x+1)\ \overline{)\,3x^2+5x+7} \\ \underline{3x^2+3x} \\ ?\ +? \end{array}$$

(c) Now focusing on $2x+7$, what do we need to multiply $x+1$ by so that it would have the same leading term as $2x+7$?

$$\begin{array}{r} 3x+? \\ (x+1)\ \overline{)\,3x^2+5x+7} \\ \underline{3x^2+3x} \\ 2x+7 \end{array}$$

(d) Now, subtract $2(x+1)$ to finish the long division.

Polynomial Long Division (PR4)

$$
\begin{array}{rr}
 & 3x+2 \\
\cline{2-2}
(x+1) & \big)\,3x^2+5x+7 \\
 & 3x^2+3x \quad \\
\cline{2-2}
 & 2x+7 \\
 & ?\;\;+? \\
\cline{2-2}
 & ?
\end{array}
$$

(e) This long division calculation has shown that

$$3x^2 + 5x + 7 = (x + 1)(3x + 2) + 5.$$

Use this to rewrite $\dfrac{3x^2 + 5x + 7}{x + 1}$ as a polynomial plus a proper rational function.

Remark 4.4.8 Note that in Activity 4.4.6 and Activity 4.4.7 we performed the same computations, but just organized our work a little differently.

Polynomial Long Division (PR4)

Activity 4.4.9 Rewrite $\dfrac{x^2+1}{x-1}$ as a polynomial plus a proper rational function.

Hint. Note that $x^2+1 = x^2+0x+1$.

Polynomial Long Division (PR4)

Activity 4.4.10 Rewrite $\dfrac{x^5 + x^3 + 2x^2 - 6x + 7}{x^2 + x - 1}$ as a polynomial plus a proper rational function.

Polynomial Long Division (PR4)

Activity 4.4.11 Rewrite $\dfrac{3x^4 - 5x^2 + 2}{x - 1}$ as a polynomial plus a proper rational function.

4.5 Zeros of Polynomial Functions (PR5)

Objectives

- Determine the zeros of a polynomial function with real coefficients.

Zeros of Polynomial Functions (PR5)

Activity 4.5.1 Consider the function $f(x) = x^3 - 7x^2 + 7x + 15$.

(a) Use polynomial division from Section 4.4 to divide $f(x)$ by $x - 2$. What is the remainder?

(b) Find $f(2)$. What do you notice?

Activity 4.5.2 Again consider the function $f(x) = x^3 - 7x^2 + 7x + 15$.

(a) Divide $f(x)$ by $x - 3$. What is the remainder?

(b) Find $f(3)$. What do you notice?

Zeros of Polynomial Functions (PR5)

Remark 4.5.3 If we know one zero, then we can divide by $x - a$ where a is a zero. After this, the quotient will have smaller degree and we can work on factoring the rest. We can "chip away" at the polynomial one zero at a time.

Activity 4.5.4 One more time consider the function $f(x) = x^3 - 7x^2 + 7x + 15$.

(a) We already know from Activity 4.5.4 that $x - 3$ is a factor of the polynomial $f(x)$. Use division to express $f(x)$ as $(x - 3) \cdot q(x)$, where $q(x)$ is a quadratic function.

A. $q(x) = x^2 - 2x - 3$

B. $q(x) = x^2 - 10x - 37$

C. $q(x) = x^2 - 4x - 5$

D. $q(x) = x^2 + 4x - 5$

(b) Notice that $q(x)$ is something we can factor. Factor this quadratic and find the remaining zeros.

A. -5

B. 5

C. 4

D. -1

E. 3

F. 1

Remark 4.5.5 We were able to find all the zeros of the polynomial in Activity 4.5.7 because we were given one of the zeros. If we don't have a zero to help us get started (or need more than one zero for a function of higher degree), we have a couple of options.

Zeros of Polynomial Functions (PR5)

Activity 4.5.6 Consider the function $f(x) = 18x^4 + 67x^3 - 81x^2 - 202x + 168$.

(a) Graph the function. According to the graph, what value(s) seem to be zeros?

(b) Use the Remainder Theorem to confirm that your guesses are actually zeros.

(c) Now use these zeros along with polynomial division to rewrite the function as $f(x) = (x-a)(x-b)q(x)$ where a and b are zeros and $q(x)$ is the remaining quadratic function.

(d) Solve the quadratic $q(x)$ to find the remaining zeros.

(e) List all zeros of $f(x)$.

(f) Rewrite $f(x)$ as a product of linear factors.

Remark 4.5.7 Using the graph to find an initial zero can be helpful, but they may not always be easy to identify.

Zeros of Polynomial Functions (PR5)

Activity 4.5.8 Consider the quadratic function $f(x) = (2x - 5)(3x - 8) = 6x^2 - 31x + 40$.

(a) What are the roots of this quadratic?

(b) What do you notice about these roots in relation to the factors of $a = 6$ and $c = 40$ in $f(x) = 6x^2 - 31x + 40$?

Remark 4.5.9 In Activity 4.5.11 we found that the roots were both factors of the constant term divided by factors of the leading coefficient. This can be extended to polynomials of larger degree.

Zeros of Polynomial Functions (PR5)

Activity 4.5.10 Consider the polynomial $f(x) = 5x^3 - 2x^2 + 20x - 8$.

(a) List the factors of the constant term.

(b) List the factors of the leading coefficient.

(c) Use parts (a) and (b) to list all the possible rational roots.

(d) Use the Remainder Theorem to determine at least one root of $f(x)$.

Zeros of Polynomial Functions (PR5)

Activity 4.5.11 Consider the polynomial $f(x) = 6x^4 + 5x^3 - 6x - 5$

(a) Use the graph and the Rational Root Theorem (Theorem 4.5.13) to find the rational zeros of $f(x)$.

(b) Use the roots, along with the Factor Theorem, to simplify the polynomial into linear and quadratic factors.

(c) Find the zeros of the quadratic factor.

(d) List the roots of the polynomial.

Remark 4.5.12 Notice that the zeros of the quadratic factor were imaginary and are related. This also occured in Activity 1.5.21.

Zeros of Polynomial Functions (PR5)

Activity 4.5.13 Consider the function $f(x) = x^5 + 3x^4 + 4x^3 + 8x^2 - 16$.

(a) Use a graphing utility to graph $f(x)$.

(b) Find *all* the zeros of $f(x)$ and their corresponding multiplicities.

Zeros of Polynomial Functions (PR5)

Activity 4.5.14 Consider the following information about a polynomial $f(x)$:

- $x = 2$ is a zero with multiplicity 1
- $x = -1$ is a zero with multiplicity 2
- $x = i$ is a zero with multiplicity 1

(a) What is the smallest possible degree of such a polynomial $f(x)$ with real coefficients?

A. 2

B. 3

C. 4

D. 5

E. 6

(b) Write an expression for such a polynomial $f(x)$ with real coefficients of smallest possible degree.

4.6 Rational Equations (PR6)

Objectives

- Solve rational equations.

Rational Equations (PR6)

Definition 4.6.1 An algebraic expression is called a **rational expression** if it can be written as the ratio of two polynomials, p and q.

An equation is called a **rational equation** if it consists of only rational expressions and constants. $\Diamond$

Observation 4.6.2 Technically, linear and quadratic equations are also rational equations. They are a special case where the denominator of the rational expressions is 1. We will focus in this section on cases where the denominator is not a constant; that is, rational equations where there are variables in the denominator.

With variables in the denominator, there will often be values that cause the denominator to be zero. This is a problem because division by zero is undefined. Thus, we need to be sure to exclude any values that would make those denominators equal to zero.

Rational Equations (PR6)

Remark 4.6.3 For this section, it might be helpful to refer back to Activity 1.1.3, where you solved a linear equation with fractions. When revisiting that activity, think about how you had to carefully choose a number to get the denominators to cancel out. Keep this in mind as you work through the next activity.

Rational Equations (PR6)

Activity 4.6.4 Which value(s) should be excluded as possible solutions to the following rational equations? Select all that apply.

(a)

$$\frac{2}{x+5} = \frac{x-3}{x-8} - 7$$

A. -7

B. -5

C. 2

D. 3

E. 8

(b)

$$\frac{x^2-6x+8}{x^2-4x+3} = 0$$

A. 0

B. 1

C. 2

D. 3

E. 4

Rational Equations (PR6)

Activity 4.6.5 Consider the rational equation

$$5 = -\frac{6}{x-2}$$

(a) What value should be excluded as a possible solution?

A. 5
B. 6
C. −6
D. 2
E. −2

(b) To solve, we begin by clearing out the fraction involved. What can we multiply each term by that will clear the fraction?

A. $x - 5$
B. $x - 6$
C. $x + 6$
D. $x - 2$
E. $x + 2$

(c) Multiply each term by the expression you chose and simplify. Which of the following linear equations does the rational equation simplify to?

A. $5(x - 5) = -6$
B. $5(x - 6) = -6$
C. $5(x + 6) = -6$
D. $5(x - 2) = -6$
E. $5(x + 2) = -6$

(d) Solve the linear equation. Check your answer using the original rational equation.

Rational Equations (PR6)

Activity 4.6.6 Consider the rational equation

$$\frac{4}{x+1} = -\frac{2}{x+6}$$

(a) What values should be excluded as possible solutions?

A. 2 and 4

B. 1 and 6

C. -1 and -6

D. 1 and 4

E. 2 and 6

(b) To solve, we'll once again begin by clearing out the fraction involved. Which of the following should we multiply each term by to clear out all of the fractions?

A. $x-2$ and $x-4$

B. $x-1$ and $x-6$

C. $x+1$ and $x+6$

D. $x-1$ and $x-4$

E. $x-2$ and $x-6$

(c) Multiply each term by the expressions you chose and simplify. Which of the following linear equations does the rational equation simplify to?

A. $4(x+1) = -2(x+6)$

B. $4(x+6) = -2(x+1)$

C. $4(x+1)(x+6) = -2(x+1)(x+6)$

D. $4(x+1) = -2(-x-6)$

E. $4(x+6) = -2(-x-1)$

(d) Solve the linear equation. Check your answer using the original rational equation.

Observation 4.6.7 In Activity 4.6.6, you may have noticed that the resulting linear equation looked like the result of cross-multiplying. This is no coincidence! Cross-multiplying is a method of clearing out fractions that works specifically when the equation is in proportional form: $\frac{a}{b} = \frac{c}{d}$.

Rational Equations (PR6)

Activity 4.6.8 Consider the rational equation

$$\frac{x}{x+2} = -\frac{2}{x+2} - \frac{2}{5}$$

(a) What value(s) should be excluded as possible solutions?

(b) To solve, we'll once again begin by clearing out the fraction involved. Which of the following should we multiply each term by to clear out all of the fractions?

A. $x+2$, $x+2$, and 5

B. $x+2$ and 5

C. $x+2$

D. 5

(c) Multiply each term by the expressions you chose and simplify. You should end up with a linear equation.

(d) Solve the linear equation. Check your answer using the original rational equation.

Observation 4.6.9 Activity 4.6.8 demonstrates why it is so important to determine excluded values and check our answers when solving rational equations. Just because a number is a solution to the *linear* equation we found, it doesn't mean it is automatically a solution to the *rational* equation we started with.

Rational Equations (PR6)

Activity 4.6.10 Consider the rational equation

$$\frac{2x}{x-1} - \frac{3}{x-3} = \frac{x^2 - 11x + 18}{x^2 - 4x + 3}$$

(a) What values should be excluded as possible solutions? Select all that apply.

A. 0
B. 1
C. 2
D. 3
E. 9

(b) To solve, we'll begin by clearing out any fractions involved. Which of the following should we multiply each term by to clear out all of the fractions?

A. $x - 1$
B. $x - 1$ and $x - 3$
C. $x - 1$, $x - 3$, and $x^2 - 4x + 3$
D. $x - 1$ and $x^2 - 4x + 3$
E. $x - 3$ and $x^2 - 4x + 3$

(c) Multiply each term by the expressions you chose and simplify. Notice that the result is a quadratic equation. Which of the following quadratic equations does the rational equation simplify to?

A. $x^2 + 2x - 15 = 0$
B. $x^2 - 11x + 18 = 0$
C. $x^2 - 9x - 9 = 0$
D. $x^2 - 13x + 21 = 0$

(d) Solve the quadratic equation. Check your answer using the original rational equation. What are the solutions to the rational equation?

A. $x = 3$ and $x = -5$
B. $x = -3$ and $x = 5$
C. $x = 3$
D. $x = -5$
E. $x = -3$
F. $x = 5$

Rational Equations (PR6)

Activity 4.6.11 Consider the rational equation

$$\frac{2x}{x-2} - \frac{x^2+21x-15}{x^2+3x-10} = \frac{-6}{x+5}$$

(a) What values should be excluded as possible solutions?

(b) What expression(s) should we multiply by to clear out all of the fractions?

(c) Multiply each term by the expressions you chose and simplify. Your result should be a quadratic equation.

(d) Solve the quadratic equation. Check your answer using the original rational equation. What are the solutions to the rational equation?

Rational Equations (PR6)

Activity 4.6.12 Solve the following rational equations.

(a) $\dfrac{4}{x} + 9 = 16$

(b) $-5 = \dfrac{2}{x-4}$

(c) $\dfrac{-3}{x-10} = \dfrac{x}{x-6}$

(d) $\dfrac{x+2}{x-3} + \dfrac{x}{2x-1} = 6$

4.7 Properties and Graphs of Rational Functions (PR7)

Objectives

- Find the domain and range, vertical and horizontal asymptotes, and intercepts of a rational function and use this information to sketch the graph.

Properties and Graphs of Rational Functions (PR7)

Definition 4.7.1 A function r is **rational** provided that it is possible to write r as the ratio of two polynomials, p and q. That is, r is rational provided that for some polynomial functions p and q, we have

$$r(x) = \frac{p(x)}{q(x)}.$$

$\diamond$

Properties and Graphs of Rational Functions (PR7)

Observation 4.7.2 Rational functions occur in many applications, so our goal in this lesson is to learn about their properties and be able to graph them. In particular we want to investigate the domain, end behavior, and zeros of rational functions.

Activity 4.7.3 Consider the rational function

$$r(x) = \frac{x^2 - 3x + 2}{x^2 - 4x + 3}.$$

(a) Find $r(1)$, $r(2)$, $r(3)$, and $r(4)$.

(b) Label each of these four values as giving us information about the DOMAIN of $r(x)$, information about the ZEROES of $r(x)$, or NEITHER.

Properties and Graphs of Rational Functions (PR7)

Observation 4.7.4 Let p and q be polynomial functions so that $r(x) = \dfrac{p(x)}{q(x)}$ is a rational function. The domain of r is the set of all real numbers except those for which $q(x) = 0$.

Recall that the **domain** of a function was defined in Definition 2.1.1.

Activity 4.7.5 Let's investigate the domain of $r(x)$ more closely. We will be using the same function from the previous activity:

$$r(x) = \frac{x^2 - 3x + 2}{x^2 - 4x + 3}.$$

(a) Rewrite $r(x)$ by factoring the numerator and denominator, but do not try to simplify any further. What do you notice about the relationship between the values that are not in the domain and how the function is now written?

(b) The function was not defined for $x = 3$. Make a table for values of $r(x)$ near $x = 3$.

Table 4.7.6

x	$r(x)$
2	
2.9	
2.99	
2.999	
3	undefined
3.001	
3.01	
3.1	

(c) Which of the following describe the behavior of the graph near $x = 3$?

A. As $x \to 3$, $r(x)$ approaches a finite number

B. As $x \to 3$ from the left, $r(x) \to \infty$

C. As $x \to 3$ from the left, $r(x) \to -\infty$

D. As $x \to 3$ from the right, $r(x) \to \infty$

E. As $x \to 3$ from the right, $r(x) \to -\infty$

(d) The function was also not defined for $x = 1$. Make a table for values of $r(x)$ near $x = 1$.

Table 4.7.7

x	$r(x)$
0	
0.9	
0.99	
0.999	
1	undefined
1.001	
1.01	
1.1	

(e) Which of the following describe the behavior of the graph near $x = 1$?

A. As $x \to 1$, $r(x)$ approaches a finite number

B. As $x \to 1$ from the left, $r(x) \to \infty$

C. As $x \to 1$ from the left, $r(x) \to -\infty$

D. As $x \to 1$ from the right, $r(x) \to \infty$

E. As $x \to 1$ from the right, $r(x) \to -\infty$

(f) The function is behaving differently near $x = 1$ than it is near $x = 3$. Can you see anything in the factored form of $r(x)$ that may help you account for the difference?

Remark 4.7.8 Features of a rational function. Let $r(x) = \dfrac{p(x)}{q(x)}$ be a rational function.

- If $p(a) = 0$ and $q(a) \neq 0$, then $r(a) = 0$, so r has a **zero** at $x = a$.
- If $q(a) = 0$ and $p(a) \neq 0$, then $r(a)$ is undefined and r has a **vertical asymptote** at $x = a$.
- If $p(a) = 0$ and $q(a) = 0$ and we can show that there is a finite number L such that $r(x) \to L$, then $r(a)$ is not defined and r has a **hole** at the point (a, L).

Properties and Graphs of Rational Functions (PR7)

Observation 4.7.9 Another property of rational functions we want to explore is the end behavior. This means we want to explore what happens to a given rational function $r(x)$ when x goes toward positive infinity or negative infinity.

Activity 4.7.10 Consider the rational function $r(x) = \dfrac{1}{x^3}$.

(a) Plug in some very large positive numbers for x to see what $r(x)$ is tending toward. Which of the following best describes the behavior of the graph as x approaches positive infinity?

A. As $x \to \infty$, $r(x) \to \infty$.

B. As $x \to \infty$, $r(x) \to -\infty$.

C. As $x \to \infty$, $r(x) \to 0$.

D. As $x \to \infty$, $r(x) \to 1$.

(b) Now let's look at $r(x)$ as x tends toward negative infinity. Plug in some very large negative numbers for x to see what $r(x)$ is tending toward. Which of the following best describes the behavior of the graph as x approaches negative infinity?

A. As $x \to -\infty$, $r(x) \to \infty$.

B. As $x \to -\infty$, $r(x) \to -\infty$.

C. As $x \to -\infty$, $r(x) \to 0$.

D. As $x \to -\infty$, $r(x) \to 1$.

Observation 4.7.11 We can generalize what we have just found to any function of the form $\frac{1}{x^n}$, where $n > 0$. Since x^n increases without bound as $x \to \infty$, we find that $\frac{1}{x^n}$ will tend to 0. In fact, the numerator can be any constant and the function will still tend to 0!

Similarly, as $x \to -\infty$, we find that $\frac{1}{x^n}$ will tend to 0 too.

Properties and Graphs of Rational Functions (PR7)

Activity 4.7.12 Consider the rational function $r(x) = \dfrac{3x^2 - 5x + 1}{7x^2 + 2x - 11}$.

Observe that the largest power of x that's present in $r(x)$ is x^2. In addition, because of the dominant terms of $3x^2$ in the numerator and $7x^2$ in the denominator, both the numerator and denominator of r increase without bound as x increases without bound.

(a) In order to understand the end behavior of r, we will start by writing the function in a different algebraic form.

Multiply the numerator and denominator of r by $\dfrac{1}{x^2}$. Then distribute and simplify as much as possible in both the numerator and denominator to write r in a different algebraic form. Which of the following is that new form?

A. $\dfrac{3x^4 - 5x^3 + x^2}{7x^4 + 2x^3 - 11x^2}$

B. $\dfrac{3 - \dfrac{5}{x} + \dfrac{1}{x^2}}{7 + \dfrac{2}{x} - \dfrac{11}{x^2}}$

C. $\dfrac{\dfrac{3x^2}{x^2} - \dfrac{5x}{x^2} + \dfrac{1}{x^2}}{\dfrac{7x^2}{x^2} + \dfrac{2x}{x^2} - \dfrac{11}{x^2}}$

D. $\dfrac{3x^2 - 5x + 1}{7x^4 + 2x^3 - 11x^2}$

(b) Now determine the end behavior of each piece of the numerator and each piece of the denominator.

Hint. Use Observation 4.7.11 to help!

(c) Simplify your work from the previous step. Which of the following best describes the end behavior of $r(x)$?

A. As $x \to \pm\infty$, $r(x)$ goes to 0.

B. As $x \to \pm\infty$, $r(x)$ goes to $\dfrac{3}{7}$.

C. As $x \to \pm\infty$, $r(x)$ goes to ∞.

D. As $x \to \pm\infty$, $r(x)$ goes to $-\infty$.

Properties and Graphs of Rational Functions (PR7)

Observation 4.7.13 If the end behavior of a function tends toward a specific value a, then we say that the function has a **horizontal asymptote** at $y = a$.

Properties and Graphs of Rational Functions (PR7)

Activity 4.7.14 Find the horizontal asymptote (if one exists) of the following rational functions. Follow the same method we used in Activity 4.7.12.

(a) $f(x) = \dfrac{4x^3 - 3x^2 + 6}{9x^3 + 7x - 5}$

A. $y = 0$

B. $y = \dfrac{4}{9}$

C. $y = -\dfrac{3}{7}$

D. $y = -\dfrac{6}{5}$

E. There is no horizontal asymptote.

(b) $g(x) = \dfrac{4x^3 - 3x^2 + 6}{9x^5 + 7x - 5}$

A. $y = 0$

B. $y = \dfrac{4}{9}$

C. $y = -\dfrac{3}{7}$

D. $y = -\dfrac{6}{5}$

E. There is no horizontal asymptote.

(c) $h(x) = \dfrac{4x^5 - 3x^2 + 6}{9x^3 + 7x - 5}$

A. $y = 0$

B. $y = \dfrac{4}{9}$

C. $y = -\dfrac{3}{7}$

D. $y = -\dfrac{6}{5}$

E. There is no horizontal asymptote.

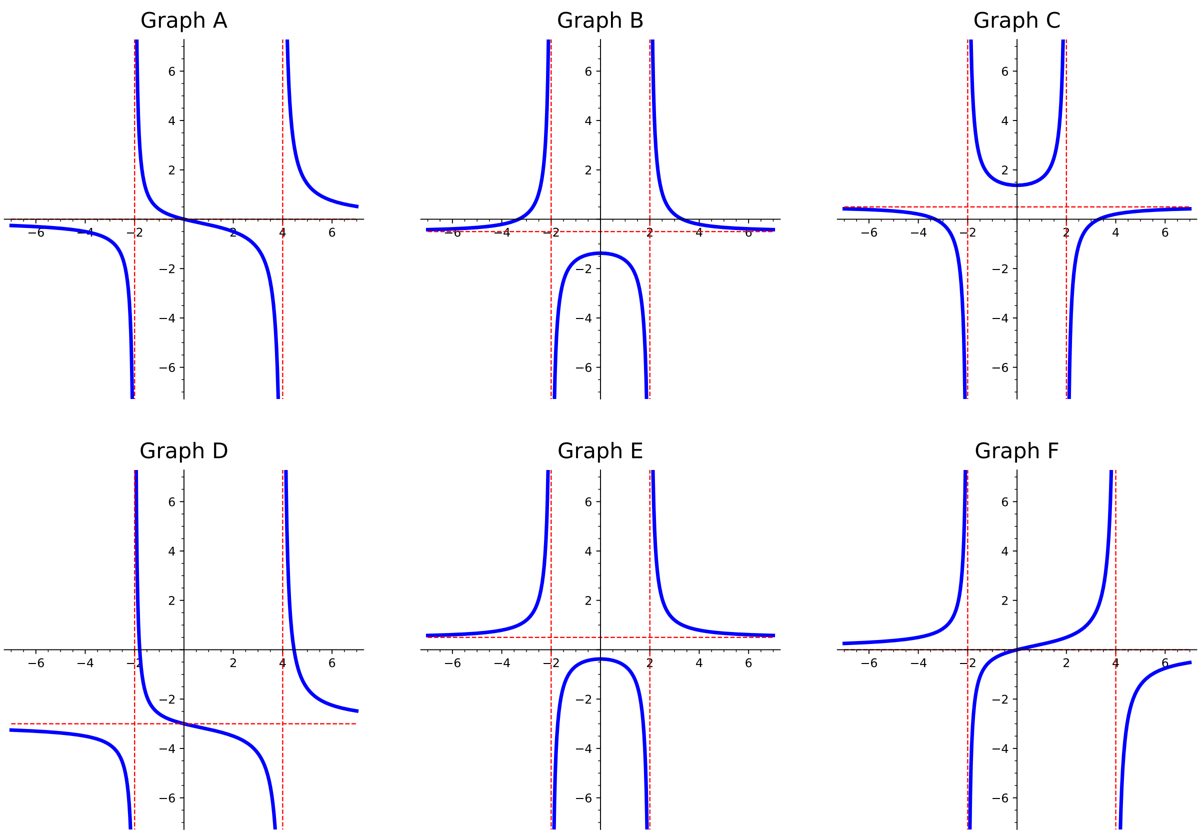
Activity 4.7.15 Some patterns have emerged from the previous problem. Fill in the rest of the sentences below to describe how to find horizontal asymptotes of rational functions.

(a) If the degree of the numerator is the same as the degree of the denominator, then...

(b) If the degree of the numerator is less than the degree of the denominator, then...

(c) If the degree of the numerator is greater than the degree of the denominator, then...

Activity 4.7.16 Consider the following six graphs of rational functions:

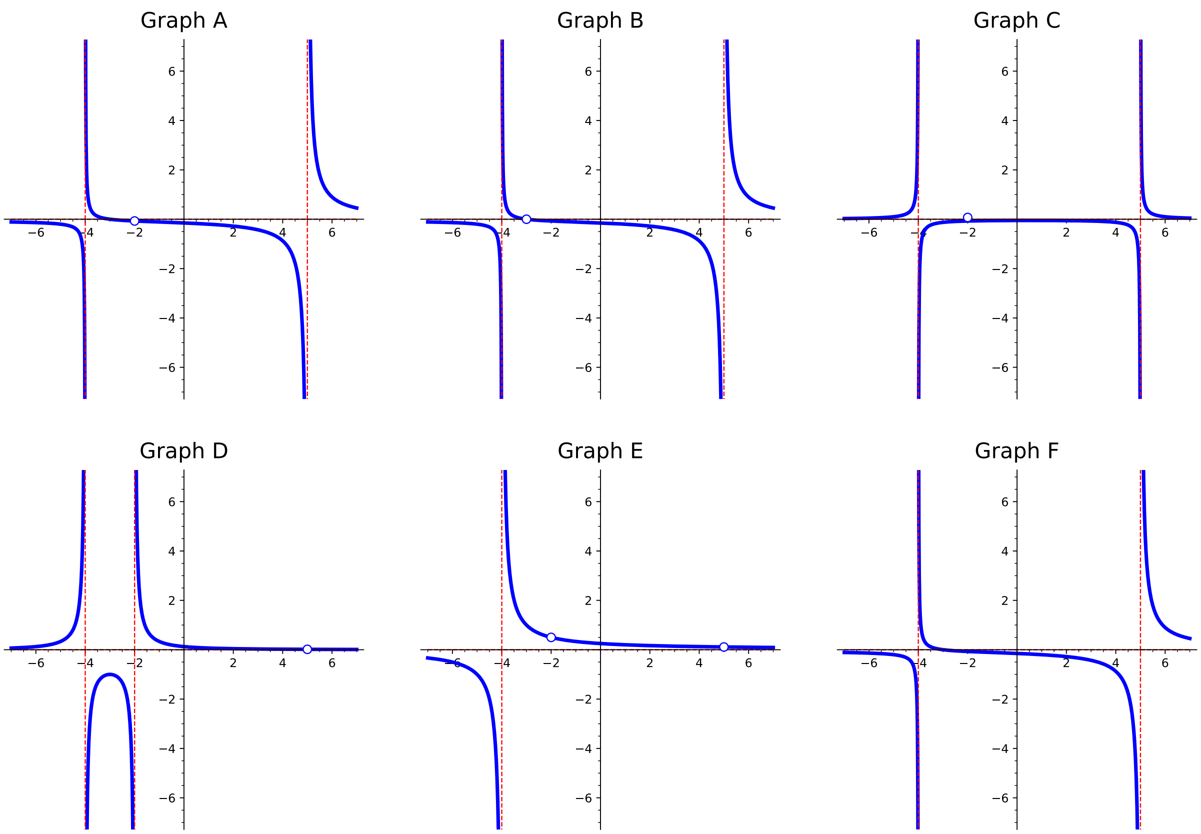


(a) Which of the graphs above represents the function $f(x) = \dfrac{2x}{x^2 - 2x - 8}$?

(b) Which of the graphs above represents the function $g(x) = \dfrac{x^2 + 3}{2x^2 - 8}$?

Properties and Graphs of Rational Functions (PR7)

Activity 4.7.17 Consider the following six graphs of rational functions:



(a) Which of the graphs above represents the function $f(x) = \dfrac{x^2+5x+6}{(x^2-3x-10)(x+4)}$?

(b) Which of the graphs above represents the function $g(x) = \dfrac{x^2-3x-10}{(x+2)(x^2-x-20)}$?

Properties and Graphs of Rational Functions (PR7)

Activity 4.7.18 Let $f(x) = \dfrac{-(x-1)(x-4)}{2\,(x+3)^2(x-1)}$.

(a) Find the roots of $f(x)$.

(b) Find the y-intercept of the graph of $f(x)$.

(c) Find any horizontal asymptotes on the graph of $f(x)$.

(d) Find any vertical asymptotes on the graph of $f(x)$.

(e) Find any holes on the graph of $f(x)$.

(f) Sketch the graph of $f(x)$.

Properties and Graphs of Rational Functions (PR7)

Activity 4.7.19 For each of the following rational functions, identify the location of any potential hole in the graph. Then, create a table of function values for input values near where the hole should be located. Use your work to decide whether or not the graph indeed has a hole, with written justification.

(a) $r(x) = \dfrac{x^2 - 16}{x + 4}$

(b) $s(x) = \dfrac{(x-2)^2(x+3)}{x^2 - 5x + 6}$

(c) $u(x) = \dfrac{(x-2)^3(x+3)}{x^2 - 5x + 6}$

(d) $w(x) = \dfrac{(x-2)(x+3)}{(x^2 - 5x + 6)^2}$

Activity 4.7.20 Suppose you are given a function $r(x) = \dfrac{p(x)}{q(x)}$, and you know that $p(3) = 0$ and $q(3) = 0$. What can you conclude about the function $r(x)$ at $x = 3$?

A. $r(x)$ has a hole at $x = 3$.

B. $r(x)$ has an asymptote at $x = 3$.

C. $r(x)$ has either a hole or an asymptote at $x = 3$.

D. $r(x)$ has neither a hole nor an asymptote at $x = 3$.

4.8 Quadratic Inequalities (PR8)

Objectives

- Solve quadratic inequalities and express the solution graphically and with interval notation.

Quadratic Inequalities (PR8)

Remark 4.8.1 In Section 1.5 we learned how to solve quadratic equations. In this section, we use these skills to solve quadratic *inequalities*.

Quadratic Inequalities (PR8)

Definition 4.8.2 A **quadratic inequality** is an inequality that can be written in one of the following forms:

- $$ax^2 + bx + c > 0$$
- $$ax^2 + bx + c < 0$$
- $$ax^2 + bx + c \geq 0$$
- $$ax^2 + bx + c \leq 0$$

where a, b, and c are real numbers and $a \neq 0$. $\Diamond$

Activity 4.8.3 Consider the graph shown below.

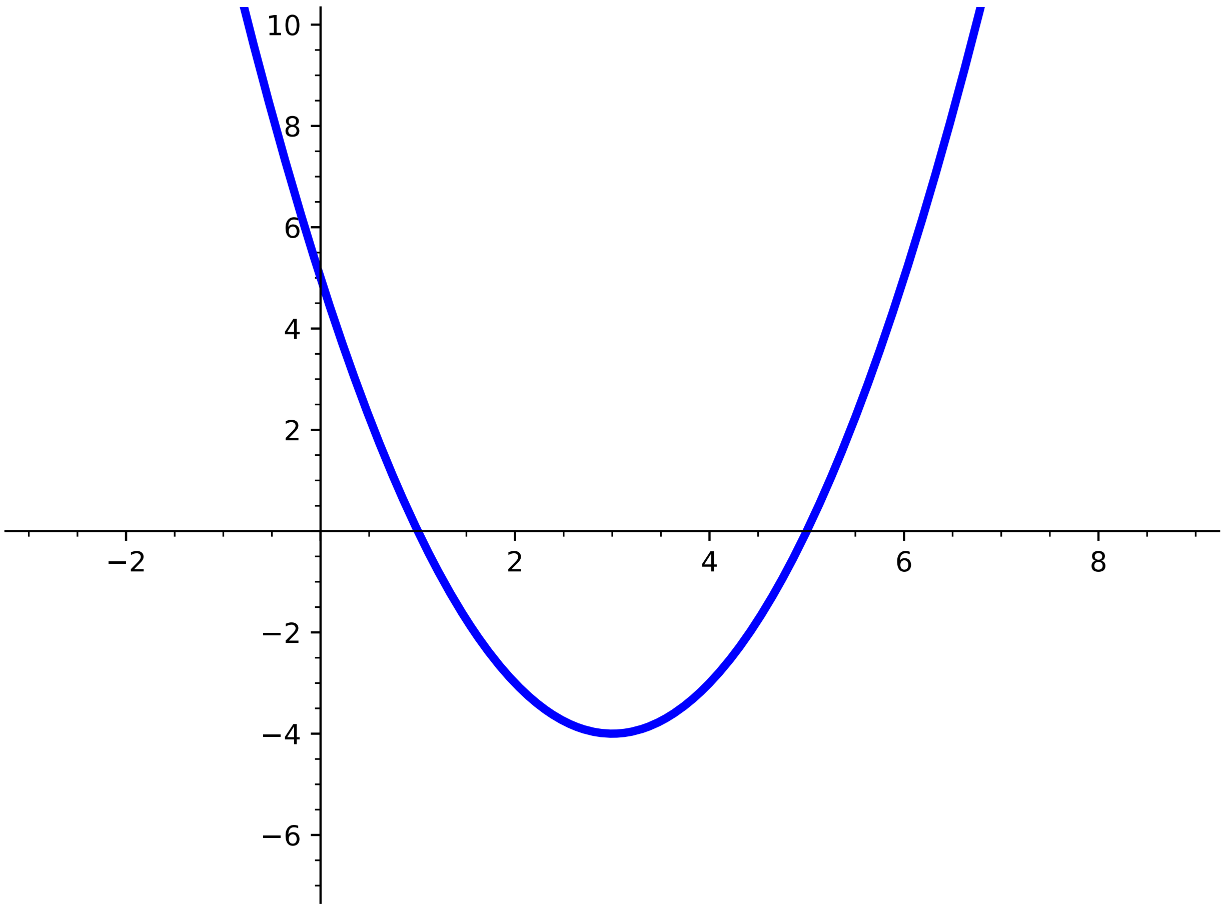


Figure 4.8.4

(a) What is the value of $f(0)$?

A. 1
B. -4
C. 5
D. 0

(b) What is the value of $f(3)$?

A. 1
B. -4
C. 5
D. 0

(c) What is the value of $f(5)$?

A. 1
B. -4
C. 5
D. 0

(d) What is the value of $f(6)$?

A. 1
B. -4
C. 5
D. 0

Quadratic Inequalities (PR8)

(e) Notice from parts (a) - (d), that $f(x)$ can either be positive, negative, or zero depending on the value of x. What are the x-intercept(s) of $f(x)$?

A. 1

B. -4

C. 5

D. 0

(f) Based on what you see on the graph (and your solutions to parts (a) - (d)), for what values of x would $f(x)$ be positive?

A. $x < 1$

B. $x > 1$

C. $x > 5$

D. $x < 5$

E. $1 < x < 5$

(g) Now use interval notation to express where $x^2 - 6x + 5 > 0$.

A. $(\infty, 1] \cup [5, \infty)$

B. $[1, 5]$

C. $(\infty, 1) \cup (5, \infty)$

D. $(1, 5)$

Remark 4.8.5 From Activity 4.8.3, we saw that a function could have y-values that are positive, negative, or zero, which can then help us find values of x to solve inequalities. Let's now look at how we can solve inequalities using algebra.

Quadratic Inequalities (PR8)

Activity 4.8.6 Use Definition 4.8.2 and your knowledge of quadratic equations (see Section 1.5) to help answer the following questions.

(a) Which of the following inequalities are quadratic inequalities?

A. $(x-1)(x+3) < 7$

B. $-4x + 3 \geq 10$

C. $2x^2 - 7x + 3 > 0$

D. $5x - 1 \leq 4x$

(b) Given the quadratic equation, $x^2 - x - 6 = 0$, determine whether -1 is a solution.

(c) Given the quadratic inequality, $x^2 - x - 6 \leq 0$, determine whether -1 is a solution.

(d) Are there any other values of x that would satisfy the inequality, $x^2 - x - 6 \leq 0$?

Remark 4.8.7 Notice from Activity 4.8.6, a solution to a quadratic inequality is a real number that will produce a true statement when substituted for the variable. Quadratic inequalities will often have an infinite number of solutions, which we will express in interval notation. However, it is also possible for the inequality to have no solution.

Quadratic Inequalities (PR8)

Activity 4.8.8 Let's look at how we can algebraically determine the solutions to a quadratic inequality using a number line. Consider the quadratic inequality

$$x^2 - 4x - 32 \geq 0.$$

(a) What is the factored form of the quadratic (left-hand side of the inequality)?

A. $(x-16)(x+2)$

B. $(x-4)(x+8)$

C. $(x+4)(x-8)$

D. $(x-4)(x-8)$

(b) Rewrite the quadratic inequality with the answer you got from part (a). What values of x would give you 0?

A. $x = 16, -2$

B. $x = 4, -8$

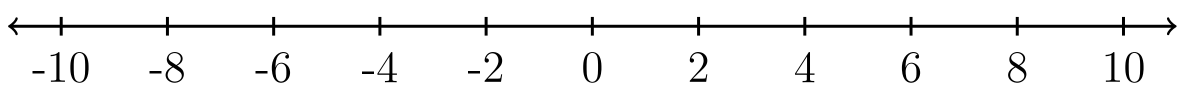
C. $x = 4, 8$

D. $x = -4, 8$

Hint. Refer back to Section 1.5 and think about this as a quadratic equation (equal to 0).

(c) These solutions (that you got in part (b)) correspond to all of the x-intercepts of the graph, and are the only spots where the y-values on either side might change from positive to negative or negative to positive. So, with our two x-intercepts, we have divided our graph into three intervals. What are these three intervals?

(d) Notice that the x-intercepts are solutions to the quadratic equation $x^2 - 4x - 32 = 0$. How should we mark these values of x on a number line?



(e) Choose a value of x within each interval and substitute that value into the equation $x^2 - 4x - 32$ (or the factored form you found in part (a)). If you get a positive value, place an "+" sign above that region on the number line. Similarly, if you get a negative value, place a "-" sign above that region on the number line.

(f) Using the number line and what you determined in part (e), shade in areas on the number line that satisfies the quadratic inequality $x^2 - 4x - 32 \geq 0$?

(g) Using your graph, express the solution of the inequality $x^2 - 4x - 32 \geq 0$ in interval notation.

A. $(-4, 8)$

B. $[-4, 8]$

C. $(-\infty, -4) \cup (8, \infty)$

D. $(-\infty, -4] \cup [8, \infty)$

Remark 4.8.9 Notice in Activity 4.8.8, we had to begin with the solutions to the quadratic equation to determine regions of the number line to then test values of x that satisfy the inequality that was given. Creating a visual can help in determining the solutions to inequalities, especially when there are many solutions.

Definition 4.8.10 A **sign chart** is a number line representing the x-axis that shows where a function is positive or negative by using a '+' or a '-' sign to indicate which regions are positive or negative. For example, a sign chart for $f(x) = x^2 - 4x - 32$ (also seen in Activity 4.8.8) is shown below.

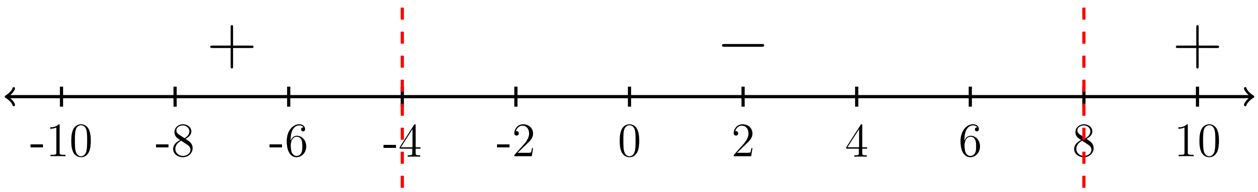


Figure 4.8.11 A sign chart for the function $f(x) = x^2 - 4x - 32$.

◊

Quadratic Inequalities (PR8)

Definition 4.8.12 In Activity 4.8.8, we saw that when we place the solutions to the quadratic equation, $x^2 - 4x - 32 = 0$, on the number line, that it divided the number line into three regions.

The values in the domain of a function that separate regions that produce positive or negative results are called **critical points** or **boundary points**. These values bound the regions where the function is positive or negative. ◊

Quadratic Inequalities (PR8)

Activity 4.8.13 For this activity, consider the quadratic inequality

$$2x^2 - 28 < 10x.$$

(a) Factor the quadratic equation $2x^2 - 28 = 10x$ using methods discussed in Section 1.5.

A. $(x-2)(x+7)$
B. $(2x+4)(x-7)$
C. $(x+2)(x-7)$
D. $2(x^2-14)$

(b) Use the factors you found in part (a) to find the critical points.

A. $x = -2, 7$
B. $x = \sqrt{14}$
C. $x = -7, 2$
D. $x = 2, -12$

(c) Plot the critical points you found in part (b) on a number line. Determine whether those should be included in your solution based on the inequality that was given.

(d) Test values of x in each region that was created by the critical points and create a sign chart to show which regions are positive and negative.

(e) Using the number line and what you determined in part (d), shade in areas on the number line that satisfies the quadratic inequality $2x^2 - 28 < 10x$?

(f) Write your solution using interval notation.

A. $(-\infty, -2) \cup (7, \infty)$
B. $(-\infty, -7) \cup (2, \infty)$
C. $(-7, 2)$
D. $(-2, 7)$

Quadratic Inequalities (PR8)

Remark 4.8.14 When solving quadratic inequalities, be sure to get all your terms to one side of the inequality first! Then, apply the methods we learned in Section 1.5 to determine the critical points (boundary points). From there, you can then create your sign chart to help determine the solution to the inequality.

Quadratic Inequalities (PR8)

Activity 4.8.15 For each of the following, determine the critical points and use a number line (and sign chart) to then find the solutions. Write your answers in interval notation.

(a) $x^2 - 1 < 0$

(b) $x^2 - 5x - 6 \geq 0$

(c) $2x^2 - 7x + 3 > 0$

(d) $4x^2 - 6x - 9 < x^2$

(e) $-2x^2 - 10x - 10 \geq 6x + 20$

4.9 Rational Inequalities (PR9)

Objectives

- Solve rational inequalities and express the solution graphically and using interval notation.

Rational Inequalities (PR9)

Remark 4.9.1 In Section 4.6 we learned how to solve rational equations. In this section, we use these skills to solve rational *inequalities*.

Rational Inequalities (PR9)

Definition 4.9.2 A **rational inequality** is an inequality that contains a rational expression. Some examples of a rational inequality are shown below:

$$\frac{3}{2x} > 1$$
$$\frac{2x-3}{x-6} \leq x$$

◊

Activity 4.9.3 Use Definition 4.9.2 and your knowledge of quadratic equations (see Section 4.6) to help answer the following questions.

(a) Which of the following inequalities are rational inequalities?

A. $\dfrac{4x+7}{5} < -1$

B. $\dfrac{4x-1}{3x+2} \geq 0$

C. $2x^2 - 7x > -3$

D. $5x - 1 \leq 4x$

(b) Given the rational equation, $\frac{x-1}{x+3} = 0$, determine whether 1 is a solution.

(c) Given the rational equation, $\frac{x-1}{x+3} = 0$, determine whether -4 is a solution.

(d) Given the rational inequality, $\frac{x-1}{x+3} \geq 0$, determine whether -4 is a solution.

(e) Given the rational equation, $\frac{x-1}{x+3} = 0$, determine whether -3 is a solution.

(f) Are there any other values of x that would satisfy the inequality, $\frac{x-1}{x+3} \geq 0$?

Remark 4.9.4 In Activity 4.9.3, we saw that there can be solutions to the rational inequality $\frac{x-1}{x+3} \geq 0$ that does NOT satisfy the rational equation $\frac{x-1}{x+3} = 0$. Just like with linear and quadratic inequalities, we can have many solutions that can satisfy rational inequalities. We do, however, need to be careful about our critical points, as we will see in the next activity.

Rational Inequalities (PR9)

Activity 4.9.5 For this activity, let's consider the inequality

$$\frac{x-1}{x+3} \geq 0.$$

(a) Focus on the rational equation $\frac{x-1}{x+3}$ (the left-hand side of the inequality). What value(s) should be excluded as possible solutions to that rational expression?

A. 1

B. -1

C. -3

D. 3

(b) The value you got in part (a) is a critical value for this rational expression because it is a value of x where the rational expression is undefined. Critical points for rational expressions also include values that make the rational expression equal to 0. What value of x would make this rational expression equal to 0?

A. 1

B. -1

C. -3

D. 3

(c) Use the critical points you found in parts (a) and (b) and plot them on a number line. Determine whether those should be included in your solution based on the inequality that was given.

(d) Notice that your number line has now been divided into three regions. Test values of x in each region that was created by the critical points and create a sign chart to show which regions are positive and negative.

Hint. Make sure you substitute values into BOTH xs in the rational expression.

(e) Using the number line and what you determined in part (d), shade in areas on the number line that satisfies the quadratic inequality $\frac{x-1}{x+3} \geq 0$?

(f) Write your solution using interval notation.

A. $(-\infty, -3) \cup (1, \infty)$

B. $(-\infty, -3) \cup [1, \infty)$

C. $(-3, 1)$

D. $(-3, 1)$

Rational Inequalities (PR9)

Remark 4.9.6 When looking for critical points of rational inequalities, remember to look for points where the rational expression will be zero or undefined.

Rational Inequalities (PR9)

Activity 4.9.7 For each of the following, determine the critical points and use a number line (and sign chart) to then find the solutions. Write your answers in interval notation.

(a) $\frac{x+5}{x-4} \leq 0$

(b) $\frac{(x+3)(x+5)}{x+2} > 0$

Hint. Notice for this rational inequality there are 3 critical points! Be sure to test all four regions on your number line.

Rational Inequalities (PR9)

Activity 4.9.8 When solving an inequality, the goal is to first get x (or whatever the variable is) on its own on one side of the inequality sign and 0 on the other side. To do this, we have to be careful of the actions we take as some actions can change the direction of the inequality. Let's revisit some of the actions we have taken previously and see how we can apply these same actions to solve the rational inequality

$$\frac{3x-10}{x-4} > 2.$$

(a) When solving rational equations in Section 4.6, we often started by "clearing the fractions" by multiplying the denominator to both sides of the equation. Suppose we have the equation $\frac{3x-10}{x-4} = 2$. What can we multiply each term by that will clear the fraction?

A. $x+4$

B. $3x-10$

C. $x-4$

D. $x-2$

(b) Multiply each term by the expression you chose and simplify. Which of the following linear equations does the rational equation simplify to?

A. $(x+4)(3x-10) = 2(x+4)$

B. $(3x-10) = 2(x-4)$

C. $2(3x-10) = (x-4)$

D. $(3x-10) = 2(x+4)$

(c) What values of x would make the denominator of the rational expression positive?

(d) What values of x would make the denominator of the rational expression negative?

(e) Now suppose we applied the same actions as we did in parts (a) and (b) to the rational inequality $\frac{3x-10}{x-4} > 2$. If the denominator of the rational expression $\frac{3x-10}{x-4}$ was positive, how would you write the inequality as a linear inequality?

A. $(x+4)(3x-10) > 2(x+4)$

B. $(3x-10) > 2(x-4)$

C. $2(3x-10) > (x-4)$

D. $(3x-10) > (x+4)$

(f) If the denominator of the rational expression $\frac{3x-10}{x-4}$ was negative, how would you write the inequality as a linear inequality?

A. $(x+4)(3x-10) < 2(x+4)$

B. $(3x-10) > 2(x-4)$

C. $2(3x-10) < (x-4)$

D. $(3x-10) < 2(x-4)$

Rational Inequalities (PR9)

Hint. What happens to the inequality symbol when multiplying or dividing by a negative number?

(g) Look at your answers in parts (e) and (f). Which inequality would help us solve $\frac{3x-10}{x-4} > 2$.

Remark 4.9.9 In Activity 4.9.9, we saw that it is not clear whether the denominator of a rational expression would yield a positive or negative value. Because we do not actually know whether the denominator is positive or negative, we cannot multiply the denominator to ”clear the fractions” as we did before when solving rational equations. In the next activity, we will look at how we can solve rational inequalities.

Rational Inequalities (PR9)

Activity 4.9.10 Let's reconsider the rational inequality

$$\frac{3x-10}{x-4} > 2.$$

(a) From Activity 4.9.9, we saw that we cannot multiply by the denominator to "clear the fraction." Our goal, however, is still to get all the xs to one side and 0 on the other side. What action can we take to get 0 on one side?

A. Add 2 to each side

B. Subtract 2 from each side

C. Multiply by 2 on each side

D. Multiply by 4 on each side

(b) What rational inequality do you now have?

A. $\frac{3x-10}{x-4} + 2 < 0$

B. $\frac{3x-10}{x-4} - 2 < 0$

C. $\frac{3x-10}{x-4} + 2 > 0$

D. $\frac{3x-10}{x-4} - 2 > 0$

(c) What is the common denominator of $\frac{3x-10}{x-4}$ and 2?

A. $x+4$

B. $x-4$

C. $3x-10$

D. $3x+10$

(d) Multiply 2 by the common denominator and simplify the rational expression. What rational inequality do you now have?

A. $\frac{x-18}{x-4} > 0$

B. $\frac{x-2}{x-4} > 0$

C. $\frac{x-2}{x-4} > 0$

D. $\frac{2x-6}{x-4} > 0$

(e) Use the inequality you got in part (d) to determine the critical points.

Hint. Remember that for a rational inequality, the critical points are the values of x that make the rational expression equal to 0 or undefined.

(f) Use the critical points you found in part (e) and plot them on a number line. Determine whether those should be included in your solution based on the inequality that was given.

(g) Use the critical points to create regions on the number line for you to test values of x and create a sign chart to show which regions are positive and negative.

(h) Using the number line and what you determined in part (d), shade in areas on the number line that satisfies the rational inequality $\frac{3x-10}{x-4} > 2$?

Rational Inequalities (PR9)

(i) Write your solution using interval notation.

A. $(-\infty, 2) \cup (4, \infty)$

B. $(-\infty, 2] \cup [4, \infty)$

C. $(2, 4)$

D. $[2, 4]$

Rational Inequalities (PR9)

Activity 4.9.11 Consider the rational inequality

$$\frac{4x+3}{x+2} > x.$$

(a) Subtract x on both sides of the inequality to get 0 on one side. Simplify $\frac{4x+3}{x+2} - x$ into a single rational expression.

A. $\frac{4x+3}{x+2}$

B. $\frac{3x+3}{x+2}$

C. $\frac{x^2+6x+3}{x+2}$

D. $\frac{-x^2+2x+3}{x+2}$

(b) What are the critical points of this rational inequality?

Hint. Factor the numerator.

(c) Use the critical points you found in part (b) and plot them on a number line. Determine whether those should be included in your solution based on the inequality that was given.

(d) Use the critical points to create regions on the number line for you to test values of x and create a sign chart to show which regions are positive and negative.

(e) Using the number line and what you determined in part (d), shade in areas on the number line that satisfies the rational inequality $\frac{4x+3}{x+2} > x$?

(f) How can we express the answers to part (e) for the rational inequality using interval notation?

A. $(1, 3)$

B. $(-2, -1) \cup (3, \infty)$

C. $(-2, -1)$

D. $(-\infty, -2) \cup (-1, 3)$

Rational Inequalities (PR9)

Activity 4.9.12 For each of the following, solve the rational inequality by bringing all x terms to one side (and 0 on the other) and simplify the rational expression. Then, use critical points and the sign chart/number line to determine the solution.

(a) $\frac{x+68}{x+8} \geq 5$

(b) $\frac{x^2-4x+4}{x-2} > 1$

(c) $\frac{x-8}{x} \leq 3 - x$

(d) $\dfrac{x+8}{x-2} \leq \dfrac{x+10}{x+5}$

Chapter 5

Exponential and Logarithmic Functions (EL)

Objectives

How do we model exponential growth?
By the end of this chapter, you should be able to...

1. Determine if a given function is exponential. Find an equation of an exponential function. Evaluate exponential functions (including base e).
2. Graph exponential functions and determine the domain, range, and asymptotes.
3. Convert between exponential and logarithmic form. Evaluate a logarithmic function, including common and natural logarithms.
4. Graph logarithmic functions and determine the domain, range, and asymptotes.
5. Use properties of logarithms to condense or expand logarithmic expressions.
6. Solve exponential and logarithmic equations.
7. Solve application problems using exponential and logarithmic equations.

5.1 Introduction to Exponentials (EL1)

Objectives

- Determine if a given function is exponential. Find an equation of an exponential function. Evaluate exponential functions (including base e).

Introduction to Exponentials (EL1)

Remark 5.1.1 Linear functions have a constant rate of change - that is a constant change in output for every change in input. Let's consider functions which do not fit this model - those which grow more rapidly and change by a varying amount for every change in input.

Introduction to Exponentials (EL1)

Activity 5.1.2 You have two job offers on the horizon. One has offered to pay you \$10,000 per month while the other is offering \$0.01 the first month, \$0.02 the second month, \$0.04 the third month and doubles every month. Which job would you rather take?

(a) Make a table representing how much money you will be paid each month for the first two years from the first job - paying \$10,000 per month.

(b) Make a table representing how much money you will be paid each month for the first two years from the second job - paying \$0.01 the first month and doubling every month after.

(c) Which job is earning more money per month after one year?

(d) Which job is earning more money per month after 18 months?

(e) According to your tables, does the second job ever earn more money per month than the first job?

Remark 5.1.3 This idea of a function that grows very rapidly by a factor, ratio, or percent each time, like the second job in Activity 5.1.2, is considered exponential growth.

Introduction to Exponentials (EL1)

Definition 5.1.4 Let a be a non-zero real number and $b \neq 1$ a positive real number. An **exponential function** takes the form

$$f(x) = ab^x$$

where a is the initial value and b is the base. $\Diamond$

Introduction to Exponentials (EL1)

Activity 5.1.5 Evaluate the following exponential functions.

(a) $f(x) = 4^x$ for $f(3)$

(b) $f(x) = \left(\frac{1}{3}\right)^x$ for $f(3)$

(c) $f(x) = 3 \cdot (5)^x$ for $f(-2)$

(d) $f(x) = -2^{3x-4}$ for $f(4)$

Remark 5.1.6 Notice that in Activity 5.1.5 part (a) the output value is larger than the base, representing an increasing function, while in part (b) the output value is smaller than the base.

Introduction to Exponentials (EL1)

Activity 5.1.7 Consider two exponential functions $f(x) = 100(2)^x$ and $g(x) = 100\left(\frac{1}{2}\right)^x$.

(a) Fill in the table of values for $f(x)$.

x	$f(x)$
0	
1	
2	
3	
4	

(b) Fill in the table of values for $g(x)$.

x	$g(x)$
0	
1	
2	
3	
4	

(c) How do the values in the tables compare?

Remark 5.1.8 In Activity 5.1.7, the only difference between the two exponential functions was the base. $f(x)$ has a base of 2, while $g(x)$ has a base of $\frac{1}{2}$. Let's use this fact to update Definition 5.1.4.

Remark 5.1.9 An exponential function of the form $f(x) = ab^x$ will grow (or increase) if $b > 1$ and decay (or decrease) if $0 < b < 1$.

Introduction to Exponentials (EL1)

Activity 5.1.10 For each year t, the population of a certain type of tree in a forest is represented by the function $F(t) = 856 \cdot (0.93)^t$.

(a) How many of that certain type of tree are in the forest initially?

(b) Is the number of trees of that type growing or decaying?

Activity 5.1.11 To begin creating equations for exponential functions using a and b, let's compare a linear function and an exponential function. The tables show outputs for two different functions r and s that correspond to equally spaced input.

x	$r(x)$
0	12
3	10
6	8
9	6

x	$s(x)$
0	12
3	9
6	6.75
9	5.0625

(a) Which function is linear?

(b) What is the initial value of the linear function?

(c) What is the slope of the linear function?

(d) What is the initial value of the exponential function?

(e) What is the ratio of consecutive outputs in the exponential function?

A. $\frac{4}{3}$

B. $\frac{3}{4}$

C. $-\frac{4}{3}$

D. $-\frac{3}{4}$

Remark 5.1.12 In a linear function the differences are constant, while in an exponential function the ratios are constant.

Introduction to Exponentials (EL1)

Activity 5.1.13 Find an equation for an exponential function passing through the points $(0, 4)$ and $(1, 6)$.

(a) Find the initial value.

A. 0

B. 4

C. 1

D. 6

(b) Find the common ratio.

A. $\frac{3}{2}$

B. 6

C. $\frac{2}{3}$

D. $\frac{1}{6}$

(c) Find the equation.

A. $f(x) = 6\left(\frac{3}{2}\right)^x$

B. $f(x) = 4 \cdot (6)^x$

C. $f(x) = 4\left(\frac{2}{3}\right)^x$

D. $f(x) = 4 \cdot \left(\frac{3}{2}\right)^x$

Remark 5.1.14 Recall the negative rule of exponents which states that for any nonzero real number a and natural number n

$$a^{-n} = \frac{1}{a^n}$$

Introduction to Exponentials (EL1)

Activity 5.1.15 Let's consider the two exponential functions $f(x) = 2^{-x}$ and $g(x) = \left(\frac{1}{2}\right)^x$.

(a) Fill in the table of values for $f(x)$.

x	$f(x)$
-2	
-1	
0	
1	
2	

(b) Fill in the table of values for $g(x)$.

x	$g(x)$
-2	
-1	
0	
1	
2	

(c) What do you notice about the two functions?

(d) Use Remark 5.1.14 and other properties of exponents to try and rewrite $f(x)$ as $g(x)$.

Remark 5.1.16 Similar to how π arises naturally in geometry, there is an irrational number called e that arises naturally when working with exponentials. We usually use the approximation $e \approx 2.718282$. e is also found on most scientific and graphing calculators.

See Activity 5.7.11 for an exploration of how e arises naturally.

Introduction to Exponentials (EL1)

Activity 5.1.17 Use a calculator to evaluate the following exponentials involving the base e.

(a) $f(x) = -2e^x - 2$ for $f(-2)$

A. -0.0366

B. -2.2707

C. -1.7293

D. -16.778

(b) $f(x) = \frac{1}{3}e^{x+1}$ for $f(-1)$

A. 1

B. 0

C. 1.122

D. $\frac{1}{3}$

5.2 Graphs of Exponential Functions (EL2)

Objectives

- Graph exponential functions and determine the domain, range, and asymptotes.

Graphs of Exponential Functions (EL2)

Activity 5.2.1 Consider the function $f(x) = 2^x$.

(a) Fill in the table of values for $f(x)$. Then plot the points on a graph.

x	$f(x)$
-2	
-1	
0	
1	
2	

(b) What seems to be happening with the graph as x goes toward infinity? Plug in large positive values of x to test your guess, then describe the end behavior.

A. As $x \to \infty$, $f(x) \to -\infty$.

B. As $x \to \infty$, $f(x) \to -2$.

C. As $x \to \infty$, $f(x) \to 0$.

D. As $x \to \infty$, $f(x) \to 2$.

E. As $x \to \infty$, $f(x) \to \infty$.

(c) What seems to be happening with the graph as x goes toward negative infinity? Plug in large negative values of x to test your guess, then describe the end behavior.

A. As $x \to -\infty$, $f(x) \to -\infty$.

B. As $x \to -\infty$, $f(x) \to -2$.

C. As $x \to -\infty$, $f(x) \to 0$.

D. As $x \to -\infty$, $f(x) \to 2$.

E. As $x \to -\infty$, $f(x) \to \infty$.

(d) Complete the graph you started in Task 5.2.1.a, connecting the points and including the end behavior you've just determined.

(e) Does your graph seem to have any asymptotes?

A. No. There are no asymptotes.

B. There is a vertical asymptote but no horizontal one.

C. There is a horizontal asymptote but no vertical one.

D. The graph has both a horizontal and vertical asymptote.

(f) What the equation for each asymptote of $f(x)$? Select all that apply.

A. There are no asymptotes.

B. $x = 0$

C. $x = 3$

D. $y = 0$

E. $y = 3$

(g) Find the domain and range of $f(x)$. Write your answers using interval notation.

(h) Find the interval(s) where $f(x)$ is increasing and the interval(s) where $f(x)$ is decreasing. Write your answers using interval notation.

Graphs of Exponential Functions (EL2)

Remark 5.2.2 The graph of an exponential function $f(x) = b^x$ where $b > 1$ has the following characteristics:

- Its domain is $(-\infty, \infty)$ and its range is $(0, \infty)$.
- It is an exponential growth function; that is it is increasing on $(-\infty, \infty)$.
- There is a horizontal asymptote at $y = 0$. There is no vertical asymptote.
- There is a y-intercept at $(0, 1)$. There is no x-intercept.

Activity 5.2.3 Consider the function $g(x) = \left(\frac{1}{2}\right)^x$.

(a) Fill in the table of values for $g(x)$. Then plot the points on a graph.

x	$g(x)$
-2	
-1	
0	
1	
2	

(b) What seems to be happening with the graph as x goes toward infinity? Plug in large positive values of x to test your guess, then describe the end behavior.

A. As $x \to \infty$, $g(x) \to -\infty$.
B. As $x \to \infty$, $g(x) \to -2$.
C. As $x \to \infty$, $g(x) \to 0$.
D. As $x \to \infty$, $g(x) \to 2$.
E. As $x \to \infty$, $g(x) \to \infty$.

(c) What seems to be happening with the graph as x goes toward negative infinity? Plug in large negative values of x to test your guess, then describe the end behavior.

A. As $x \to -\infty$, $g(x) \to -\infty$.
B. As $x \to -\infty$, $g(x) \to -2$.
C. As $x \to -\infty$, $g(x) \to 0$.
D. As $x \to -\infty$, $g(x) \to 2$.
E. As $x \to -\infty$, $g(x) \to \infty$.

(d) Complete the graph you started in Task 5.2.3.a, connecting the points and including the end behavior you've just determined.

(e) What are the equations of the asymptote(s) of the graph?

(f) Find the domain and range of $f(x)$. Write your answers using interval notation.

(g) Find the interval(s) where $f(x)$ is increasing and the interval(s) where $f(x)$ is decreasing. Write your answers using interval notation.

Graphs of Exponential Functions (EL2)

Activity 5.2.4 Consider the two exponential functions we've just graphed: $f(x) = 2^x$ and $g(x) = \left(\frac{1}{2}\right)^x$.

(a) How are the graphs of $f(x)$ and $g(x)$ similar?

(b) How are the graphs of $f(x)$ and $g(x)$ different?

Graphs of Exponential Functions (EL2)

Remark 5.2.5 We can now update Remark 5.2.2 so that it includes all values of the base of an exponential function.

The graph of an exponential function $f(x) = b^x$ has the following characteristics:

- Its domain is $(-\infty, \infty)$ and its range is $(0, \infty)$.
- Remember, exponential functions are only defined when $b > 0$ and $b \neq 1$ as we saw in Definition 5.1.4.

  If $b > 1$, $f(x)$ is increasing on $(-\infty, \infty)$ and is an exponential growth function. If $0 < b < 1$, $f(x)$ is decreasing on $(-\infty, \infty)$ and is an exponential decay function.
- There is a horizontal asymptote at $y = 0$. There is no vertical asymptote.
- There is a y-intercept at $(0, 1)$. There is no x-intercept.

Graphs of Exponential Functions (EL2)

Activity 5.2.6 Let's look at a third exponential function, $h(x) = 2^{-x}$.

(a) Before plotting any points or graphing, what do you think the graph might look like? What sort of characteristics might it have?

(b) Fill in the table of values for $h(x)$. Then plot the points on a graph.

x	$h(x)$
-2	
-1	
0	
1	
2	

(c) This function $h(x)$ looks to be the same as a function we looked at previously. Use properties of exponents to rewrite $h(x)$ in a different way.

(d) In addition to plotting points, we can use transformations to graph. If we consider $f(x) = 2^x$ to be the parent function, what transformation is needed to graph $h(x) = 2^{-x}$?

A. A vertical stretch.

B. A horizontal stretch.

C. A reflection over the x-axis.

D. A reflection over the y-axis.

Graphs of Exponential Functions (EL2)

Remark 5.2.7 For a reminder of transformations, see Section 2.4 and the following definitions:

- Definition 2.4.5
- Definition 2.4.9
- Definition 2.4.15
- Definition 2.4.16
- Definition 2.4.24
- Definition 2.4.25

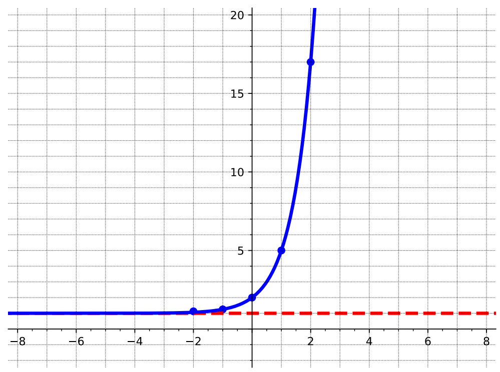
Graphs of Exponential Functions (EL2)

Activity 5.2.8 Let $f(x) = 4^x$.

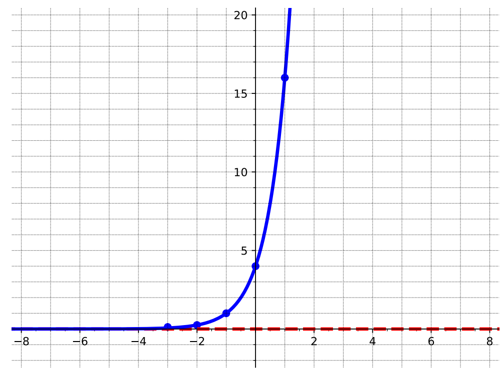
(a) Graph $f(x)$.

(b) Match the following functions to their graphs.

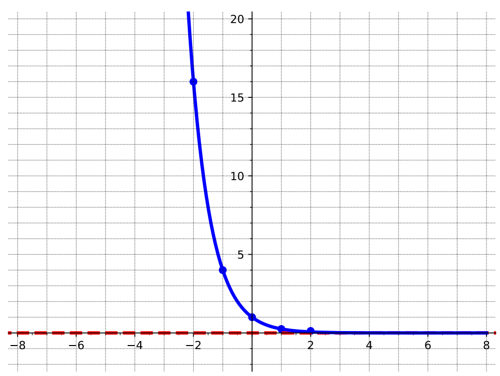
- $g(x) = -4^x$
- $h(x) = 4^{-x}$
- $j(x) = 4^{x+1}$
- $k(x) = 4^x + 1$



A.



C.



B.

D.

(c) Find the domain, range, and equation of the asymptote for the parent function $(f(x))$ and each of the four transformations $(g(x), h(x), j(x), \text{ and } k(x))$.

(d) Which of the transformations affected the domain of the exponential function? Select all that apply.

A. A vertical shift.

B. A horizontal shift.

C. A reflection over the x-axis.

D. A reflection over the y-axis.

E. None of these.

(e) Which of the transformations affected the range of the exponential function? Select all that apply.

A. A vertical shift.

B. A horizontal shift.

C. A reflection over the x-axis.

D. A reflection over the y-axis.

E. None of these.

(f) Which of the transformations affected the asymptote of the exponential function? Select all that apply.

A. A vertical shift.

B. A horizontal shift.

C. A reflection over the x-axis.

D. A reflection over the y-axis.

E. None of these.

Graphs of Exponential Functions (EL2)

Activity 5.2.9 Consider the function $f(x) = e^x$.

(a) Graph $f(x) = e^x$. First find $f(0)$ and $f(1)$. Then use what you know about the characteristics of exponential graphs to sketch the rest. Then state the domain, range, and equation of the asymptote. (Recall that $e \approx 2.72$ to help estimate where to put your points.)

(b) Sketch the graph of $g(x) = e^{x-2}$ using transformations. State the transformation(s) used, the domain, the range, and the equation of the asymptote.

(c) Sketch the graph of $h(x) = -3e^x$ using transformations. State the transformation(s) used, the domain, the range, and the equation of the asymptote.

(d) Sketch the graph of $g(x) = e^{-x} - 4$ using transformations. State the transformation(s) used, the domain, the range, and the equation of the asymptote.

Graphs of Exponential Functions (EL2)

Activity 5.2.10 Graph each of the following exponential functions. Include any asymptotes with a dotted line. State the domain, the range, the equation of the asymptote, and whether it is growth or decay.

(a) $f(x) = 3^x$

(b) $f(x) = 6^{-x}$

(c) $f(x) = \frac{1}{5}^{x-2}$

(d) $f(x) = \frac{1}{3}^{x} + 4$

5.3 Introduction to Logarithms (EL3)

Objectives

- Convert between exponential and logarithmic form. Evaluate a logarithmic function, including common and natural logarithms.

Introduction to Logarithms (EL3)

Activity 5.3.1 Let $P(t)$ be the function given by $P(t) = 10^t$.

(a) Fill in the table of values for $P(t)$.

t	$y = P(t)$
-3	
-2	
-1	
0	
1	
2	
3	

(b) Do you think P will have an inverse function? Why or why not?

(c) Since P has an inverse function, we know there exists some other function, say L, such that $y = P(t)$ represent the same relationship between t and y as $t = L(y)$. In words, this means that L reverses the process of raising to the power of 10, telling us the *power* to which we need to raise 10 to produce a desired result. Fill in the table of values for $L(y)$.

y	$L(y)$
10^{-3}	
10^{-2}	
10^{-1}	
10^{0}	
10^{1}	
10^{2}	
10^{3}	

(d) What are the domain and range of P?

(e) What are the domain and range of L?

Remark 5.3.2 The powers of 10 function $P(t)$ has an inverse L. This new function L is called the base 10 logarithm. But, we could have done a similar procedure with any base, which leads to the following definition.

Introduction to Logarithms (EL3)

Definition 5.3.3 The **base** b **logarithm** of a number is the exponent we must raise b to get that number. We represent this function as $y = \log_b(x)$.

We read the logarithmic expression as "The logarithm with base b of x is equal to y," or "log base b of x is y." $\Diamond$

Remark 5.3.4 There are some logarithms that occur so often, we sometimes write them without noting the base. They are the common logarithm and the natural logarithm.

The **common logarithm** is a logarithm with base 10 and is written without a base.

$$\log_{10}(x) = \log(x)$$

The **natural logarithm** is a logarithm with base e and has its own notation.

Recall the number e was introduced in Remark 5.1.16

$$\log_e(x) = \ln(x)$$

Remark 5.3.5 We can use Definition 5.3.3 to express the relationship between logarithmic form and exponential form as follows:

$$\log_b(x) = y \iff b^y = x$$

whenever $b > 0, b \neq 1$.

Introduction to Logarithms (EL3)

Activity 5.3.6 Write the following logarithmic equations in exponential form.

(a) $\log_7(\sqrt{7}) = \frac{1}{2}$

A. $7^{\frac{1}{2}} = \sqrt{7}$

B. $7^{\sqrt{7}} = \frac{1}{2}$

C. $\sqrt{7}^{\frac{1}{2}} = 7$

D. $\frac{1}{2}^7 = \sqrt{7}$

(b) $\log_3(m) = r$

A. $3^m = r$

B. $r^3 = m$

C. $3^r = m$

D. $m^r = 3$

(c) $\log_2(x) = 6$

A. $2^x = 6$

B. $6^2 = x$

C. $x^x = 6$

D. $2^6 = x$

Introduction to Logarithms (EL3)

Activity 5.3.7 Write the following exponential equations in logarithmic form.

(a) $5^2 = 25$

A. $\log_5(2) = 25$

B. $\log_5(25) = 2$

C. $\log_{25}(5) = 2$

D. $\log_2(25) = 5$

(b) $3^{-1} = \frac{1}{3}$

A. $\log_3(-1) = \frac{1}{3}$

B. $\log_{\frac{1}{3}}(-1) = 3$

C. $\log_3\left(\frac{1}{3}\right) = -1$

D. $\log_{-1}\left(\frac{1}{3}\right) = 3$

(c) $10^a = n$

A. $\log_{10}(n) = a$

B. $\log_{10}(a) = n$

C. $\log_n(10) = a$

D. $\log_a(n) = 10$

Activity 5.3.8 We can use the idea of converting a logarithm to an exponential to evaluate logarithms.

(a) Consider the logarithm $\log_3(9)$. If we want to evaluate this, which question should you try and solve?

A. To what exponent must 9 be raised in order to get 3?

B. What exponent must be raised to the third in order to get 9?

C. To what exponent must 3 be raised in order to get 9?

D. What exponent must be raised to the ninth in order to get 3?

(b) Evaluate the logarithm, $\log_3(9)$, by answering the question from part (a).

Introduction to Logarithms (EL3)

Activity 5.3.9 Evaluate the following logarithms.

(a) $\log_2(8)$

A. 4
B. $\frac{1}{4}$
C. -3
D. 3

(b) $\log_{144}(12)$

A. $\frac{1}{2}$
B. -2
C. 2
D. $-\frac{1}{2}$

(c) $\log_{10}\left(\frac{1}{1000}\right)$

A. $\frac{1}{3}$
B. -3
C. 3
D. $-\frac{1}{3}$

(d) $\log_e\left(e^3\right)$

A. 3
B. e^3
C. -3
D. $\frac{1}{3}$

(e) $\log_7(1)$

A. 7
B. $\frac{1}{7}$
C. 0
D. 1

Remark 5.3.10 Consider the results of Activity 5.3.9 part (d) and (e). Using the rules of exponents and the fact that exponents and logarithms are inverses, the following properties hold for any base:

- $\log_b(b^x) = x$
- $b^{\log_b(x)} = x$

In particular, note the following special cases:

- $\log_b(1) = 0$ (and therefore $\ln(1) = 0$)
- $\log_b(b) = 1$ (and therefore $\ln(e) = 1$)

Introduction to Logarithms (EL3)

Activity 5.3.11 Evaluate the following logarithms. Some may be done by inspection and others may require a calculator.

(a) $\log_4\left(\frac{1}{64}\right)$

(b) $\ln(1)$

(c) $\ln(12)$

(d) $\log(100)$

(e) $\log_5(32)$

(f) $\log_5\left(\sqrt{5}\right)$

(g) $\log(-10)$

Remark 5.3.12 Notice that in Activity 5.3.11 part (g) you were unable to evaluate the logarithm. Given that exponentials and logarithms are inverses, their domain and range are related. The range of an exponential function is $(0, \infty)$ which becomes the domain of a logarithmic function. This means that the argument of any logarithmic function must be greater than zero.

Activity 5.3.13 Find the domain of the function $\log_3(2x - 4)$.

(a) Set up an inequality that you must solve to find the domain.

(b) Solve the inequality to find the domain. Write your answer in interval notation.

5.4 Graphs of Logarithmic Functions (EL4)

Objectives

- Graph logarithmic functions and determine the domain, range, and asymptotes.

Graphs of Logarithmic Functions (EL4)

Activity 5.4.1 Consider the function $g(x) = \log_2 x$.

(a) Since we are familiar with graphing exponential functions, we'll use that to help us graph logarithmic ones. Rewrite $g(x)$ in exponential form, replacing $g(x)$ with y.

A. $x^y = 2$

B. $2^y = x$

C. $y^2 = x$

D. $2^x = y$

E. $x^2 = y$

(b) Fill in the table of values. Notice you are given y-values, not x-values to plug in since those are easier in the equivalent exponential form. Then plot the points on a graph.

x	y
	-2
	-1
	0
	1
	2

(c) What seems to be happening with the graph as x goes toward infinity? Plug in large positive values of x to test your guess, then describe the end behavior.

A. As $x \to \infty$, $y \to -\infty$.

B. As $x \to \infty$, $y \to 0$.

C. As $x \to \infty$, $y \to 6$.

D. As $x \to \infty$, $y \to \infty$.

E. The graph isn't defined as $x \to \infty$.

(d) What seems to be happening with the graph as x goes toward negative infinity? Plug in large negative values of x to test your guess, then describe the end behavior.

A. As $x \to -\infty$, $y \to -\infty$.

B. As $x \to -\infty$, $y \to 0$.

C. As $x \to -\infty$, $y \to 6$.

D. As $x \to -\infty$, $y \to \infty$.

E. The graph isn't defined as $x \to -\infty$.

(e) What seems to be happening with the graph as we approach x-values closer and closer to zero from the positive direction?

A. As $x \to 0$ from the positive direction, $y \to -\infty$.

B. As $x \to 0$ from the positive direction, $y \to 0$.

C. As $x \to 0$ from the positive direction, $y \to \infty$.

D. As $x \to 0$ from the positive direction, the graph isn't defined.

(f) What seems to be happening with the graph as we approach x-values closer and closer to zero from the negative direction?

A. As $x \to 0$ from the negative direction, $y \to -\infty$.

B. As $x \to 0$ from the negative direction, $y \to 0$.

C. As $x \to 0$ from the negative direction, $y \to \infty$.

D. As $x \to 0$ from the negative direction, the graph isn't defined.

(g) Complete the graph you started in Task 5.4.1.a, connecting the points and including the end behavior and behavior near zero that you've just determined.

(h) Does your graph seem to have any asymptotes?

A. No. There are no asymptotes.

B. There is a vertical asymptote but no horizontal one.

C. There is a horizontal asymptote but no vertical one.

D. The graph has both a horizontal and vertical asymptote.

(i) What the equation for each asymptote of $f(x)$? Select all that apply.

A. There are no asymptotes.

B. $x = 0$

C. $x = 6$

D. $y = 0$

E. $y = 6$

(j) Find the domain and range of $g(x)$. Write your answers using interval notation.

(k) Find the interval(s) where $g(x)$ is increasing and the interval(s) where $g(x)$ is decreasing. Write your answers using interval notation.

Activity 5.4.2 The function we've just graphed, $g(x) = \log_2 x$, and the function $f(x) = 2^x$ (which we graphed in Activity 5.2.1) are inverse functions.

(a) How are the graphs of $f(x)$ and $g(x)$ similar?

(b) How are the graphs of $f(x)$ and $g(x)$ different?

Graphs of Logarithmic Functions (EL4)

Remark 5.4.3 The graph of a logarithmic function $g(x) = \log_b x$ where $b > 0$ and $b \neq 1$ has the following characteristics:

- Its domain is $(0, \infty)$ and its range is $(-\infty, \infty)$.
- There is a vertical asymptote at $x = 0$. There is no horizontal asymptote.
- There is an x-intercept at $(1, 0)$. There is no y-intercept.

Graphs of Logarithmic Functions (EL4)

Remark 5.4.4 Just as with other types of functions, we can use transformations to graph logarithmic functions. For a reminder of these transformations, see Section 2.4 and the following definitions:

- Definition 2.4.5
- Definition 2.4.9
- Definition 2.4.15
- Definition 2.4.16
- Definition 2.4.24
- Definition 2.4.25

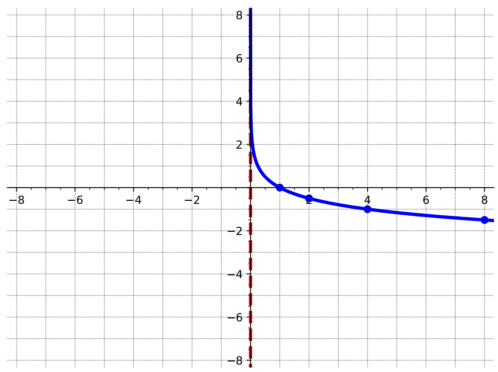
Graphs of Logarithmic Functions (EL4)

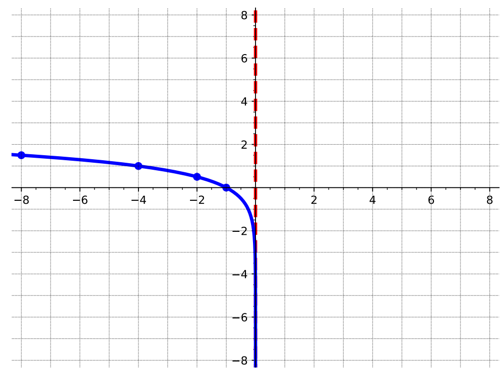
Activity 5.4.5 Let $f(x) = \log_4 x$.

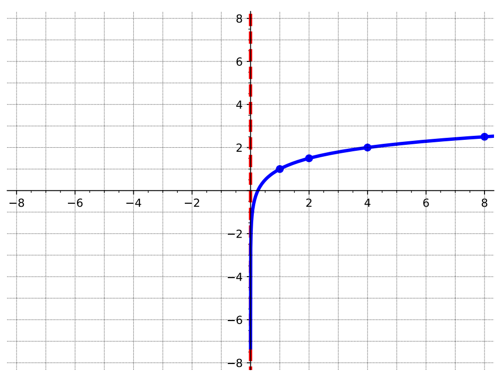
(a) Graph $f(x)$.

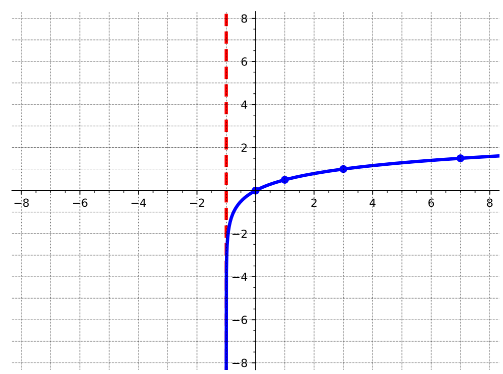
(b) Match the following functions to their graphs.

- $g(x) = -\log_4 x$
- $h(x) = \log_4(-x)$
- $j(x) = \log_4(x+1)$
- $k(x) = \log_4(x) + 1$

A.


C.


B.


D.


(c) Find the domain, range, and equation of the asymptote for the parent function $(f(x))$ and each of the four transformations $(g(x), h(x), j(x),$ and $k(x))$.

(d) Which of the transformations affected the domain of the logarithmic function? Select all that apply.

A. A vertical shift.

B. A horizontal shift.

C. A reflection over the x-axis.

D. A reflection over the y-axis.

E. None of these.

(e) Which of the transformations affected the range of the logarithmic function? Select all that apply.

A. A vertical shift.

B. A horizontal shift.

C. A reflection over the x-axis.

D. A reflection over the y-axis.

E. None of these.

(f) Which of the transformations affected the asymptote of the logarithmic function? Select all that apply.

A. A vertical shift.

B. A horizontal shift.

C. A reflection over the x-axis.

D. A reflection over the y-axis.

E. None of these.

Graphs of Logarithmic Functions (EL4)

Activity 5.4.6 Consider the function $f(x) = \ln(x)$.

(a) Graph $f(x) = \ln(x)$. First find $f(1)$ and $f(e)$. Then use what you know about the characteristics of logarithmic graphs to sketch the rest. Then state the domain, range, and equation of the asymptote. (Recall that $e \approx 2.72$ to help estimate where to put your points.)

(b) Sketch the graph of $g(x) = \ln(x-3)$ using transformations. State the transformation(s) used, the domain, the range, and the equation of the asymptote.

(c) Sketch the graph of $h(x) = 3\ln(x)$ using transformations. State the transformation(s) used, the domain, the range, and the equation of the asymptote.

Graphs of Logarithmic Functions (EL4)

Activity 5.4.7 Graph each of the following logarithmic functions. Include any asymptotes with a dotted line. State the domain, the range, and the equation of the asymptote.

(a) $f(x) = \log_3 x$

(b) $f(x) = \log_6(-x)$

(c) $f(x) = \log_{\frac{1}{5}} x$

(d) $f(x) = \log_{\frac{1}{3}} x + 2$

5.5 Properties of Logarithms (EL5)

Objectives

- Use properties of logarithms to condense or expand logarithmic expressions.

Properties of Logarithms (EL5)

Remark 5.5.1 Recall from Remark 5.3.5 that we can convert between exponential and logarithmic forms.

$$\log_b x = y$$

is equivalent to

$$b^y = x.$$

Properties of Logarithms (EL5)

Activity 5.5.2 Suppose you are given two equations $\log_b M = x$ and $\log_b N = x$.

(a) Rewrite each logarithmic equation into an exponential equation.

(b) Look at the exponential equations you found in part (a). What conclusion can you make about M and N?

(c) Given that both $\log_b M$ and $\log_b N$ are both equal to x, what can you say about $\log_b M$ and $\log_b N$?

(d) If given $\log_b M = \log_b N$, what can you say about M and N? (Refer back to the previous parts of this activity.)

Properties of Logarithms (EL5)

Fact 5.5.3 *For any values* $M > 0$ *and* $N > 0$*, and for any* $b > 0$ *with* $b \neq 1$*,*

$$\log_b M = \log_b N$$

if and only if

$$M = N.$$

This is called the ***one-to-one property of logarithms****.*

Observation 5.5.4 Notice this fact will eventually help us solve logarithmic equations. If we have an equation where each side is expressed as a single logarithm with matching bases (such as $\log_b M = \log_b N$), then it follows that the arguments (M and N) are also equal to each other.

Properties of Logarithms (EL5)

Remark 5.5.5 Recall that exponential functions and logarithmic functions are inverses. We know that $\log_b(b^k) = k$ and according to the law of exponents, we know that:

$$x^a \cdot x^b = x^{a+b}$$

$$\frac{x^a}{x^b} = x^{a-b}$$

$$(x^a)^b = x^{a \cdot b}$$

Consider all these as you move through the activities in this section.

Properties of Logarithms (EL5)

Activity 5.5.6 Let's begin with the law of exponents to see if we can understand the product property of logs. According to the law of exponents, we know that $10^x \cdot 10^y = 10^{x+y}$. Start with this equation as you move through this activity.

(a) Let $a = 10^x$ and $b = 10^y$. How could you rewrite the left side of the equation $10^x \cdot 10^y$?

A. $a + b$

B. $a - b$

C. 10^{x+y}

D. $a \cdot b$

(b) Recall from Fact 5.5.3 that $\log_b M = \log_b N$ if and only if $M = N$. Use this property to apply the logarithm to both sides of the rewritten equation from part (a). What is that equation?

A. $\log_{10}(a+b) = \log_{10}\left(10^{x+y}\right)$

B. $\log_{10}(a \cdot b) = \log_{10}\left(10^{x+y}\right)$

C. $\log_{10}(a \cdot b) = \log_{10}\left(10^{a+b}\right)$

D. $\log_{10}(a+b) = \log_{10}\left(10^{a+b}\right)$

(c) Knowing that $\log_b(b^k) = k$, how could you simplify the right side of the equation?

A. $\log_{10}(a+b)$

B. $\log_{10}(x+y)$

C. $a + b$

D. $x + y$

(d) Recall in part (a), we defined $10^x = a$ and $10^y = b$. What would these look like in logarithmic form?

A. $\log_{10} a = x$

B. $\log_x a = 10$

C. $\log_{10} b = y$

D. $\log_y b = 10$

(e) Using your solutions in part (d), how can we rewrite the right side of the equation?

A. 10^{a+b}

B. $\log_{10} a - \log_{10} b$

C. $\log_{10} a + \log_{10} b$

D. $10^x + 10^y$

(f) Combining parts (a) and (d), which equation represents $10^x \cdot 10^y = 10^{x+y}$ in terms of logarithms?

A. $\log_{10}(a+b) = 10^{a+b}$

B. $\log_{10}(a \cdot b) = \log_{10} a - \log_{10} b$

C. $\log_{10}(a \cdot b) = \log_{10} a + \log_{10} b$

D. $\log_{10}(a \cdot b) = 10^x + 10^y$

Properties of Logarithms (EL5)

Activity 5.5.7 According to the law of exponents, we know that $\dfrac{10^x}{10^y} = 10^{x-y}$. Start with this equation as you move through this activity.

(a) Let $a = 10^x$ and $b = 10^y$. How could you rewrite the left side of the equation $\dfrac{10^x}{10^y}$?

A. $a + b$

B. $a - b$

C. 10^{x+y}

D. $a \cdot b$

(b) Use the one-to-one property of logarithms to apply the logarithm to both sides of the rewritten equation from part (a). What is that equation?

A. $\log_{10}(a - b) = \log_{10}\left(10^{x-y}\right)$

B. $\log_{10}\left(\dfrac{a}{b}\right) = \log_{10}\left(10^{x-y}\right)$

C. $\log_{10}\left(\dfrac{a}{b}\right) = \log_{10}\left(10^{a+b}\right)$

D. $\log_{10}(a - b) = \log_{10}\left(10^{a-b}\right)$

(c) Knowing that $\log_b(b^k) = k$, how could you simplify the right side of the equation?

A. $\log_{10}(a - b)$

B. $\log_{10}(x - y)$

C. $x - y$

D. $a - b$

(d) Recall in part (a), we defined $10^x = a$ and $10^y = b$. What would these look like in logarithmic form?

A. $\log_{10} a = x$

B. $\log_x a = 10$

C. $\log_{10} b = y$

D. $\log_y b = 10$

(e) Using your solutions in part (d), how can we rewrite the right side of the equation?

A. 10^{a+b}

B. $\log_{10} a - \log_{10} b$

C. $\log_{10} a - \log_{10} b$

D. 10^{x-y}

(f) Combining parts (a) and (d), which equation represents $\dfrac{10^x}{10^y} = 10^{x-y}$ in terms of logarithms?

A. $\log_{10}(a - b) = 10^{a+b}$

B. $\log_{10}(a - b) = \log_{10} a - \log_{10} b$

C. $\log_{10}\left(\dfrac{a}{b}\right) = \log_{10} a - \log_{10} b$

D. $\log_{10}\left(\dfrac{a}{b}\right) = 10^{x-y}$

Properties of Logarithms (EL5)

Remark 5.5.8 In Activity 5.5.6 and Activity 5.5.7, you explored an example of two common properties of logarithms!

Properties of Logarithms (EL5)

Definition 5.5.9 The **product property of logarithms** states

$$\log_a(m \cdot n) = \log_a m + \log_a n.$$

The **quotient property of logarithms** states

$$\log_a\left(\frac{m}{n}\right) = \log_a m - \log_a n.$$

Notice that for each of these properties, the base has to be the same. ◊

Activity 5.5.10 There is still one more property to consider. This activity will investigate the power property. Suppose you are given

$$\log(x^3) = \log(x \cdot x \cdot x).$$

(a) By applying Definition 5.5.9, how could you rewrite the right-hand side of the equation?

A. $\log(3x)$

B. $\log(x) - \log(x) - \log(x)$

C. $\log(x^3)$

D. $\log(x) + \log(x) + \log(x)$

(b) By combining "like" terms, you can simplify the right-hand side of the equation further. What equation do you have after simplifying the right-hand side?

A. $\log(x^3) = \log(3x)$

B. $\log(x^3) = -3\log(x)$

C. $\log(x^3) = \log(x^3)$

D. $\log(x^3) = 3\log(x)$

Properties of Logarithms (EL5)

Definition 5.5.11 The **power property of logarithms** states

$$\log_a(m^n) = n \cdot \log_a(m).$$

$\Diamond$

Properties of Logarithms (EL5)

Activity 5.5.12 Apply Definition 5.5.9 and Definition 5.5.11 to expand the following. (Note: When you are asked to expand logarithmic expressions, your goal is to express a single logarithmic expression into many individual parts or components.)

(a)

$$\log_3\left(\frac{6}{19}\right)$$

A. $\log_3 6 + \log_3 19$

B. $\log_3(6 - 19)$

C. $\log_3 6 - \log_3 19$

D. $\log_3(6 + 19)$

(b)

$$\log\left((a \cdot b)^2\right)$$

A. $2(\log a + \log b)$

B. $2\log a - \log b$

C. $\log a + 2\log b$

D. $2\log a + 2\log b$

(c)

$$\ln\left(\frac{x^3}{y}\right)$$

A. $3\ln x + \ln y$

B. $3\ln x - \ln y$

C. $3(\ln x - \ln y)$

D. $\ln x^3 - \ln y$

(d)

$$\log(x \cdot y \cdot z^3)$$

A. $\log x + \log y + 3\log z$

B. $\log x + \log y + \log z^3$

C. $\log x - \log y - 3\log z$

D. $3(\log x + y + z)$

Properties of Logarithms (EL5)

Activity 5.5.13 Apply Definition 5.5.9 and Definition 5.5.11 to condense into a single logarithm.

(a)

$$6 \log_6 a + 3 \log_6 b$$

A. $6(\log_6 a) + 3(\log_6 b)$

B. $\log_6 a^6 + \log_6 b^3$

C. $(\log_6 a)^6 + (\log_6 b)^3$

D. $\log_6 \left(a^6 \cdot b^3\right)$

(b)

$$\ln x - 4 \ln y$$

A. $\ln x - \ln y^4$

B. $\ln \left(\frac{x}{y^4}\right)$

C. $\ln \left(\frac{x}{y}\right)^4$

D. $\ln(x \cdot y^4)$

(c)

$$2(\log(2x) - \log y)$$

A. $\log \left(\frac{4x^2}{y^2}\right)$

B. $\log \left(\frac{2x^2}{y^2}\right)$

C. $2 \cdot \log \left(\frac{2x}{y}\right)$

D. $\log \left(\frac{4x^2}{y}\right)$

(d)

$$\log 3 + 2 \log 5$$

A. $\log(3 \cdot 5^2)$

B. $2 \cdot \log 75$

C. $\log 75$

D. $\log 225$

Properties of Logarithms (EL5)

Remark 5.5.14 You might have noticed that a scientific calculator has only ”log” and ”ln” buttons (because those are the most common bases we use), but not all logs have base 10 or e as their bases.

Properties of Logarithms (EL5)

Activity 5.5.15 Suppose you wanted to find the value of $\log_5 3$ in your calculator but you do not know how to input a base other than 10 or e (i.e., you only have the "log" and "ln" buttons on your calculator). Let's explore another helpful tool that can help us find the value of $\log_5 3$.

(a) Let's start with the general statement, $\log_b a = x$. How can we rewrite this logarithmic equation into an exponential equation?

(b) Now take the log of both sides of your equation and apply the power property of logarithms to bring the exponent down. What equation do you have now?

(c) Solve for x. What does x equal?

(d) Recall that when we started, we defined $x = \log_b a$. Substitute $\log_b a$ into your equation you got in part c for x. What is the resulting equation?

(e) Apply what you got in part d to find the value of $\log_5 3$. What is the approximate value of $\log_5 3$?

Remark 5.5.16 Notice in Activity 5.5.15, we were able to calculate $\log_5 3$ using logs of base 10. You should now be able to find the value of a logarithm of any base!

Properties of Logarithms (EL5)

Definition 5.5.17 The **change of base formula** is used to write a logarithm of a number with a given base as the ratio of two logarithms each with the same base that is different from the base of the original logarithm.

$$\log_b a = \frac{\log a}{\log b}$$

◊

Properties of Logarithms (EL5)

Activity 5.5.18 Apply Definition 5.5.17 and a calculator to approximate the value of each logarithm.

(a) $\log_2 30$

A. 0.204

B. 4.907

C. $\dfrac{\log 30}{\log 2}$

D. $\dfrac{\log 2}{\log 30}$

(b) $\ln 183$

A. $\dfrac{\ln 183}{\ln e}$

B. 67.32

C. $\dfrac{\ln e}{\ln 183}$

D. 5.209

5.6 Solving Exponential and Logarithmic Equations (EL6)

Objectives

- Solve exponential and logarithmic equations.

Solving Exponential and Logarithmic Equations (EL6)

Remark 5.6.1 Recall that we can convert between exponential and logarithmic forms.

$$\log_b x = y$$

is equivalent to

$$b^y = x$$

Solving Exponential and Logarithmic Equations (EL6)

Activity 5.6.2 This activity will investigate ways we can solve logarithmic equations using properties and definitions from previous sections.

(a) Given that $\log_3 9 = x$, how can we rewrite this into an exponential equation?

A. $9^x = 3$

B. $\log\left(\frac{9}{3}\right)$

C. $3^x = 9$

D. $x^3 = 9$

(b) Now that $\log_3 9 = x$ is rewritten as an exponential equation, what is the value of x?

A. -2

B. 2

C. $-\frac{1}{2}$

D. $\frac{1}{2}$

Remark 5.6.3 Notice in Activity 5.6.2, you were able to solve a logarithmic equation by converting it into an exponential equation. This is one method in solving logarithmic equations.

Activity 5.6.4 For each of the following, solve the logarithmic equations by first converting them to exponential equations.

(a) $\log_{10}(1{,}000{,}000) = x$

A. -6

B. 100,000

C. 6

D. 0.00001

(b) $\log_3(x+3) = 0$

A. 0

B. -2

C. 2

D. -1

(c) $\log_5(2x+4) = 2$

A. 14

B. $\frac{29}{2}$

C. 16

D. $\frac{21}{2}$

Solving Exponential and Logarithmic Equations (EL6)

Activity 5.6.5 Not all logarithmic equations can be solved by converting to exponential equations. In this activity, we will explore another way to solve logarithmic equations.

(a) Suppose you are given the equation,

$$\log(4x - 5) = \log(2x - 1),$$

and you brought all the logs to one side to get

$$\log(4x - 5) - \log(2x - 1) = 0.$$

Using the quotient property of logs, how could you condense the left side of the equation?

A. $\log\left(\dfrac{4x-5}{2x-1}\right)$

B. $\log\left(\dfrac{2x-1}{4x-5}\right)$

C. $\dfrac{\log(4x-5)}{\log(2x-1)}$

D. $\dfrac{\log(2x-1)}{\log(4x-5)}$

(b) Now that you have a single logarithm, convert this logarithmic form into exponential form. What does this new equation look like?

A. $\dfrac{4x-5}{2x-1} = 10^0$

B. $\dfrac{2x-1}{4x-5} = 1$

C. $\dfrac{2x-1}{4x-5} = 10^0$

D. $\dfrac{4x-5}{2x-1} = 1$

(c) Notice that the "log" has disappeared and you now have an equation with just the variable x. Which of the following is equivalent to the equation you got in part (b)?

A. $4x - 5 = 2x - 1$

B. $4x - 5 = 0$

C. $2x - 1 = 0$

(d) Compare the answer you got in part (c) to the original equation given $\log(4x - 5) = \log(2x - 1)$. What do you notice?

(e) Solve the equation you got in part (d) to find the value of x.

A. -3

B. $\dfrac{5}{4}$

C. $\dfrac{1}{2}$

D. 2

Remark 5.6.6 Notice in Activity 5.6.5, that you did not have to convert the logarithmic equation into an exponential equation. A faster method, when you have a log on both sides of the equals sign, is to ”drop” the logs and set the arguments equal to one another. Be careful though - you can only have one log on each side before you can ”drop” them!

Activity 5.6.7 Apply the one-to-one property of logarithms (see Fact 5.5.3) and other properties of logarithms (i.e., product, quotient, and power) to solve the following logarithmic equations.

(a) $\log(-2a + 9) = \log(7 - 4a)$

(b) $\log_9(x + 6) - \log_9 x = \log_9 2$

(c) $\log_8 2 + \log_8(4x^2) = 1$

(d) $\ln(4x + 1) - \ln 3 = 5$

Activity 5.6.8 In some cases, you will get equations with logs of different bases. Apply properties of logarithms to solve the following logarithmic equations.

(a) $\log_3(x - 6) = \log_9 x$

Hint. Use the change of base formula to rewrite $\log_9 x$ so that it has a base of 3.

(b) $\log_2 x = \log_8(4x)$

Solving Exponential and Logarithmic Equations (EL6)

Activity 5.6.9 Now that we've looked at how to solve logarithmic equations, let's see how we can apply similar methods to solving exponential equations.

(a) Suppose you are given the equation $2^x = 48$. There is no whole number value we can raise 2 to to get 48. What two whole numbers must x be between?

(b) We'll use logarithms to isolate the variable in the exponent. How can we convert $2^x = 48$ into a logarithmic equation?

A. $x = \log_{48} 2$

B. $x = \log_2 48$

C. $48 = \log_2 x$

D. $2 = \log_{48} x$

(c) Notice that the answer you got in part (b) is an exact answer for x. There will be times, though, that it will be helpful to also have an approximation for x. Which of the following is a good approximation for x?

A. $x \approx 5.585$

B. $x \approx 0.179$

C. $x \approx 24$

D. $x \approx \dfrac{1}{24}$

Remark 5.6.10 Notice in Activity 5.6.9 we started with an exponential equation and then solved by converting the equation into a logarithmic equation. Logarithms can help us get the variable out of the exponent.

Solving Exponential and Logarithmic Equations (EL6)

Activity 5.6.11 Although rewriting an exponential equation into a logarithmic equation is helpful at times, it is not the only method in solving exponential equations. In this activity, we will explore what happens when we take the log of both sides of an exponential equation and use the properties of logarithms to solve in another way.

(a) Suppose you are given the equation:

$$3^x = 7.$$

Take the log of both sides. What equation do you now have?

(b) Apply the power property of logarithms (Definition 5.5.11) to bring down the exponent. What equation do you now have?

(c) Solve for x. What does x equal?

(d) Using the change-of-base formula (Definition 5.5.17), rewrite your answer from part (c) so that x is written as a single logarithm. What is the exact value of x?

(e) If you were to solve $3^x = 7$ by converting it into a logarithmic equation, what would it look like?

(f) What do you notice about your answer from parts (d) and (e)?

Solving Exponential and Logarithmic Equations (EL6)

Activity 5.6.12 Use the method of taking the log of both sides (as you saw in Activity 5.6.11) to solve $5^{2x+3} = 8$.

(a) Take the log of both sides and use the power property of logarithms to bring down the exponent. What equation do you have now?

A. $2x + 3\log 5 = \log 8$

B. $2x + (3\log 5) = \log 8$

C. $(2x + 3) \cdot \log 5 = \log 8$

D. $2x\log 5 + 3 = \log 8$

(b) Solve for x.

A. $x = \dfrac{\log_5 8 - 3}{2}$

B. $x = \dfrac{\log 8 - 3\log 5}{2}$

C. $x = \dfrac{\log 8}{3\log 5} - 2$

D. $x = \dfrac{\log 8 - 3}{2\log 5}$

(c) What is the approximate value of x?

A. $x \approx -0.85$

B. $x \approx -1.5$

C. $x \approx -0.60$

D. $x \approx -1.57$

Solving Exponential and Logarithmic Equations (EL6)

Activity 5.6.13 In this activity, we will explore other types of exponential equations, which will require other methods of solving.

(a) Suppose you are given the equation

$$5^{3x} = 5^{7x-2}$$

and you decide to take the log of both sides as your first step to get

$$\log 5^{3x} = \log 5^{7x-2}$$

What would you use next to solve this equation?

A. Quotient property of logarithms

B. Power property of logarithms

C. Product property of logarithms

D. Change of base formula

E. One-to-one property of logarithms

(b) Applying the property you chose in part a, what would the resulting equation be?

A. $3x \log 5 = 7x - 2 \log 5$

B. $3x \log 5 = 7x \log 5 - 2 \log 5$

C. $\log 5^{3x} = \log 5^{7x-2}$

D. $(3x) \log 5 = (7x - 2) \log 5$

(c) Now that you have a logarithmic equation, divide both sides by $\log 5$ to begin to isolate the variable x. After dividing by $\log 5$, what equation do you now have?

A. $3x = \dfrac{7x}{\log 5} - 2$

B. $3x = 7x - 2$

C. $5^{3x} = 5^{7x-2}$

D. $\dfrac{3x \log 5}{\log 5} = \dfrac{(7x-2)\log 5}{\log 5}$

(d) Compare the equation you got in part (c) to the original equation given. What do you notice?

(e) Solve for x.

A. 2

B. $\dfrac{1}{2}$

C. -2

D. $\dfrac{1}{2}$

Remark 5.6.14 Notice in Activity 5.6.13, it is much faster to set the exponents equal to one another. Make sure to check that the bases are equal *before* you set the exponents equal! And if the bases are not equal, you might have to use properties of exponents to help you get the bases to be the same.

Solving Exponential and Logarithmic Equations (EL6)

Definition 5.6.15 When you are given an exponential equation with the same bases on both sides, you can simply set the exponents equal to one another and solve. This is known as the **one-to-one property of exponentials**. ◊

Solving Exponential and Logarithmic Equations (EL6)

Activity 5.6.16 When an exponential equation has the same base on each side, the exponents can be set equal to one another. If the bases aren't the same, we can rewrite them using properties of exponents and use the one-to-one property of exponentials.

(a) Suppose you are given

$$5^x = 625,$$

how could you rewrite this equation so that both sides have a base of 5?

(b) Suppose you are given

$$4^x = 32,$$

how could you rewrite this equation so that both sides have a base of 2?

(c) Suppose you are given

$$3^{1-x} = \frac{1}{27},$$

how could you rewrite this equation so that both sides have a base of 3? (Hint: you many need to revisit properties of exponents)

(d) Suppose you are given

$$6^{\frac{x-3}{4}} = \sqrt{6},$$

how could you rewrite this equation so that both sides have a base of 6? (Hint: you many need to revisit properties of exponents)

Solving Exponential and Logarithmic Equations (EL6)

Activity 5.6.17 For each of the following, use properties of exponentials and logarithms to solve.

(a) $6^{-2x} = 6^{2-3x}$

A. 2

B. $-\frac{2}{5}$

C. -2

D. $-\frac{5}{2}$

(b) $\log_2 256 = x$

A. -8

B. 254

C. 8

D. 128

(c) $5\ln(9x) = 20$

A. e^4

B. $\frac{e^4}{9}$

C. $e^{\left(\frac{4}{9}\right)}$

D. $\frac{4}{\ln 9}$

(d) $10^x = 4.23$

A. $\log 4.23$

B. 42.3

C. 1.44

D. 0.63

(e) $\log_6(5x-5) = \log_6(3x+7)$

A. 2

B. 0

C. 6

D. 1

Solving Exponential and Logarithmic Equations (EL6)

Activity 5.6.18 For each of the following, use properties of exponentials and logarithms to solve.

(a) $\log_8 2 + \log_8 4x^2 = 1$

(b) $5^{x+7} = 3$

(c) $8^{\frac{x-6}{6}} = \sqrt{8}$

(d) $\log_6(x+1) - \log_6 x = \log_6 29$

5.7 Applications of Exponential and Logarithmic Functions (EL7)

Objectives

- Solve application problems using exponential and logarithmic equations.

Applications of Exponential and Logarithmic Functions (EL7)

Remark 5.7.1 Now that we have explored multiple methods for solving exponential and logarithmic equations, let's put those in to practice using some real-world application problems.

Activity 5.7.2 A coffee is sitting on Mr. Abacus's desk cooling. It cools according to the function $T = 70(0.80)^x + 20$, where x is the time elapsed in minutes and T is the temperature in degrees Celsius.

(a) What is the initial temperature of Mr. Abacus's coffee?

A. $20°C$

B. $70°C$

C. $0.80°C$

D. $90°C$

(b) What is the temperature of Mr. Abacus's coffee after 10 minutes?

A. $7.5°C$

B. $20°C$

C. $27.5°C$

D. $76°C$

(c) According to the function given, if Mr. Abacus leaves his coffee on his desk all day, what will the coffee eventually cool to?

A. $90°C$

B. $70°C$

C. $20°C$

D. $0°C$

Hint. Think about what happens to the function as $x \to \infty$.

Applications of Exponential and Logarithmic Functions (EL7)

Activity 5.7.3 A video posted on YouTube initially had 80 views as soon as it was posted. The total number of views to date has been increasing exponentially according to the exponential growth function $y = 80e^{0.2t}$, where t represents time measured in days since the video was posted.

(a) How many views will the video have after 3 days? Round to the nearest whole number.

A. 293 views

B. 146 views

C. 98 views

D. 82 views

(b) How many days does it take until 2,500 people have viewed this video?

A. 18 days

B. 156 days

C. 39 days

D. 17 days

Applications of Exponential and Logarithmic Functions (EL7)

Activity 5.7.4 In 2006, 80 deer were introduced into a wildlife refuge. By 2012, the population had grown to 180 deer. The population was growing exponentially. Recall that the general form of an exponential equation is $f(x) = a \cdot b^x$, where a is the initial value, b is the growth/decay factor, and t is time. We want to write a function $N(t)$ to represent the deer population after t years.

(a) What is the initial value for the deer population?

(b) We are not given the growth factor, so we must solve for it. Write an exponential equation using the initial population, the 2012 population, and the time elapsed.

(c) Take your equation in part b and solve for b using logs.

(d) Now that you have found the growth factor (from part b), what is the equation, in terms of $N(t)$ and t that represents the deer population?

(e) If the growth continues according to this exponential function, when will the population reach 250?

Applications of Exponential and Logarithmic Functions (EL7)

Activity 5.7.5 The concentration of salt in ocean water, called salinity, varies as you go deeper in the ocean. Suppose $f(x) = 28.9 + 1.3\log(x+1)$ models salinity of ocean water to depths of 1000 meters at a certain latitude, where x is the depth in meters and $f(x)$ is in grams of salt per kilogram of seawater. (Note that salinity is expressed in the unit g/kg, which is often written as ppt (part per thousand) or ‰ (permil).)

(a) At 50 meters, what is the salinity of the seawater?

A. 31.1 ppt

B. 32.1 ppt

C. 51.6 ppt

D. 30.6 ppt

(b) Approximate the depth (to the nearest tenth of a meter) where the salinity equals 33 ppt.

A. -1.0 meters

B. 1426.1 meters

C. -0.9 meters

D. 1424.1 meters

Applications of Exponential and Logarithmic Functions (EL7)

Activity 5.7.6 The first key on a piano keyboard (called A_0) corresponds to a pitch with a frequency of 27.5 cycles per second. With every successive key, going up the black and white keys, the pitch multiplies by a constant. The formula for the frequency, f of the pitch sounded when the nth note up the keyboard is played is given by

$$n = 1 + 12\log_2 \frac{f}{27.5}$$

(a) A note has a frequency of 220 cycles per second. How many notes up the piano keyboard is this?

A. 37 notes

B. 39 notes

C. 12 notes

D. 96 notes

(b) What frequency does the 12th note have? Round to the nearest tenth.

A. 0.1 cycles per second

B. 227.0 cycles per second

C. 51.9 cycles per second

D. 49.4 cycles per second

Remark 5.7.7 Another application of exponential equations is compound interest. Savings instruments in which earnings are continually reinvested, such as mutual funds and retirement accounts, use compound interest. The term compounding refers to interest earned not only on the original value, but on the accumulated value of the account.

Compound interest can be calculated by using the formula

$$A(t) = P\left(1 + \frac{r}{n}\right)^{nt},$$

where $A(t)$ is the account value, t is measured in years, P is the starting amount of the account (also known as the principal), r is the annual percentage rate (APR) written as a decimal, and n is the number of compounding periods in one year.

Activity 5.7.8 Before we can apply the compound interest formula, we need to understand what "compounding" means. Recall that compounding refers to interest earned not only on the original value, but on the accumulated value of the account. This amount is calculated a certain number of times in a given year.

(a) Suppose an account is compounded quarterly, what value of n would we use in the formula?

A. 1

B. 2

C. 4

D. 12

E. 365

(b) Suppose an account is compounded daily, what value of n would we use in the formula?

A. 1

B. 2

C. 4

D. 12

E. 365

(c) Suppose an account is compounded semi-annually, what value of n would we use in the formula?

A. 1

B. 2

C. 4

D. 12

E. 365

(d) Suppose an account is compounded monthly, what value of n would we use in the formula?

A. 1

B. 2

C. 4

D. 12

E. 365

(e) Suppose an account is compounded annually, what value of n would we use in the formula?

A. 1

B. 2

C. 4

D. 12

E. 365

Activity 5.7.9 A 529 Plan is a college-savings plan that allows relatives to invest money to pay for a child's future college tuition; the account grows tax-free. Lily currently has \$10,000 and opens a 529 account that will earn 6% compounded semi-annually.

(a) Which equation could we use to determine how much money Lily will have for her granddaughter after t years?

A. $A(t) = 10{,}000\left(1+\frac{6}{2}\right)^{2t}$

B. $A(t) = 10{,}000\left(1+\frac{0.06}{2}\right)^{2t}$

C. $A(t) = 10{,}000\left(1+\frac{6}{\frac{1}{2}}\right)^{\frac{1}{2}t}$

D. $A(t) = 10{,}000\left(1+\frac{0.06}{2}\right)^{18}$

(b) To the nearest dollar, how much will Lily have in the account in 10 years?

A. \$106,090

B. \$103,000

C. \$13,439

D. \$18,061

(c) How many years will it take Lily to have \$40,000 in the account for her granddaughter? Round to the nearest tenth.

A. 23.4 years

B. 46.9 years

C. 52.1 years

D. 25.6 years

Applications of Exponential and Logarithmic Functions (EL7)

Activity 5.7.10 For each of the following, determine the appropriate equation to use to solve the problem.

(a) Kathy plans to purchase a car that depreciates (loses value) at a rate of 14% per year. The initial cost of the car is \$21,000. Which equation represents the value, v, of the car after 3 years?

A. $v = 21{,}000(0.14)^3$

B. $v = 21{,}000(0.86)^3$

C. $v = 21{,}000(1.14)^3$

D. $v = 21{,}000(0.86)(3)$

(b) Mr. Smith invested \$2,500 in a savings account that earns 3% interest compounded annually. He made no additional deposits or withdrawals. Which expression can be used to determine the number of dollars in this account at the end of 4 years?

A. $2{,}500(1 + 0.03)^4$

B. $2{,}500(1 + 0.3)^4$

C. $2{,}500(1 + 0.04)^3$

D. $2{,}500(1 + 0.4)^3$

Activity 5.7.11 Suppose you want to invest \$100 in a banking account that has a 100% interest rate. Let's investigate what would happen to the amount of money you have at the end of one year in the account with varying compounding periods. Use the formula, $A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$, to help you solve the following problems.

(a) Suppose the account is compounded annually (i.e., $n = 1$). How much money would you have in the account at the end of the year?

(b) By what factor did the money in your account grow in part a?

(c) Suppose the account is compounded semi-annually. How much money would you have in the account at the end of the year?

(d) By what factor did the money in your account grow in part b?

(e) Let's investigate as the compounding periods increase. Fill in the following table.

n	$A(t)$	**Growth Factor**
1		
2		
5		
10		
100		
1,000		
10,000		
100,000		

(f) What do you notice as the value of n increases?

Observation 5.7.12 Activity 5.7.11n2.71e2.718

Remark 5.7.13 For many real-world phenomena, e is used as the base for exponential functions. Exponential models that use e as the base are called continuous growth or decay models. We see these models in finance, computer science, and most of the sciences, such as physics, toxicology, and fluid dynamics.

For all real numbers t, and all positive numbers a and r, continuous growth or decay is represented by the formula

$$A(t) = ae^{rt},$$

where a is the initial value, r is the continuous growth rate per of unit time, and t is the elapsed time. If $r > 0$, then the formula represents continuous growth. If $r < 0$, then the formula represents continuous decay.

For business applications, the continuous growth formula is called the continuous compounding formula and takes the form

$$A(t) = Pe^{rt},$$

where P is the principal or the initial invested, r is the growth or interest rate per of unit time, and t is the period or term of the investment.

Applications of Exponential and Logarithmic Functions (EL7)

Activity 5.7.14 Use the continuous formulas to answer the following questions.

(a) A person invested \$1,000 in an account earning 10% per year compounded continuously. How much was in the account at the end of one year?

(b) Radon-222 decays at a continuous rate of 17.3% per day. How much will 100 mg of Radon-222 decay to in 3 days?

Chapter 6

Trigonometric Functions (TR)

Objectives

How can we use right triangles to make sense of angles?
By the end of this chapter, you should be able to...

1. Convert between degrees and radians. Draw angles in standard position.
2. Identify and find coterminal angles. Find the length of a circular arc.
3. Use a right triangle to evaluate trigonometric functions.
4. Find exact values of trigonometric functions of special angles (30, 45, and 60).
5. Use reference angles, signs and definitions to determine exact values of trigonometric functions.

6.1 Degree and Radian Measure (TR1)

Objectives

- Convert between degrees and radians. Draw angles in standard position.

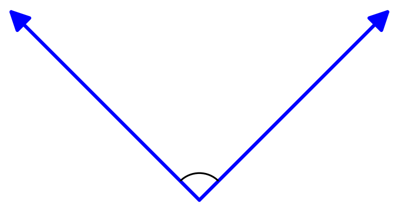
Degree and Radian Measure (TR1)

Definition 6.1.1 An **angle** is formed by joining two rays at their starting points. The point where they are joined is called the **vertex** of the angle. The measure of an angle describes the amount of rotation between the two rays. $\Diamond$

Degree and Radian Measure (TR1)

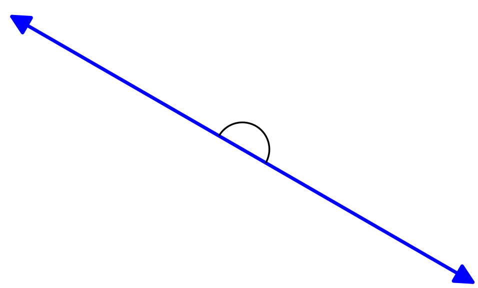
Activity 6.1.2 An angle that is rotated all the way around back to its starting point measures 360°, like a circle. Use this to estimate the measure of the given angles.

(a)



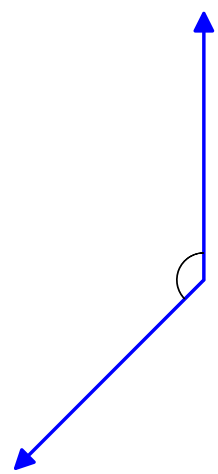
A. 45° B. 90° C. 135° D. 180°

(b)



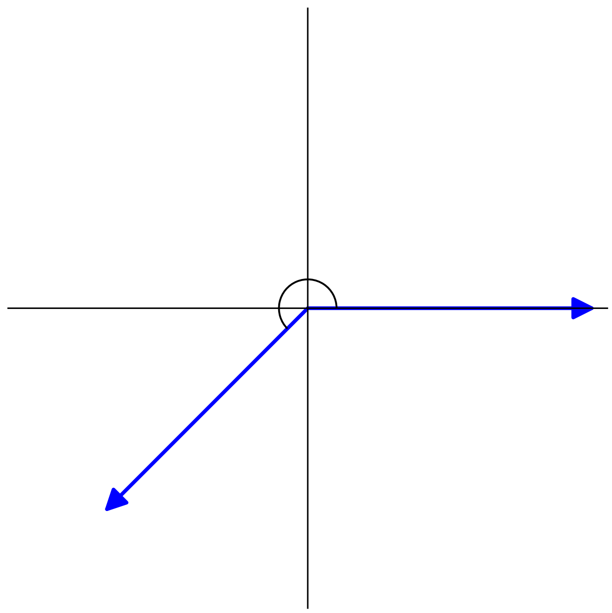
A. 45° B. 90° C. 135° D. 180°

(c)

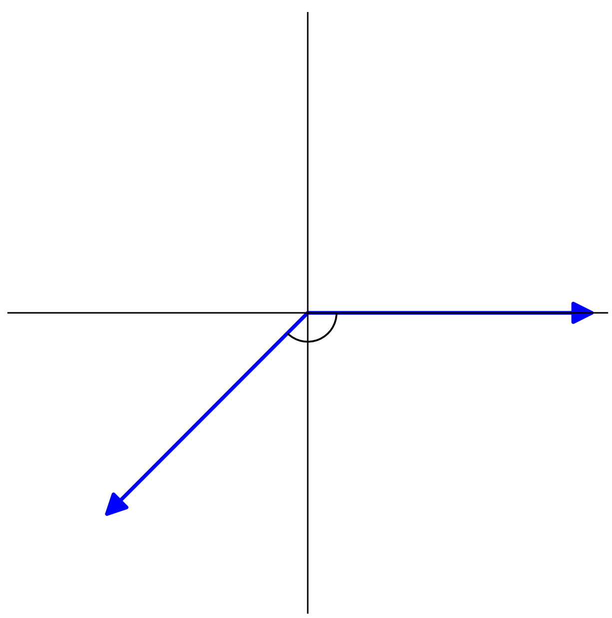


A. 45°   B. 90°   C. 135°   D. 180°

Definition 6.1.3 An angle is in **standard position** if its vertex is located at the origin and its initial side extends along the positive x-axis.



An angle measured counterclockwise from the initial side has a positive measure, while an angle measured clockwise from the initial side has a negative measure.

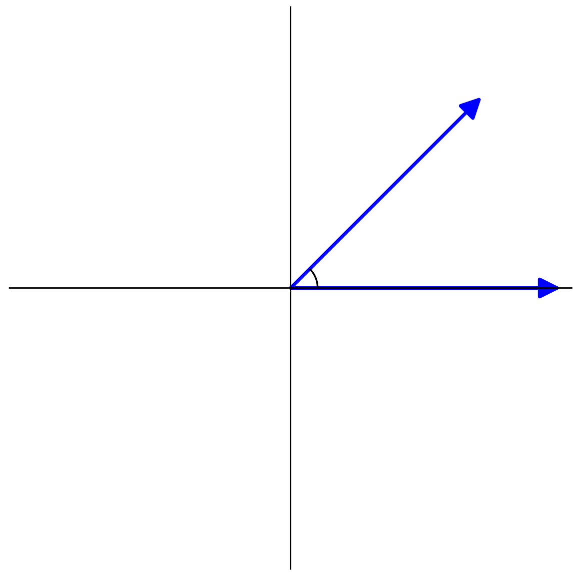


$\diamond$

Degree and Radian Measure (TR1)

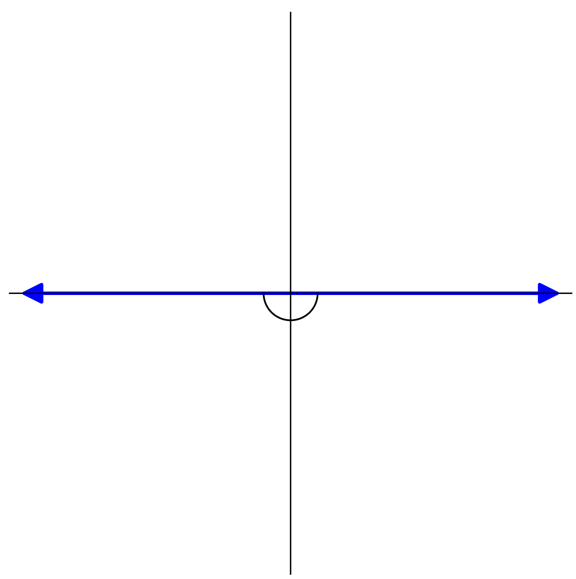
Activity 6.1.4 Estimate the measure of the angles drawn in standard position.

(a)



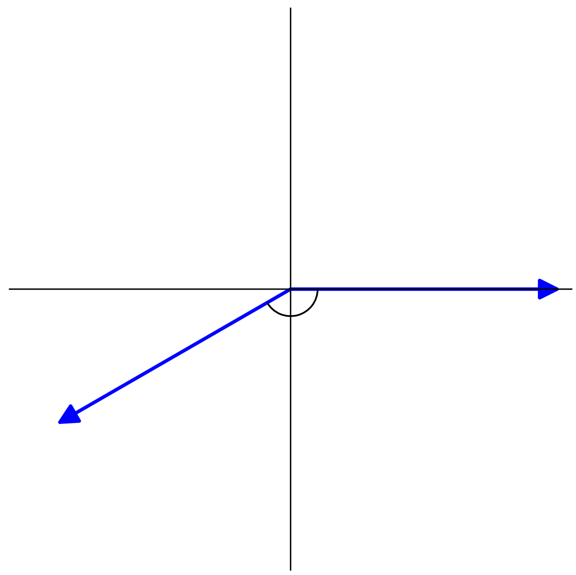
A. 45° B. 90° C. 135° D. 180°

(b)



A. 180° B. 90° C. -180° D. -90°

(c)



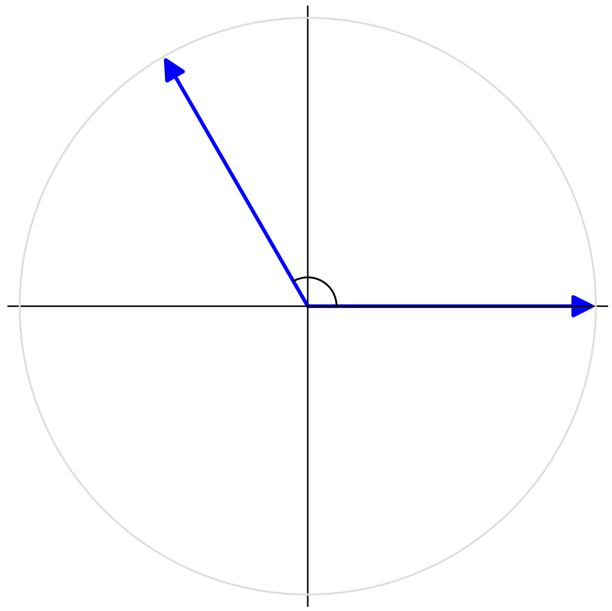
A. 30° B. −150° C. −210° D. 210°

(d) Draw an angle of measure −225° in standard position.

Degree and Radian Measure (TR1)

Remark 6.1.5 Degrees are not the only way to measure an angle. We can also describe the angle's measure by the amount of the circumference of the circle that the angle's rotation created. We'll need to define a few terms to help us come up with this new measurement.

Definition 6.1.6 A **central angle** is an angle whose vertex is at the center of a circle.



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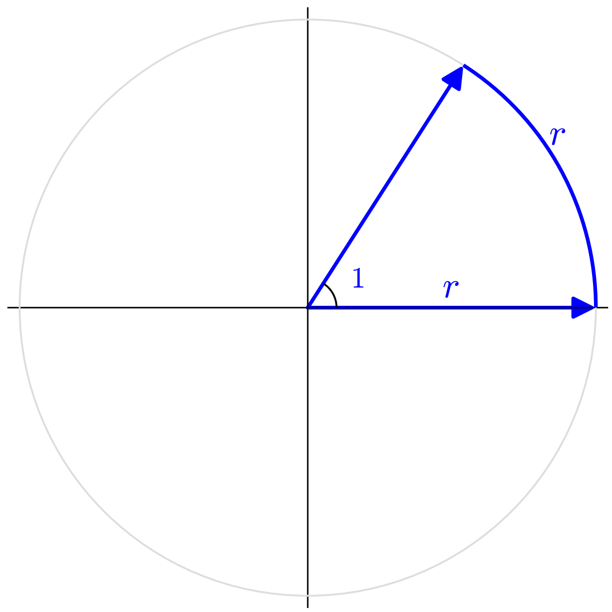
Definition 6.1.7 One **radian** is the measure of a central angle of a circle that intersects an arc the same length as the radius.



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Observation 6.1.8 Recall that the circumference of a circle is given by $C = 2\pi r$, where r is the radius of the circle. That means if we rotate through an entire circle, the circumference is $2\pi r$ which implies that the angle was 2π radians. Thus 2π radians is the same measure as 360°.

Degree and Radian Measure (TR1)

Activity 6.1.9 We now know that one turn around the circle measures 360° and also 2π radians. Use this information to set up a proportion to find the equivalent radian measure of the following angles that are given in degrees.

(a) 180°

A. $\frac{\pi}{4}$ B. π C. $\frac{3\pi}{4}$ D. $\frac{\pi}{2}$

Hint. Try setting up a proportion! $\frac{180°}{360°} = \frac{x}{2\pi}$

(b) 45°

A. $\frac{\pi}{4}$ B. π C. $\frac{3\pi}{4}$ D. $\frac{\pi}{2}$

Degree and Radian Measure (TR1)

Activity 6.1.10 Continue using the fact that one turn around the circle measures $360°$ and also 2π radians. Use this information to set up a proportion to find the equivalent degree measure of the following angles that are given in radians.

(a) $\dfrac{\pi}{2}$

A. $45°$ B. $90°$ C. $180°$ D. $360°$

Hint. Try setting up a proportion! $\dfrac{x}{360°} = \dfrac{\frac{\pi}{2}}{2\pi}$

(b) $\frac{3\pi}{4}$

A. $45°$ B. $90°$ C. $135°$ D. $180°$

Degree and Radian Measure (TR1)

Activity 6.1.11 We'll now use the proportions from before to come up with a way to convert between degrees and radians for any given angle. We'll call a the angle's measure in degrees and b the angle's measure in radians. So, we have the following proportion that must hold:

$$\frac{a}{360^\circ} = \frac{b}{2\pi}$$

(a) Let's say we know an angle's measure in degrees, a, and need to find the angle's measure in radians, b. Solve for b in the proportion.

(b) Use the formula you just developed to convert 60° to radians. Leave your answer in terms of π. Do not approximate!

(c) Now let's assume we know an angle's measure in radians, b, and need to find the angle's measure in degrees, a. Solve for a in the proportion.

(d) Use the formula you just developed to convert $\frac{\pi}{6}$ to degrees.

Remark 6.1.12 We now have a way to convert back and forth between the two types of measurements. If we know the angle's measure in degrees, we multiply it by $\dfrac{\pi}{180^\circ}$ to find the measure in radians. If we know the angle's measure in radians, we multiply it by $\dfrac{180^\circ}{\pi}$ to find the measure in degrees.

Degree and Radian Measure (TR1)

Activity 6.1.13 Convert each of the following angles.

(a) $\frac{2\pi}{3}$ radians to degrees

(b) $\frac{7\pi}{6}$ radians to degrees

(c) 240° to radians

(d) 315° to radians

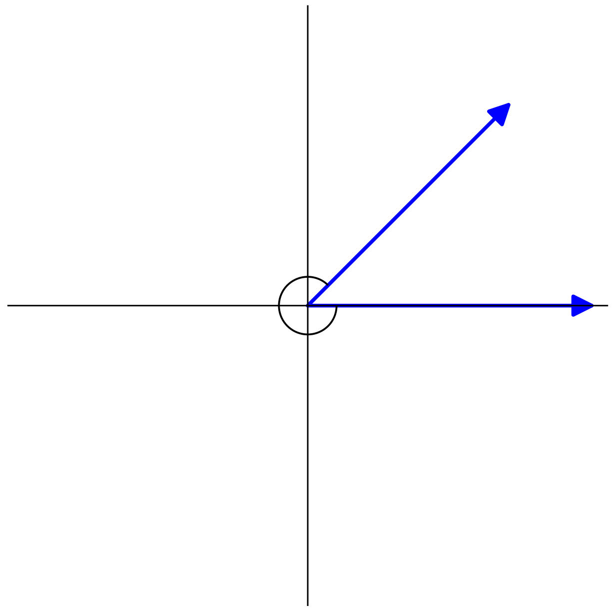
6.2 Angle Position and Arc Length (TR2)

Objectives

- Identify and find coterminal angles. Find the length of a circular arc.

Angle Position and Arc Length (TR2)

Activity 6.2.1 Consider the angle given below:



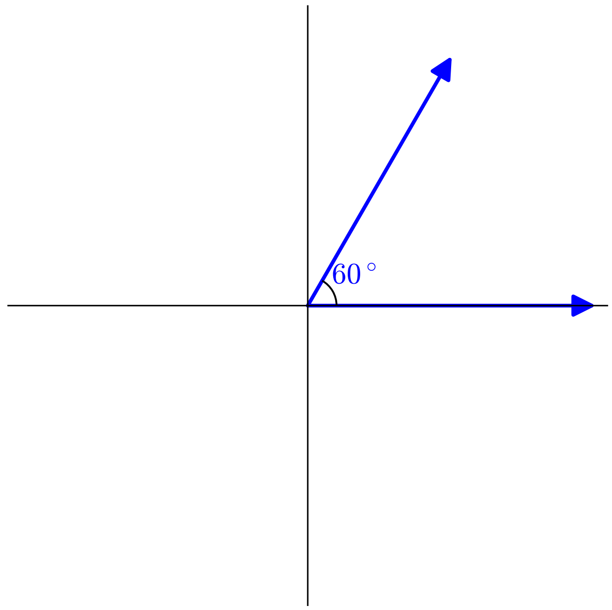
Which of the following angles describe the plotted angle?

A. $-45°$ B. $-135°$ C. $-225°$ D. $-315°$

Definition 6.2.2 Two angles are called **coterminal angles** if they have the same terminal side when drawn in standard position. ◊

Angle Position and Arc Length (TR2)

Activity 6.2.3 Consider the angle given below:



(a) Find two angles larger than $60°$ that are coterminal to $60°$.

(b) Find two angles smaller than $60°$ that are coterminal to $60°$.

Angle Position and Arc Length (TR2)

Observation 6.2.4 For any angle θ, the angle $\theta + k \cdot 360°$ is coterminal to θ for any integer k.

Remark 6.2.5 Since there are many coterminal angles for any given angle, it is convenient to systematically choose one for every angle. For a given angle, we typically choose the smallest positive coterminal angle to work with instead.

Angle Position and Arc Length (TR2)

Definition 6.2.6 If θ is an angle, there is a unique angle α with $0 \leq \alpha < 360°$ (or $0 \leq \alpha < 2\pi$) such that α and θ are coterminal. This angle α is called the **principal angle** of θ. $\Diamond$

Angle Position and Arc Length (TR2)

Activity 6.2.7 Find the principal angles for each of the following angles.

(a) $540°$

(b) $-600°$

(c) $\dfrac{11\pi}{3}$

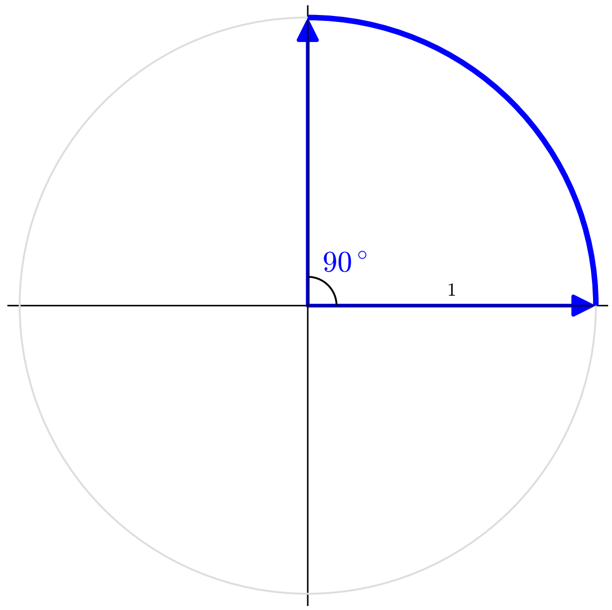
(d) $\dfrac{-7\pi}{5}$

Angle Position and Arc Length (TR2)

Remark 6.2.8 Recall that the circumference of a circle of radius r is $2\pi r$. We will use this to determine the lengths of arcs on a circle.

Angle Position and Arc Length (TR2)

Activity 6.2.9 Consider the portion of a circle of radius 1 graphed below.



(a) What is the circumference of an entire circle of radius 1?

A. $\frac{\pi}{2}$ B. π C. $\frac{3\pi}{2}$ D. 2π

(b) What portion of the entire circle is the sector graphed above?

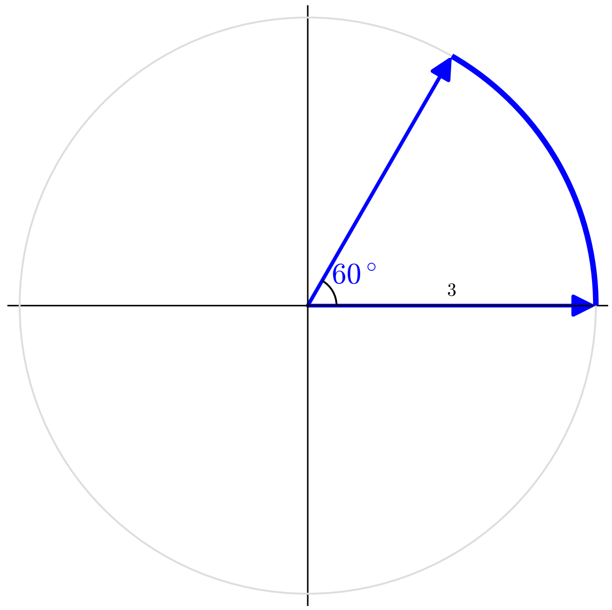
A. $\frac{1}{4}$ B. $\frac{1}{3}$ C. $\frac{1}{2}$ D. $\frac{3}{4}$

(c) Use proportions to determine the length of the arc displayed above.

A. $\frac{\pi}{2}$ B. π C. $\frac{3\pi}{2}$ D. 2π

Angle Position and Arc Length (TR2)

Activity 6.2.10 Consider the portion of a circle of radius 3 graphed below.



(a) What is the circumference of an entire circle of radius 3?

A. π B. 3π C. 6π D. 9π

(b) What portion of the entire circle is the sector graphed above?

A. $\frac{1}{12}$ B. $\frac{1}{6}$ C. $\frac{1}{4}$ D. $\frac{1}{3}$

(c) Use proportions to determine the length of the arc displayed above.

A. $\frac{\pi}{2}$ B. π C. 2π D. 3π

Angle Position and Arc Length (TR2)

Observation 6.2.11 For a sector of angle θ and radius r, we can use proportions to find the length of the arc, s. If θ is measured in degrees, we have $s = \frac{\theta}{360^\circ}(2\pi r)$, which simplifies to

$$s = \frac{\theta}{180^\circ}\pi r.$$

In radians, the formula is even nicer: $s = \frac{\theta}{2\pi}(2\pi r)$, which simplifies to

$$s = \theta r.$$

Angle Position and Arc Length (TR2)

Activity 6.2.12 Find the lengths of the arcs described below.

(a) The length of the arc of a sector of measure $120°$ of a circle of radius 4.

(b) The length of the arc of a sector of measure $270°$ of a circle of radius 2.

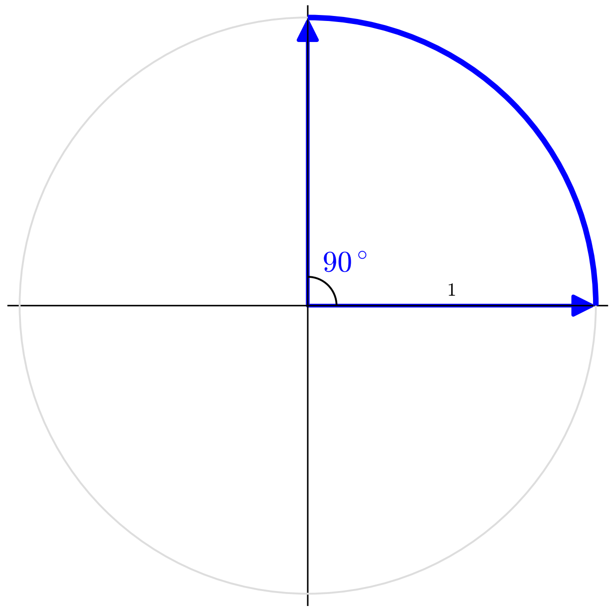
(c) The length of the arc of a sector of measure $\frac{5\pi}{6}$ of a circle of radius 3.

(d) The length of the arc of a sector of measure $\frac{11\pi}{12}$ of a circle of radius 6.

Remark 6.2.13 Recalling that the area of a circle of radius r is πr^2, we can use this same idea of proportions to find the area of a sector of a circle.

Angle Position and Arc Length (TR2)

Activity 6.2.14 Consider the portion of a circle of radius 1 graphed below.



(a) What is the area of an entire circle of radius 1?

A. $\frac{\pi}{2}$ B. π C. $\frac{3\pi}{2}$ D. 2π

(b) What portion of the entire circle is the sector graphed above?

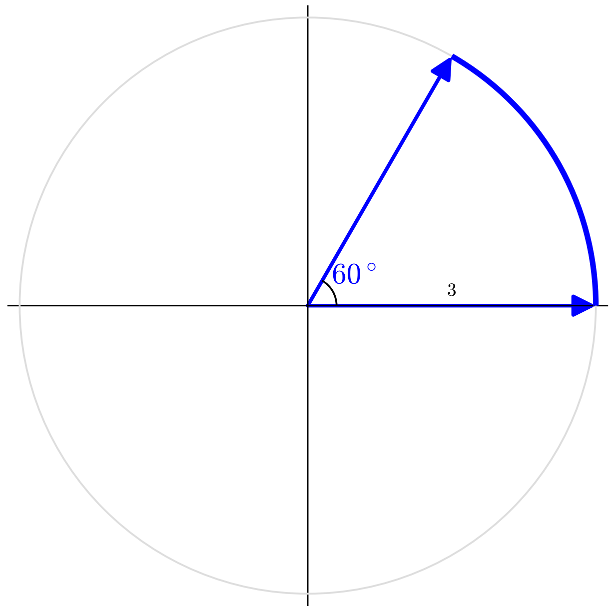
A. $\frac{1}{4}$ B. $\frac{1}{3}$ C. $\frac{1}{2}$ D. $\frac{3}{4}$

(c) Use proportions to determine the area of the arc displayed above.

A. $\frac{\pi}{4}$ B. $\frac{\pi}{2}$ C. π D. $\frac{3\pi}{2}$

Angle Position and Arc Length (TR2)

Activity 6.2.15 Consider the portion of a circle of radius 3 graphed below.



(a) What is the area of an entire circle of radius 3?

A. π B. 3π C. 6π D. 9π

(b) What portion of the entire circle is the sector graphed above?

A. $\frac{1}{12}$ B. $\frac{1}{6}$ C. $\frac{1}{4}$ D. $\frac{1}{3}$

(c) Use proportions to determine the area of the sector displayed above.

A. $\frac{\pi}{2}$ B. π C. $\frac{3\pi}{2}$ D. 2π

Observation 6.2.16 For a sector of angle θ and radius r, we can use proportions to find the area of the arc. If θ is measured in degrees, we have $A = \frac{\theta}{360°}\left(\pi r^2\right)$. In radians, the formula is even nicer: $A = \frac{\theta}{2\pi}\left(\pi r^2\right)$, which simplifies to

$$A = \frac{1}{2}\theta r^2.$$

Activity 6.2.17 Find the areas of each sector described below.

(a) The sector with central angle $120°$ in a circle of radius 4.

(b) The sector with central angle $270°$ in a circle of radius 2.

(c) The sector with central angle $\frac{5\pi}{6}$ in a circle of radius 3.

(d) The sector with central angle $\frac{11\pi}{12}$ in a circle of radius 6.

6.3 Trigonometric Ratios (TR3)

Objectives

- Use a right triangle to evaluate trigonometric functions.

Trigonometric Ratios (TR3)

Remark 6.3.1 In this section, we will learn how to use right triangles to evaluate trigonometric functions. Before doing that, however, let's review some key concepts of right triangles that can be helpful when solving.

Trigonometric Ratios (TR3)

Definition 6.3.2 Given a right triangle with legs of length a and b and hypotenuse of length c, the following equation holds:

$$a^2 + b^2 = c^2$$

This is called the **Pythagorean Theorem**. $\Diamond$

Remark 6.3.3 The Pythagorean Theorem is helpful because if we know the lengths of any two sides of a right triangle, we can always find the length of the third side.

Trigonometric Ratios (TR3)

Activity 6.3.4 Suppose two legs of a right triangle measure 3 inches and 4 inches.

(a) Draw a picture of this right triangle and label the sides. Use x to refer to the missing side.

(b) What is the value of x (i.e., the length of the third side)?

A. 5 inches

B. $\sqrt{5}$ inches

C. 25 inches

D. 16 inches

E. $\sqrt{7}$ inches

Trigonometric Ratios (TR3)

Activity 6.3.5 Suppose the hypotenuse of a right triangle is 13 cm long and one of the legs is 5 cm long.

(a) Draw a picture of this right triangle and label the sides. Use x to refer to the missing side.

(b) What is the value of x (i.e., the length of the third side)?

A. 144 B. 12 C. 194 D. 14

Definition 6.3.6 Pythagorean triples are integers a, b, and c that satisfy the Pythagorean Theorem. Activity 6.3.4 and Activity 6.3.5 highlight some of the most common types of Pythagorean triples: $3-4-5$ and $5-12-13$. All triangles similar to the $3-4-5$ triangle will also have side lengths that are multiples of $3-4-5$ (like $6-8-10$). Similarly, this is true for all triangles similar to the $5-12-13$ triangle. $\diamondsuit$

Trigonometric Ratios (TR3)

Activity 6.3.7 Suppose you are given a right triangle where the hypotenuse is 11 cm long and one of the interior angles is $60°$.

(a) Draw a picture of this right triangle and label the parts of the triangle given.

(b) What is the measure of the third angle?

A. $90°$ B. $60°$ C. $30°$ D. $180°$

(c) Suppose you are asked to find one of the sides of the right triangle. What additional information would you need to find the length of another side of the triangle?

Remark 6.3.8 When working with right triangles, it is often helpful to refer to specific angles and sides. One way this is done is by using letters, such as A and a to show that these are an angle-side pair because every angle has a side opposite the angle in a triangle. Note that the capital letter indicates the angle, and the lower case letter indicates the side.

Another way to label the sides of a triangle is to use the relationships between a given angle within a triangle and the sides.

- The **hypotenuse** of a right triangle is always the side opposite the right angle. This side also happens to be the longest side of the triangle.
- The **opposite side** is the non-hypotenuse side across from a given angle.
- The **adjacent side** is the non-hypotenuse side that is next to a given angle.

When given an angle, all sides of a triangle can be labeled from that angle's perspective. For example, from angle A's perspective, the sides of a right triangle are labeled as:

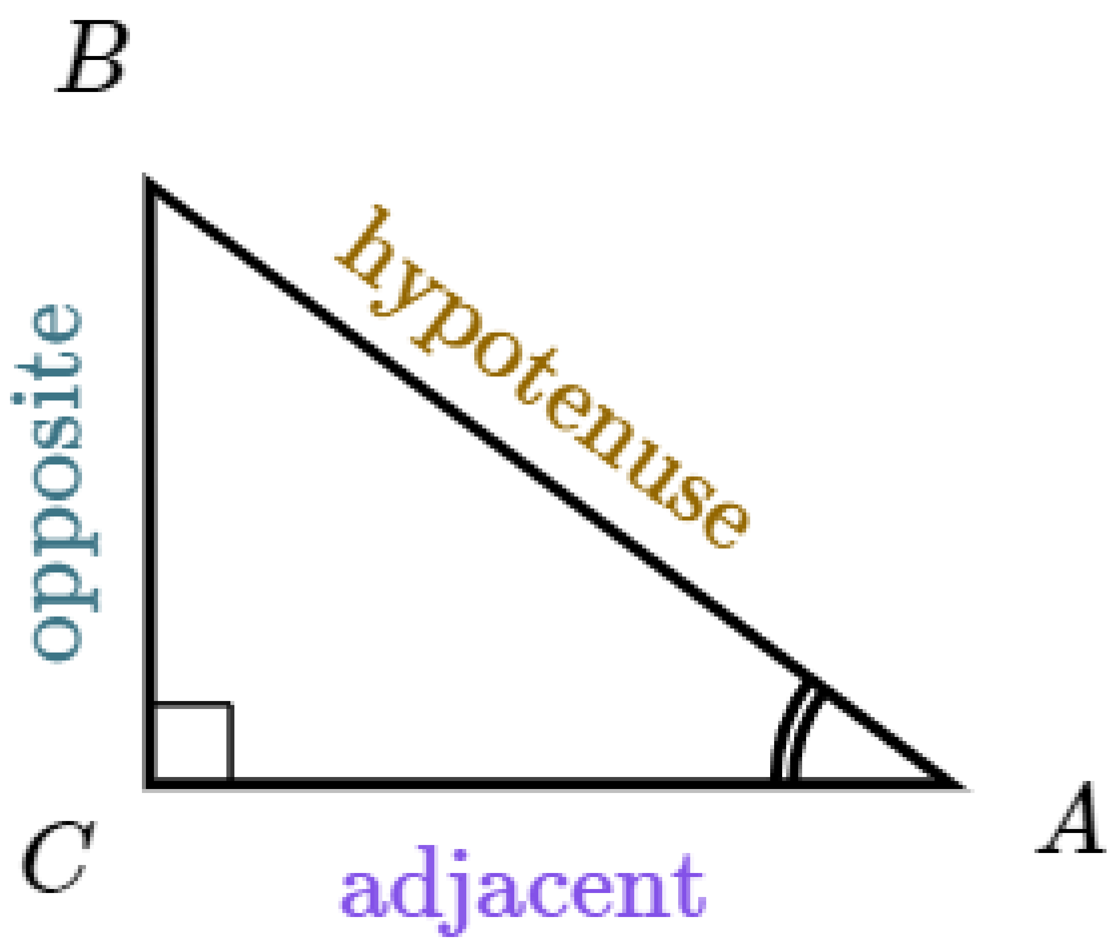


Figure 6.3.9 From the perspective of angle A, all sides of a right triangle can be labeled.

Trigonometric Ratios (TR3)

Definition 6.3.10 We will define **trigonometric functions** for a given angle θ as ratios between the side lengths of a right triangle. The first three trigonometric functions we will discuss are the sine function, the cosine function, and the tangent function.

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$$
$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$$
$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

Notice that these are defined according to the sides of a triangle from the perspective of an angle θ. This is why it is important to be able to label the triangle correctly! $\diamondsuit$

Trigonometric Ratios (TR3)

Activity 6.3.11 Let ABC be a right triangle with lengths $a = 35$, $b = 12$, and $c = 37$.

(a) Draw a right triangle and label it with angles A, B, and C, with c being the hypotenuse of the triangle.

(b) Find $\sin A$.

A. $\dfrac{12}{37}$ B. $\dfrac{35}{37}$ C. $\dfrac{35}{12}$

(c) Find $\cos A$.

A. $\dfrac{12}{37}$ B. $\dfrac{35}{37}$ C. $\dfrac{35}{12}$

(d) Find $\tan A$.

A. $\dfrac{12}{37}$ B. $\dfrac{35}{37}$ C. $\dfrac{35}{12}$

Trigonometric Ratios (TR3)

Definition 6.3.12 The three trigonometric ratios we have worked with so far (sine, cosine, and tangent) are referred to as the **basic trigonometric functions**. There are three additional functions, **cosecant**, **secant**, and **cotangent** that are found by taking the reciprocal of the basic trigonometric functions. These three ratios are referred to as the **reciprocal trigonometric functions** and can be defined as follows:

$$\csc\theta = \frac{\text{hypotenuse}}{\text{opposite}} = \frac{1}{\sin\theta}$$
$$\sec\theta = \frac{\text{hypotenuse}}{\text{adjacent}} = \frac{1}{\cos\theta}$$
$$\cot\theta = \frac{\text{adjacent}}{\text{opposite}} = \frac{1}{\tan\theta}$$

◊

Trigonometric Ratios (TR3)

Activity 6.3.13 Suppose you are given triangle ABC, where $a = 24$, $b = 32$, and $c = 40$, with c being the hypotenuse of the triangle.

(a) Find $\sin A$.

A. $\frac{4}{5}$ B. $\frac{3}{5}$ C. $\frac{5}{3}$ D. $\frac{5}{4}$

(b) Find $\csc A$.

A. $\frac{4}{5}$ B. $\frac{3}{5}$ C. $\frac{5}{3}$ D. $\frac{5}{4}$

(c) Find $\cos B$.

A. $\frac{4}{5}$ B. $\frac{3}{5}$ C. $\frac{5}{3}$ D. $\frac{5}{4}$

(d) Find $\sec B$.

A. $\frac{4}{5}$ B. $\frac{3}{5}$ C. $\frac{5}{3}$ D. $\frac{5}{4}$

(e) Find $\cot B$.

A. $\frac{4}{5}$ B. $\frac{4}{3}$ C. $\frac{5}{4}$ D. $\frac{3}{4}$

Trigonometric Ratios (TR3)

Activity 6.3.14 Suppose $\cos\theta = \dfrac{12}{13}$.

(a) Draw a triangle and label one of the angles θ. Then, label each side as ”opposite,” ”adjacent,” and ”hypotenuse.”

(b) Use the given information to determine the length of each side of the triangle.

(c) Find $\csc\theta$.

(d) Find $\cot\theta$

Trigonometric Ratios (TR3)

Activity 6.3.15 For each triangle given, determine which trigonometric function would be the most helpful in determining the length of the side of a triangle. Be sure to draw a picture of the triangle to help you determine the relationship between the given angle and sides. In each case assume angle C is the right angle.

(a) In triangle ABC, $B = 37°$ and $a = 11$. Which trigonometric function will best help determine the length of side c?

A. sine B. cosine C. tangent

(b) In triangle ABC, $A = 32°$ and $b = 13$. Which trigonometric function will best help determine the length of side a?

A. sine B. cosine C. tangent

(c) In triangle ABC, angle $A = 24°$ and the hypotenuse has length 14. Which trigonometric function will best help determine the length of side a?

A. sine B. cosine C. tangent

Trigonometric Ratios (TR3)

Activity 6.3.16 In triangle ABC, $B = 37°$, $a = 11$, and c is the hypotenuse.

(a) Draw a right triangle and label the angles/sides with the information given.

(b) Which trigonometric equation will best help determine the length of side b?

A. $\sin 37° = \dfrac{b}{11}$ B. $\tan 37° = \dfrac{b}{11}$ C. $\cos 37° = \dfrac{b}{11}$

(c) Use the equation from part (b) and solve for b. Round to the nearest tenth.

Hint. Multiply by 11 on both sides and make sure you are in degree mode on your calculator.

Trigonometric Ratios (TR3)

Activity 6.3.17 Sarah is standing 500 meters from the base of the Eiffel Tower. She looks at the top of the tower at an angle of 31°.

(a) Draw a diagram to represent the situation. Use x to refer to the missing side.

(b) Which trig function could we use to find the height of the tower?

A. sine

B. cosine

C. tangent

(c) How could we correctly set up the trigonometric function to find the height of the Eiffel Tower?

A. $\sin 31° = \frac{x}{500}$

B. $\cos 31° = \frac{500}{x}$

C. $\cos 31° = \frac{500}{x}$

D. $\tan 31° = \frac{x}{500}$

(d) Take your equation in part (c) and solve for x.

(e) Find the height of the tower to the nearest hundredth.

A. 428.58 meters

B. 300.43 meters

C. 220.85 meters

D. 257.52 meters

6.4 Special Right Triangles (TR4)

Objectives

- Find exact values of trigonometric functions of special angles (30, 45, and 60).

Special Right Triangles (TR4)

Remark 6.4.1 Recall from the previous section that we can find values of trigonometric functions of any angle. In this section, we can take what we know about right triangles and find exact values of trigonometric functions of special angles.

Special Right Triangles (TR4)

Definition 6.4.2 There are two special right triangle relationships that will continually appear when speaking about special angles.

- $45 - 45 - 90$ triangle
- $30 - 60 - 90$ triangle

$\diamondsuit$

Special Right Triangles (TR4)

Activity 6.4.3 Let's explore the relationship of the $45-45-90$ special right triangle.

(a) Draw a right triangle and label the angles to have $45°$, $45°$, and $90°$.

(b) In a $45-45-90$ triangle, two angles are the same size. If those two angles are the same size, what do we know about the sides opposite those angles?

(c) Suppose one of the legs of the right triangle is of length 1, how long is the other leg?

(d) Now that we know two sides of the right triangle, use that information and the Pythagorean Theorem to find the length of the third side.

Special Right Triangles (TR4)

Definition 6.4.4 From Activity 6.4.3, we saw that a $45-45-90$ triangle is an isosceles right triangle, which means that two of the sides of the triangle are equal. The ratio of its legs and hypotenuse is expressed as follows:

$$\text{Leg} : \text{Leg} : \text{Hypotenuse} = 1 : 1 : \sqrt{2}.$$

In terms of x, this ratio can be expressed as

$$x : x : x\sqrt{2}.$$

Therefore, the **45-45-90 triangle rule** states that the three sides of the triangle are in the ratio $x : x : x\sqrt{2}$. ◊

Special Right Triangles (TR4)

Activity 6.4.5 Suppose you are given a right triangle ABC, where $C = 45°$, $a = 6$ cm, and b is the hypotenuse.

(a) Draw a picture of this right triangle and label the sides.

(b) If we apply Definition 6.4.4, what would the length of c be?

(c) If we apply Definition 6.4.4, what would the length of the hypotenuse be?

Special Right Triangles (TR4)

Activity 6.4.6 For each of the following, use Definition 6.4.4 to find the missing side.

(a) In triangle ABC, $A = 90°$, $c = 3$, and $B = 45°$. What is the length of a?

A. 6 B. 5 C. 3 D. $3\sqrt{2}$

(b) In triangle ABC, $B = 90°$, $b = 6\sqrt{2}$, and $C = 45°$. What is the length of c?

A. 6 B. 5 C. 4 D. 12

(c) In triangle ABC, $B = 90°$, $b = 4$, and $A = 45°$. What is the length of c?

A. $\sqrt{6}$ B. $2\sqrt{2}$ C. 4 D. $4\sqrt{2}$

Special Right Triangles (TR4)

Activity 6.4.7 Suppose you are given an equilateral triangle, which has three equal sides and three equal angles ($60°$).

(a) Draw an equilateral triangle and then draw the height from the base of the triangle to the top angle. What kind of triangles did you just create when drawing the height?

(b) What do you notice about the relationships of the sides of the equilateral triangle to that of the $30 - 60 - 90$ triangles?

(c) Label the angles to have $30°$, $60°$, and $90°$ and the side opposite of the $30°$ angle as having a length of 1.

(d) Given that the side opposite the $30°$ angle has a length of 1, how long is the length of one side of the equilateral triangle?

(e) Now that you know the length of two sides of the $30-60-90$ triangles, find the length of the third side using the Pythagorean Theorem.

Special Right Triangles (TR4)

Definition 6.4.8 From Activity 6.4.7, we saw that if a triangle has angle measures 30°, 60°, and 90°, then the sides are in the ratio:

$$1 : \sqrt{3} : 2.$$

In terms of x, this ratio can be expressed as

$$x : x\sqrt{3} : 2x.$$

Therefore, the **30-60-90 triangle rule** states that the three sides of the triangle are in the ratio $x : x\sqrt{3} : 2x$. Note that the shorter leg is always x, the longer leg is always $x\sqrt{3}$, and the hypotenuse is always $2x$. ◊

Special Right Triangles (TR4)

Activity 6.4.9 Suppose you are given a right triangle ABC, where $C = 30°$, $c = 7$ cm, and b is the hypotenuse.

(a) Draw a picture of this right triangle and label the sides.

(b) If we apply Definition 6.4.8, what would the length of a be?

(c) If we apply Definition 6.4.8, what would the length of the hypotenuse be?

Special Right Triangles (TR4)

Activity 6.4.10 For each of the following, use Definition 6.4.8 to find the missing side.

(a) In triangle ABC, $B = 90,°$, $c = 6$, and $C = 30°$. What is the length of a?

A. $6\sqrt{3}$ B. 4 C. 3 D. $6\sqrt{2}$

(b) In triangle ABC, $C = 90,°$, $a = 4\sqrt{3}$, and $B = 30°$. What is the length of b?

A. 3 B. 6 C. $4\sqrt{2}$ D. 4

(c) In triangle ABC, $B = 90,°$, $a = 8\sqrt{3}$, and $A = 60°$. What is the length of b?

A. 4 B. 16 C. 12 D. $8\sqrt{2}$

Special Right Triangles (TR4)

Remark 6.4.11 Recall that we can find values of trigonometric functions of angles of right triangles. We can use the same idea to find values of trigonometric functions of angles of special right triangles.

Special Right Triangles (TR4)

Activity 6.4.12 Suppose you are given triangle ABC, where $C = 45°$, $a = 5$, and b is the hypotenuse.

(a) Find the measures of sides b and c.

(b) Now that you have found all the sides and angles of triangle ABC, find the ratio that represents $\tan A$.

A. $\dfrac{5}{5\sqrt{2}}$ B. $\dfrac{5\sqrt{2}}{5}$ C. $\sqrt{2}$ D. 1

(c) Find the ratio that represents $\sin A$.

A. $\dfrac{5}{5\sqrt{2}}$ B. $\dfrac{5\sqrt{2}}{5}$ C. $\dfrac{1}{\sqrt{2}}$ D. $\dfrac{\sqrt{2}}{2}$

(d) What is the approximate value of $\sin A$?

Remark 6.4.13

Some textbooks insist on **rationalizing the denominator** and will always convert to expressions like $\dfrac{\sqrt{2}}{2}$. However, it is useful to be able to use both, and to recognize that $\dfrac{1}{\sqrt{2}}$ and $\dfrac{\sqrt{2}}{2}$ are the same value.

In Activity 6.4.12, notice that when finding a value of a trigonometric ratio, sometimes the values are decimal approximations. For example, the ratio for $\sin A$ was $\frac{1}{\sqrt{2}}$ (or $\frac{\sqrt{2}}{2}$), which gives an approximate value of 0.707.

Special Right Triangles (TR4)

Definition 6.4.14 The **exact values of trigonometric functions** are values of trigonometric functions of certain angles that can be expressed exactly using expressions containing real numbers and roots of real numbers. When finding trigonometric ratios, we often give an exact value, rather than an approximation. ◊

Special Right Triangles (TR4)

Activity 6.4.15 Find the exact value of the trigonometric function you are asked to find for each of the following.

(a) Given the triangle, ABC, with $B = 90°$, $C = 45°$, and $a = 7$, find $\cos A$.

A. $\dfrac{7}{7\sqrt{2}}$ B. $\dfrac{7\sqrt{2}}{7}$ C. $\dfrac{1}{\sqrt{2}}$ D. $\dfrac{\sqrt{2}}{2}$

(b) Using your calculator, find the value of $\cos 45°$ to the nearest thousandth.

(c) Which of the following is equivalent to the value of $\cos 45°$ that you found in part (b)?

A. $\dfrac{7}{7\sqrt{2}}$ B. $\dfrac{7\sqrt{2}}{7}$ C. $\dfrac{1}{\sqrt{2}}$ D. $\dfrac{\sqrt{2}}{2}$

(d) How can using the $45 - 45 - 90$ triangle help us find the value of $\sin 45°$?

Special Right Triangles (TR4)

Remark 6.4.16 Notice that if you know the relationships of the sides of special right triangles, it can help you find the exact value of special angles (i.e., 30°, 45°, and 60°). In addition, notice that the trig values of the angles were the same regardless of the size of the triangle.

Special Right Triangles (TR4)

Activity 6.4.17 For each of the following, find the exact value of the trigonometric function. Use the $45-45-90$ and $30-60-90$ trigonometric rules to help you:

$$1 : 1 : \sqrt{2}$$

$$1 : \sqrt{3} : 2$$

(a) What is $\tan 45°$?

A. $\dfrac{1}{\sqrt{2}}$ B. $\sqrt{2}$ C. 1 D. $\dfrac{\sqrt{2}}{2}$

(b) What is $\cos 60°$?

A. $\dfrac{1}{2}$ B. $\sqrt{3}$ C. 2 D. $\dfrac{\sqrt{3}}{2}$

(c) What is $\sin 30°$?

A. $\dfrac{\sqrt{3}}{2}$ B. $\sqrt{3}$ C. 2 D. $\dfrac{1}{2}$

(d) What is $\cos 45°$?

A. $\dfrac{1}{\sqrt{2}}$ B. $\sqrt{2}$ C. 1 D. $\dfrac{\sqrt{2}}{2}$

(e) Now that you have determined $\tan 45°$ (part a), what is $\cot 45°$?

(f) What is $\csc 30°$?

(g) What is $\sec 30°$?

6.5 The Unit Circle (TR5)

Objectives

- Use reference angles, signs and definitions to determine exact values of trigonometric functions.

The Unit Circle (TR5)

Remark 6.5.1 In Section 6.4, we learned how to find the exact values of the six trigonometric ratios for the special acute angles 30°, 45°, and 60°. In this section, we will use that knowledge and expand to finding the exact trig values of any multiple of those angles.

The Unit Circle (TR5)

Definition 6.5.2 The **unit circle** is the circle of radius 1 centered at the origin on the coordinate plane.

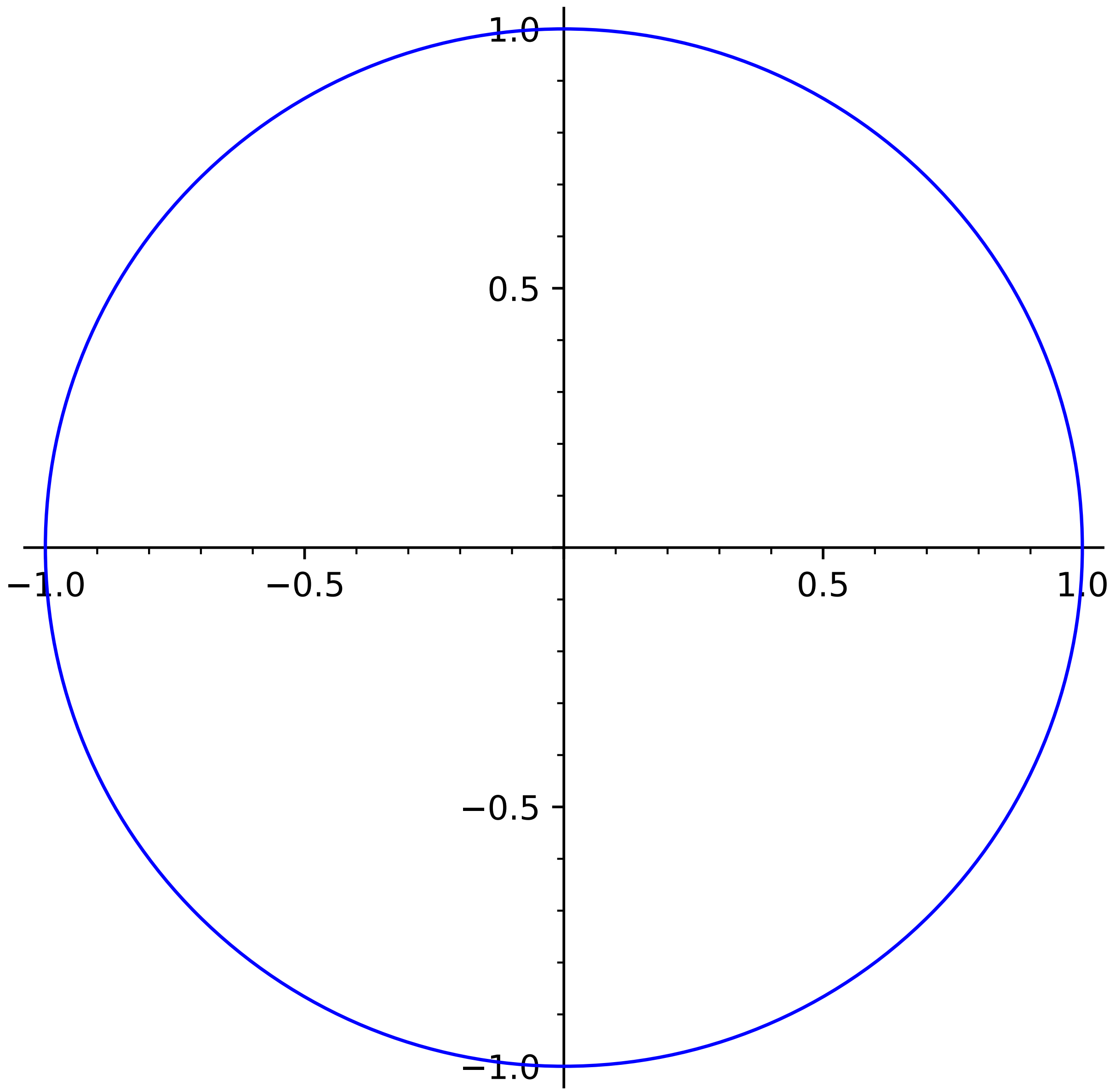
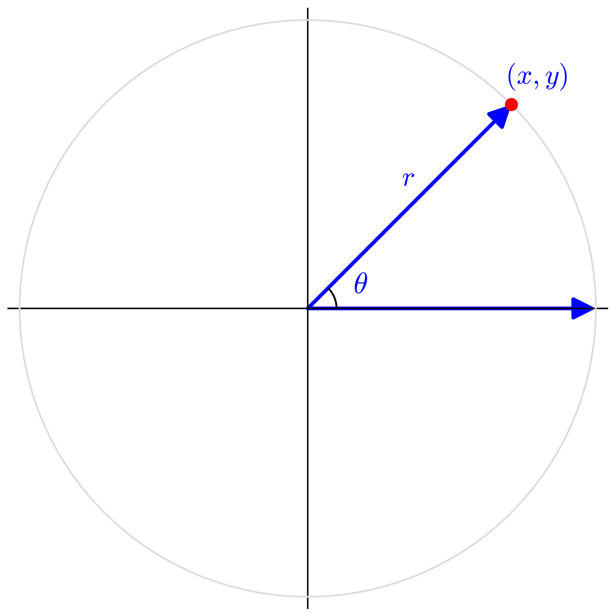


Figure 6.5.3

◊

The Unit Circle (TR5)

Activity 6.5.4 Let θ be the angle shown below in standard form. Notice that the terminal side intersects with the unit circle. (Note: We will assume a circle drawn in this context is the unit circle unless told otherwise.) We will label that point of intersection as (x, y).



(a) What is the length of line segment r, whose endpoints are the origin and the point (x, y) ?

A. 1

B. 2

C. 3

D. cannot be determined

(b) We will now create a right triangle using the previous line segment r as the hypotenuse. Draw in a line segment of length x and another of length y to create such a triangle.

(c) Using the triangle you've just created, find $\cos\theta$.

A. $\frac{x}{y}$ B. $\frac{1}{x}$ C. $\frac{x}{1}$ D. $\frac{1}{y}$ E. $\frac{y}{1}$

(d) Using that same triangle, find $\sin\theta$.

A. $\frac{x}{y}$ B. $\frac{1}{x}$ C. $\frac{x}{1}$ D. $\frac{1}{y}$ E. $\frac{y}{1}$

(e) Solve for x in one of the equations you've found above to determine an expression for the x-value of the point (x, y) .

A. $y\cos\theta$ B. $y\sin\theta$ C. $\cos\theta$ D. $\sin\theta$ E. $\frac{1}{\cos\theta}$ F. $\frac{1}{\sin\theta}$

The Unit Circle (TR5)

(f) Solve for y in one of the equations you've found above to determine an expression for the y-value of the point (x, y) .

A. $y \cos\theta$ B. $y \sin\theta$ C. $\cos\theta$ D. $\sin\theta$ E. $\dfrac{1}{\cos\theta}$ F. $\dfrac{1}{\sin\theta}$

The Unit Circle (TR5)

Remark 6.5.5 From the previous activity, we have found a connection between the sine and cosine values of an angle θ and the coordinates (x, y) of the point at which that angle intersects the unit circle. Namely,

$$x = \cos\theta \text{ and } y = \sin\theta$$

The Unit Circle (TR5)

Activity 6.5.6 Consider each angle θ given below. Find the coordinates (x, y) for the point at which θ intersects the unit circle.

(a) $\theta = \frac{\pi}{4}$

A. $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

B. $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

C. $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$

D. $(0, 1)$

E. $(1, 0)$

(b) $\theta = 30°$

A. $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

B. $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

C. $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$

D. $(0, 1)$

E. $(1, 0)$

(c) $\theta = \frac{\pi}{3}$

A. $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

B. $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

C. $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$

D. $(0, 1)$

E. $(1, 0)$

(d) $\theta = 0°$

A. $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

B. $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

C. $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$

D. $(0, 1)$

E. $(1, 0)$

(e) $\theta = \frac{\pi}{2}$

A. $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

B. $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

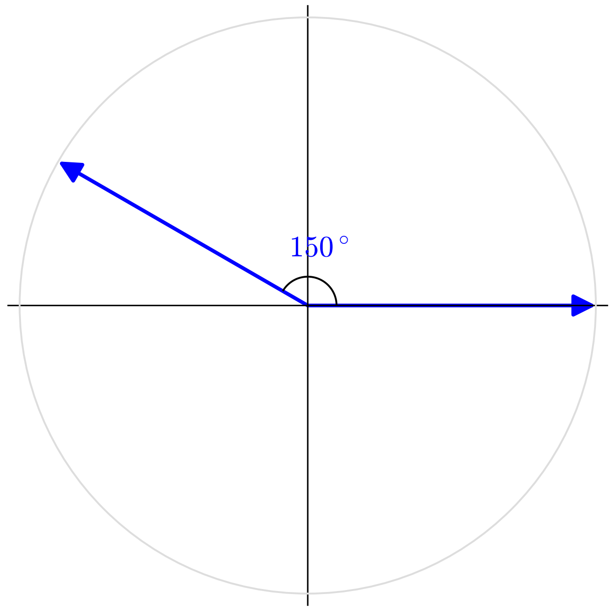
C. $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$

D. $(0, 1)$

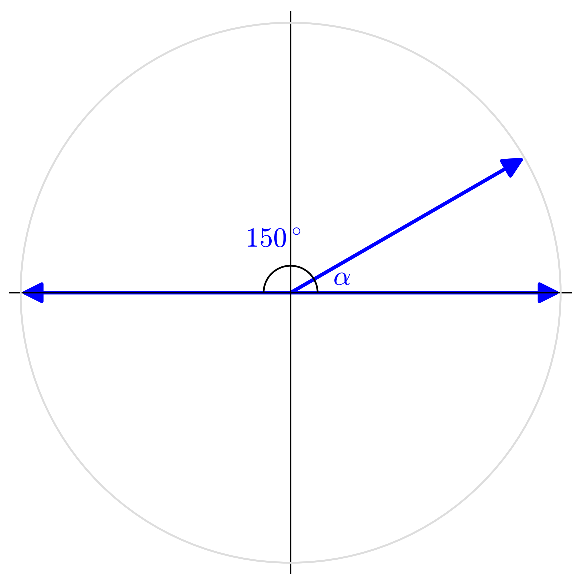
E. $(1, 0)$

Remark 6.5.7 In Activity 6.5.6, we found (x, y)-coordinates (and thus the sine and cosine values) for angles that terminated either in Quadrant 1 or on an axis adjacent to Quadrant 1. We'll now expand to angles that terminate elsewhere, using our knowledge of the cosine and sine values of angles in the first quadrant along with how reflections over the x and y axes affect the signs of the coordinates. (See Section 2.4 for a reminder on how these reflections work.)

Activity 6.5.8 Let's consider the angle $\theta = 150°$, drawn below with the unit circle.



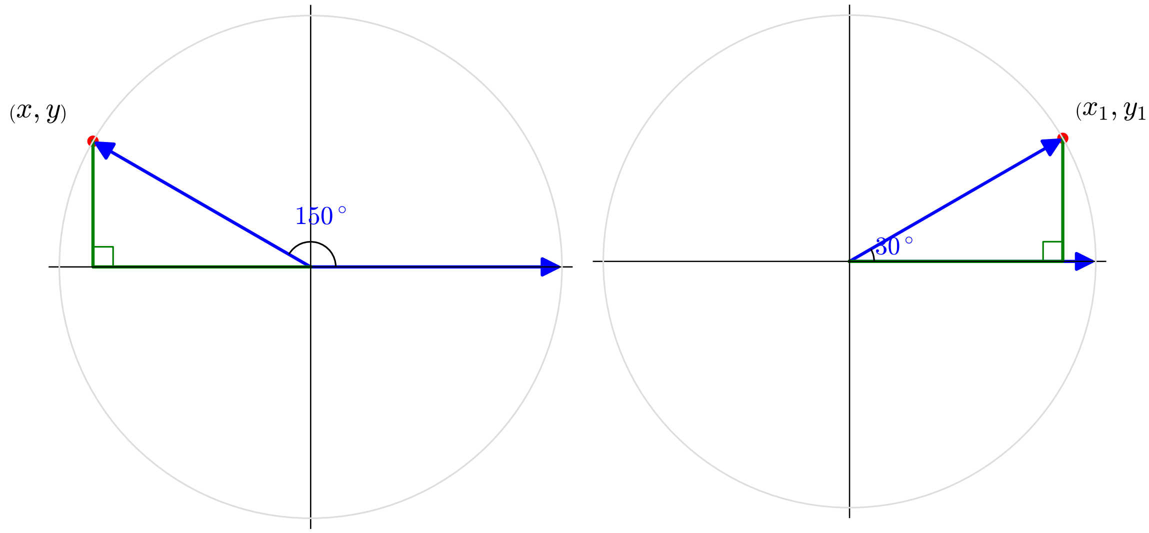
(a) If we reflect this angle across the y-axis, we can obtain an angle α in the first quadrant. What is the measure of α?



A. 0° B. 30° C. 45° D. 60° E. 75°

(b) We can find the sine and cosines values of our original angle, $\theta = 150°$, by using the angle $\alpha = 30°$ to help. Find the point (x_1, y_1), where the terminal side of the 30° angle intersects the unit circle.

The Unit Circle (TR5)



A. $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

B. $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$

C. $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

D. $\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$

E. $\left(\frac{-\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

(c) How does the point you've just found compare with the point (x, y), where the terminal edge of $\theta = 150°$ intersects the unit circle?

A. The x-values and the y-values are switched with each other.

B. The x-values will be the same, but the y-values will have opposite signs.

C. The y-values will be the same, but the x-values will have opposite signs.

D. The x-values and the y-values will both have opposite signs.

(d) What are the cosine and sine values of $\theta = 150°$?

A. $\cos 150° = \frac{\sqrt{3}}{2}$ and $\sin 150° = \frac{1}{2}$

B. $\cos 150° = \frac{1}{2}$ and $\sin 150° = \frac{\sqrt{3}}{2}$

C. $\cos 150° = \frac{1}{2}$ and $\sin 150° = -\frac{\sqrt{3}}{2}$

D. $\cos 150° = -\frac{\sqrt{3}}{2}$ and $\sin 150° = \frac{1}{2}$

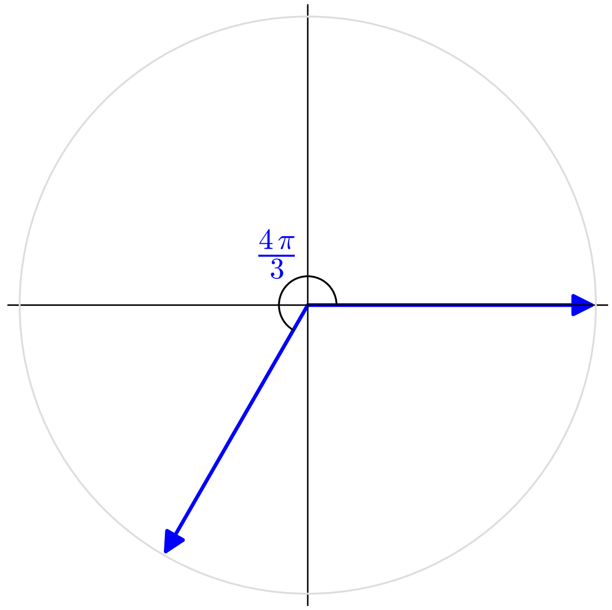
E. $\cos 150° = \frac{\sqrt{3}}{2}$ and $\sin 150° = -\frac{1}{2}$

The Unit Circle (TR5)

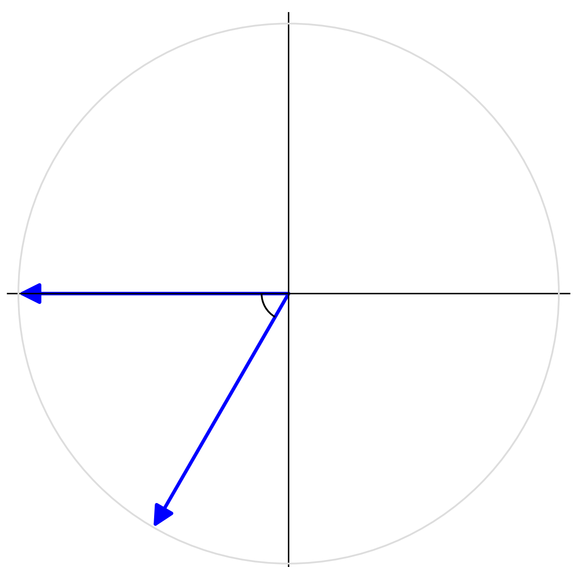
Definition 6.5.9 The **reference angle** for a given angle θ is the angle in the first quadrant obtained from reflecting θ. Equivalently, it is the smallest angle between the terminal side of θ and the x-axis. $\diamondsuit$

The Unit Circle (TR5)

Activity 6.5.10 Let's consider the angle $\theta = \dfrac{4\pi}{3}$, drawn below with the unit circle.



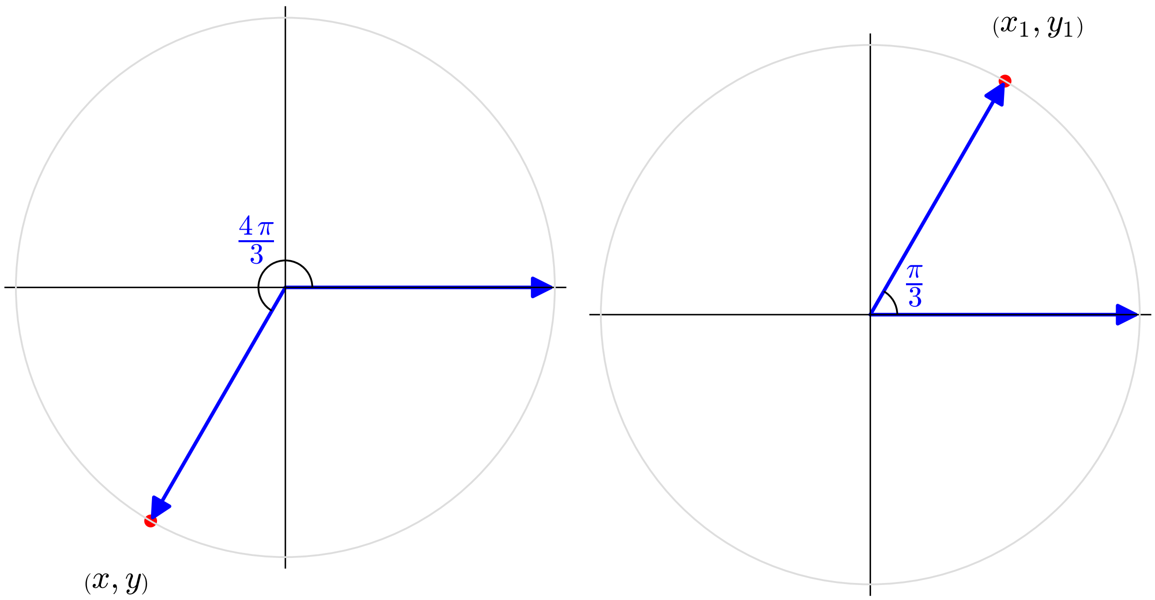
(a) The angle below represents the reference angle for $\theta = \dfrac{4\pi}{3}$, which is the smallest angle between the terminal side of θ and the x-axis. What is the measure of this reference angle?



A. $\dfrac{\pi}{2}$ B. $\dfrac{\pi}{3}$ C. $\dfrac{\pi}{4}$ D. $\dfrac{\pi}{5}$ E. $\dfrac{\pi}{6}$

(b) We can find the sine and cosines values of our original angle, $\theta = \dfrac{4\pi}{3}$, by using the reference angle to help. Find the point (x_1, y_1), where the terminal side of the angle $\dfrac{\pi}{3}$ intersects the unit circle.

The Unit Circle (TR5)



A. $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

B. $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$

C. $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

D. $\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$

E. $\left(\frac{-\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

(c) How does the point you've just found compare with the point (x, y), where the terminal edge of $\theta = \dfrac{4\pi}{3}$ intersects the unit circle?

A. The x-values and the y-values are switched with each other.

B. The x-values will be the same, but the y-values will have opposite signs.

C. The y-values will be the same, but the x-values will have opposite signs.

D. The x-values and the y-values will both have opposite signs.

(d) What are the cosine and sine values of $\theta = \dfrac{4\pi}{3}$?

A. $\cos\frac{4\pi}{3} = \frac{\sqrt{3}}{2}$ and $\sin\frac{4\pi}{3} = \frac{1}{2}$

B. $\cos\frac{4\pi}{3} = -\frac{1}{2}$ and $\sin\frac{4\pi}{3} = -\frac{\sqrt{3}}{2}$

C. $\cos\frac{4\pi}{3} = \frac{1}{2}$ and $\sin\frac{4\pi}{3} = -\frac{\sqrt{3}}{2}$

D. $\cos\frac{4\pi}{3} = -\frac{\sqrt{3}}{2}$ and $\sin\frac{4\pi}{3} = \frac{1}{2}$

E. $\cos\frac{4\pi}{3} = \frac{\sqrt{3}}{2}$ and $\sin\frac{4\pi}{3} = -\frac{1}{2}$

The Unit Circle (TR5)

Activity 6.5.11 Find $\sin\theta$ and $\cos\theta$ for each angle given.

(a) $\theta = \dfrac{\pi}{4}$

(b) $\theta = \dfrac{2\pi}{3}$

(c) $\theta = \dfrac{11\pi}{6}$

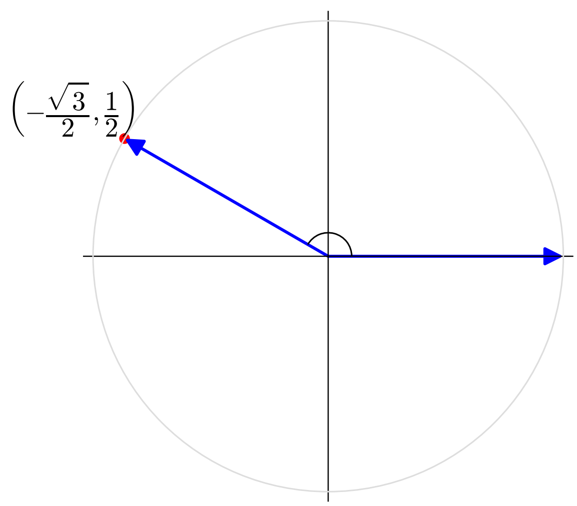
(d) $\theta = 135^\circ$

The Unit Circle (TR5)

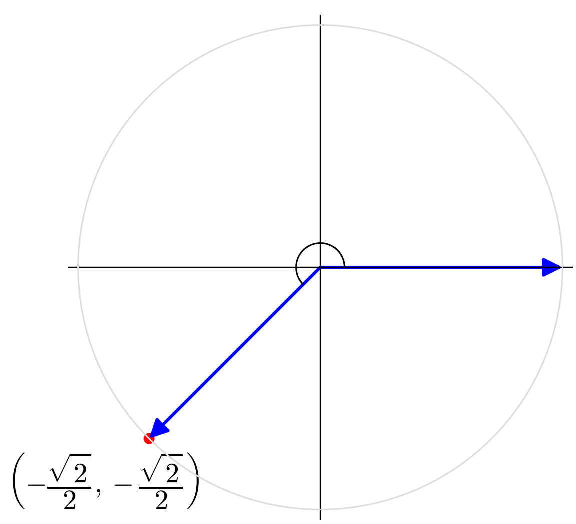
Activity 6.5.12 Find the following for each angle graphed below.

- θ in radians and degrees
- $\sin\theta$
- $\cos\theta$
- $\tan\theta$
- $\sec\theta$
- $\csc\theta$
- $\cot\theta$

(a)



(b)



The Unit Circle (TR5)

Remark 6.5.13 So far we've only dealt with angles that are part of a special right triangle (30-60-90 or 45-45-90) or are a multiple of one of these angles, but we can extend to other angles as well.

The Unit Circle (TR5)

Activity 6.5.14 A point (x, y) lies on the unit circle in Quadrant IV. Its x-coordinate is $\frac{3}{4}$.

(a) Draw a sketch of the angle θ whose terminal side intersects the unit circle as described above.

(b) What sign will the y-coordinate be?

A. positive

B. negative

(c) Find the exact value of the y-coordinate.

A. $\frac{7}{16}$

B. $-\frac{7}{16}$

C. $\frac{\sqrt{7}}{4}$

D. $-\frac{\sqrt{7}}{4}$

Hint. Use the Pythagorean Theorem to help.

(d) Find $\sin\theta$, $\cos\theta$, $\tan\theta$, $\sec\theta$, $\csc\theta$, and $\cot\theta$.

The Unit Circle (TR5)

Activity 6.5.15 Let θ be the angle whose terminal side intersects the unit circle at the point described in each situation below. Find $\sin\theta$, $\cos\theta$, $\tan\theta$, $\sec\theta$, $\csc\theta$, and $\cot\theta$.

(a) The point $\left(\frac{4}{5}, y\right)$ that lies on the unit circle in Quadrant I.

(b) The point $\left(-\frac{35}{37}, \frac{12}{37}\right)$.

(c) The point $\left(x, -\frac{\sqrt{11}}{6}\right)$ that lies on the unit circle in Quadrant III.

(d) The point $\left(\frac{12}{13}, -\frac{5}{13}\right)$.

Chapter 7

Periodic Functions (PF)

Objectives

How can we understand trigonometric ratios as functions?
By the end of this chapter, you should be able to...

1. Determine the basic properties of the graphs of sine and cosine, including amplitude, period, and phase shift.
2. Graph trigonometric functions including tangent, cotangent, secant, and cosecant functions and transformations of these functions, and determine the domain and range.
3. Determine the inverse sine, cosine, and tangent values; graph inverse trig functions and determine the limitations on the domain and range.

7.1 Properties of Sine and Cosine Graphs (PF1)

Objectives

- Determine the basic properties of the graphs of sine and cosine, including amplitude, period, and phase shift.

Properties of Sine and Cosine Graphs (PF1)

Remark 7.1.1 In the last module, we learned about finding values of trigonometric functions. Now, we will learn about the graphs of these functions.

Properties of Sine and Cosine Graphs (PF1)

Activity 7.1.2 We'll begin with the graph of the sine function, $f(x) = \sin x$.

(a) Fill in the missing values in the table below for $f(x) = \sin x$. Find the exact values, then express as a decimal, approximated to two decimal places if needed. (Notice that the values in the table are all the standard angles found on the unit circle!)

x	$\sin x$ **(exact)**	$\sin x$ **(as a decimal)**
0		
$\frac{\pi}{6}$	$\frac{1}{2}$	
$\frac{\pi}{4}$		≈ 0.71
$\frac{\pi}{3}$		
$\frac{\pi}{2}$		1
$\frac{2\pi}{3}$	$\frac{\sqrt{3}}{2}$	
$\frac{3\pi}{4}$		
$\frac{5\pi}{6}$		0.5
π		
$\frac{7\pi}{6}$	$-\frac{1}{2}$	
$\frac{5\pi}{4}$	$\frac{\sqrt{2}}{2}$	
$\frac{4\pi}{3}$		≈ -0.87
$\frac{3\pi}{2}$		
$\frac{5\pi}{3}$	$-\frac{\sqrt{3}}{2}$	
$\frac{7\pi}{4}$		
$\frac{11\pi}{6}$		-0.5
2π		

(b) Plot these values on a coordinate plane to approximate the graph of $f(x) = \sin x$. Then sketch in the graph of the sine curve using the points as a guide.

(c) What is the range of the function?

Properties of Sine and Cosine Graphs (PF1)

Activity 7.1.3 Let's change our function a bit and look at $g(x) = 3\sin x$.

(a) Fill in the table below.

x	$f(x) = \sin x$	$g(x) = 3\sin x$
0		
$\frac{\pi}{6}$		
$\frac{\pi}{4}$		
$\frac{\pi}{3}$		
$\frac{\pi}{2}$		

(b) Which of the following best describes how $g(x)$ is related to $f(x) = \sin x$?

A. The x-values in $g(x)$ are three times the x-values of $f(x)$.

B. The x-values in $g(x)$ are one third of the x-values of $f(x)$.

C. The y-values in $g(x)$ are three times the y-values of $f(x)$.

D. The y-values in $g(x)$ are one third of the y-values of $f(x)$.

(c) What is the range of $g(x)$?

Properties of Sine and Cosine Graphs (PF1)

Definition 7.1.4 The **amplitude** of a sine curve is vertical distance from the center of the curve to the maximum (or minimum) value.

We can also think of the **amplitude** as the value of the vertical stretch or compression.

When written as a function $f(x) = A \sin x$, the **amplitude** is $|A|$. $\Diamond$

Properties of Sine and Cosine Graphs (PF1)

Activity 7.1.5

(a) We only found $f(x) = \sin x$ for some values of x in the table in Activity 7.1.2, but those did not represent the entire domain. For which values of x can you find $\sin x$? (That is, what is the domain of $f(x) = \sin x$?)

(b) Coterminal angles will have the same sine values. How do we know if two angles are coterminal?

A. The difference between them is a multiple of $\frac{\pi}{2}$.

B. The difference between them is a multiple of π.

C. The difference between them is a multiple of $\frac{3\pi}{2}$.

D. The difference between them is a multiple of 2π.

(c) How often will the sine values repeat?

A. Every $\frac{\pi}{2}$ radians.

B. Every π radians.

C. Every $\frac{2\pi}{2}$ radians.

D. Every 2π radians.

(d) Extend the graph you made in Activity 7.1.2 in both the positive and negative direction to show the repeated sine values.

Properties of Sine and Cosine Graphs (PF1)

Definition 7.1.6 The **period** of a sine function is the minimum value for which the y-values begin repeating.

The **period** for $f(x) = \sin x$, the standard sine curve, is 2π. $\Diamond$

Activity 7.1.7 Now let's look at $h(x) = \sin 2x$.

(a) Think back to the types of transformations a function can have. (See Section 2.4 if you need a reminder!) What kind of transformation is happening in $h(x)$ compared the parent function $f(x) = \sin x$?

A. A vertical stretch/compression.

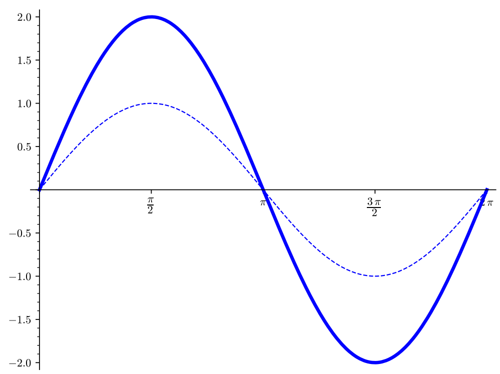
B. A horizontal stretch/compression.

C. A vertical shift.

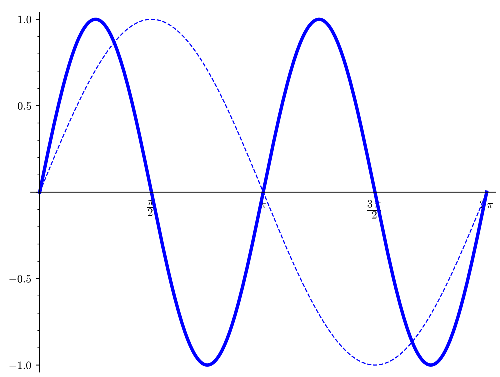
D. A horizontal shift.

(b) Which of the following graphs represents one cycle of $h(x) = \sin 2x$. (To help compare the functions, one cycle of $f(x) = \sin x$ is shown as a dashed line on each graph.)

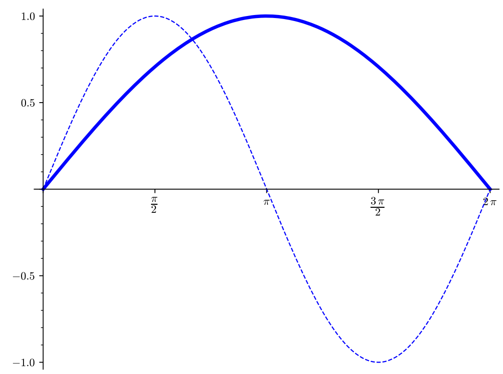
A.



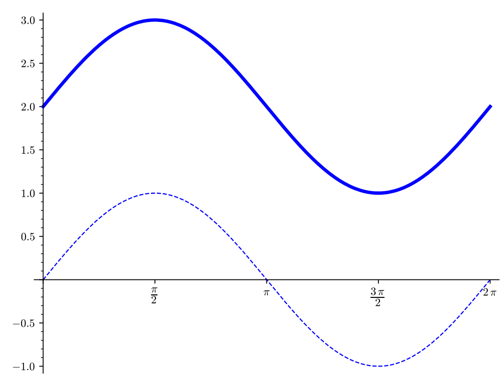
B.



C.



D.



Properties of Sine and Cosine Graphs (PF1)

Activity 7.1.8 Consider $j(x) = \sin \frac{1}{2}x$.

(a) What type of transformation is happening in $j(x)$ compared the parent function $f(x) = \sin x$?

A. A vertical stretch/compression.

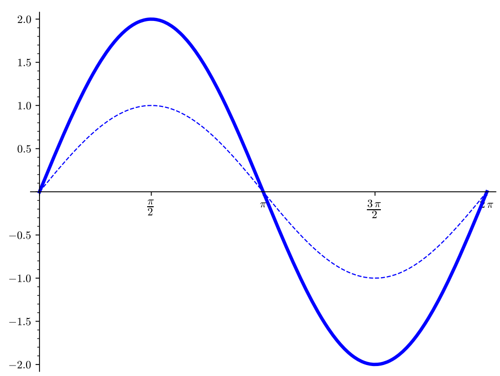
B. A horizontal stretch/compression.

C. A vertical shift.

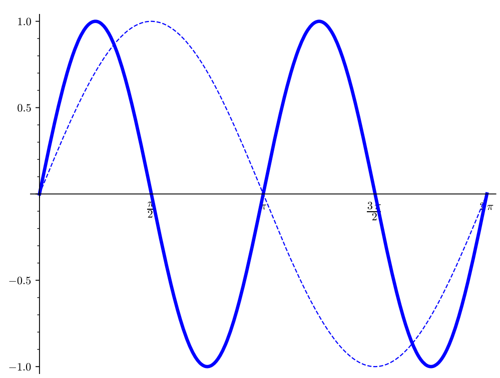
D. A horizontal shift.

(b) Which of the following graphs represents one cycle of $j(x) = \sin \frac{1}{2}x$. (To help compare the functions, one cycle of $f(x) = \sin x$ is shown as a dashed line on each graph.)

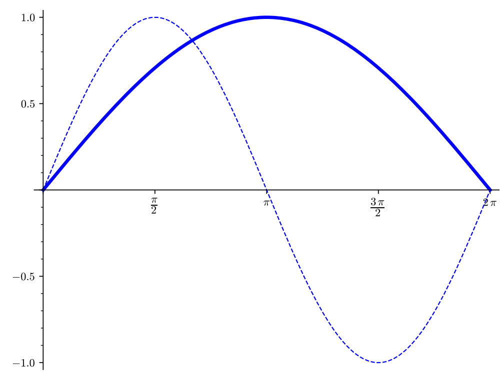
A.



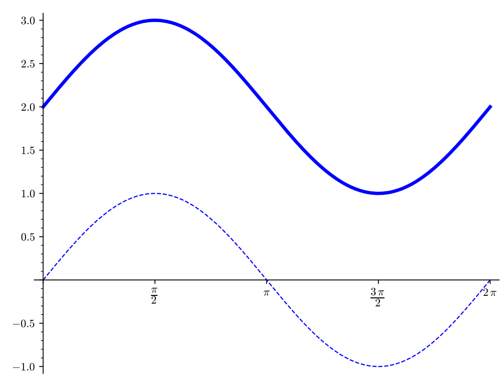
B.



C.



D.



Properties of Sine and Cosine Graphs (PF1)

Remark 7.1.9 When written as a function $f(x) = \sin Bx$, the **period** is $\dfrac{2\pi}{|B|}$.

Properties of Sine and Cosine Graphs (PF1)

Activity 7.1.10 Consider $k(x) = \sin\left(x + \frac{\pi}{2}\right)$.

(a) What type of transformation is happening in $k(x)$ compared the parent function $f(x) = \sin x$?

A. A vertical stretch/compression.

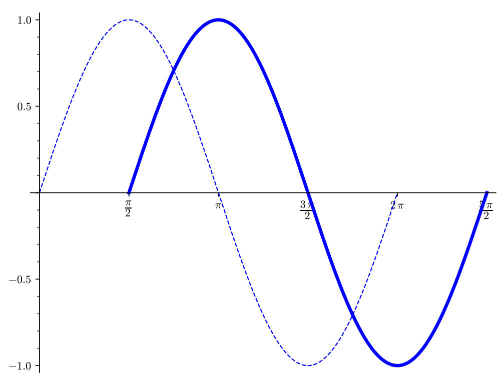
B. A horizontal stretch/compression.

C. A vertical shift.

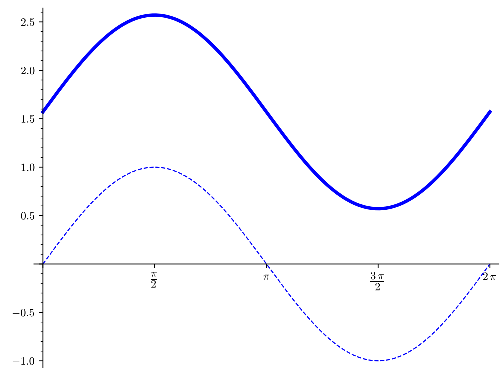
D. A horizontal shift.

(b) Which of the following graphs represents one cycle of $k(x) = \sin\left(x + \frac{\pi}{2}\right)$. (To help compare the functions, one cycle of $f(x) = \sin x$ is shown as a dashed line on each graph.)

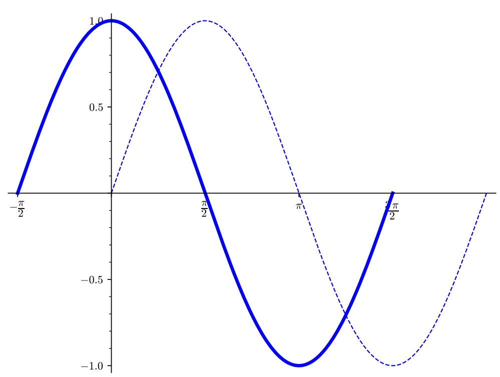
A.



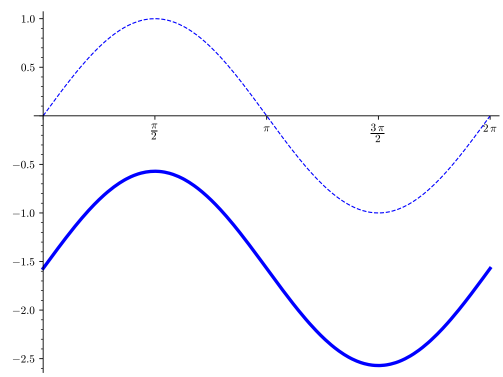
C.



B.



D.



Definition 7.1.11 The **phase shift** is the amount which a sine function is shifted horizontally from the standard sine curve.

The **phase shift** for $f(x) = \sin(x - C)$ is C, or C units to the right. The **phase shift** for $f(x) = \sin(x + C)$ is $-C$, or C units to the left.

A function can have both a horizontal shift and a change in period. In that case, it could be written as $f(x) = \sin(Bx - C)$. Here the phase shift would be $\dfrac{C}{B}$. You can think of solving the equation $Bx - C = 0$ for x. A positive value would represent a shift to the right and a negative value would represent a shift to the left. $\Diamond$

Activity 7.1.12 Let's now turn our focus to the cosine function, $f(x) = \cos x$.

(a) Fill in the missing values in the table below for $f(x) = \cos x$. Find the exact values, then express as a decimal, approximated to two decimal places if needed. (Notice that the values in the table are all the standard angles found on the unit circle!)

x	$\cos x$ **(exact)**	$\cos x$ **(as a decimal)**
0		
$\frac{\pi}{6}$	$\frac{\sqrt{3}}{2}$	
$\frac{\pi}{4}$		≈ 0.71
$\frac{\pi}{3}$		
$\frac{\pi}{2}$		0
$\frac{2\pi}{3}$	$\frac{-1}{2}$	
$\frac{3\pi}{4}$		
$\frac{5\pi}{6}$		$\approx -.87$
π		
$\frac{7\pi}{6}$	$-\frac{\sqrt{3}}{2}$	
$\frac{5\pi}{4}$	$\frac{-\sqrt{2}}{2}$	
$\frac{4\pi}{3}$		-0.5
$\frac{3\pi}{2}$		
$\frac{5\pi}{3}$	$\frac{1}{2}$	
$\frac{7\pi}{4}$		
$\frac{11\pi}{6}$		≈ 0.87
2π		

(b) Plot these values on a coordinate plane to approximate the graph of $f(x) = \cos x$. Then sketch in the graph of the cosine curve using the points as a guide.

(c) What is the range of the function?

Properties of Sine and Cosine Graphs (PF1)

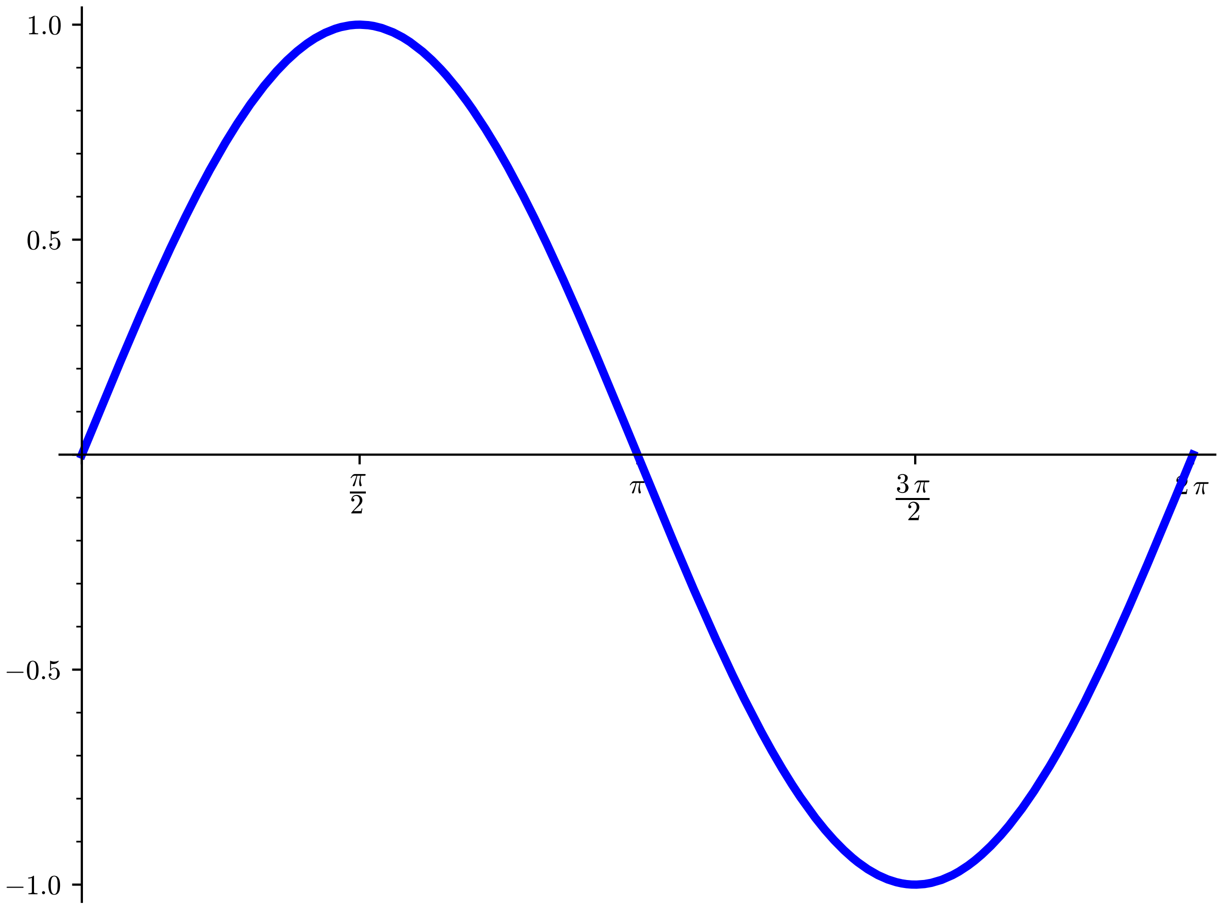
Observation 7.1.13 The cosine function, $f(x) = \cos x$, is equivalent to the sine function shifted to the left $\frac{\pi}{2}$ units, $g(x) = \sin\left(x + \frac{\pi}{2}\right)$.

Because of this, all of the methods we used to find amplitude, period, and phase shift for the sine function apply to the cosine function as well.

Properties of Sine and Cosine Graphs (PF1)

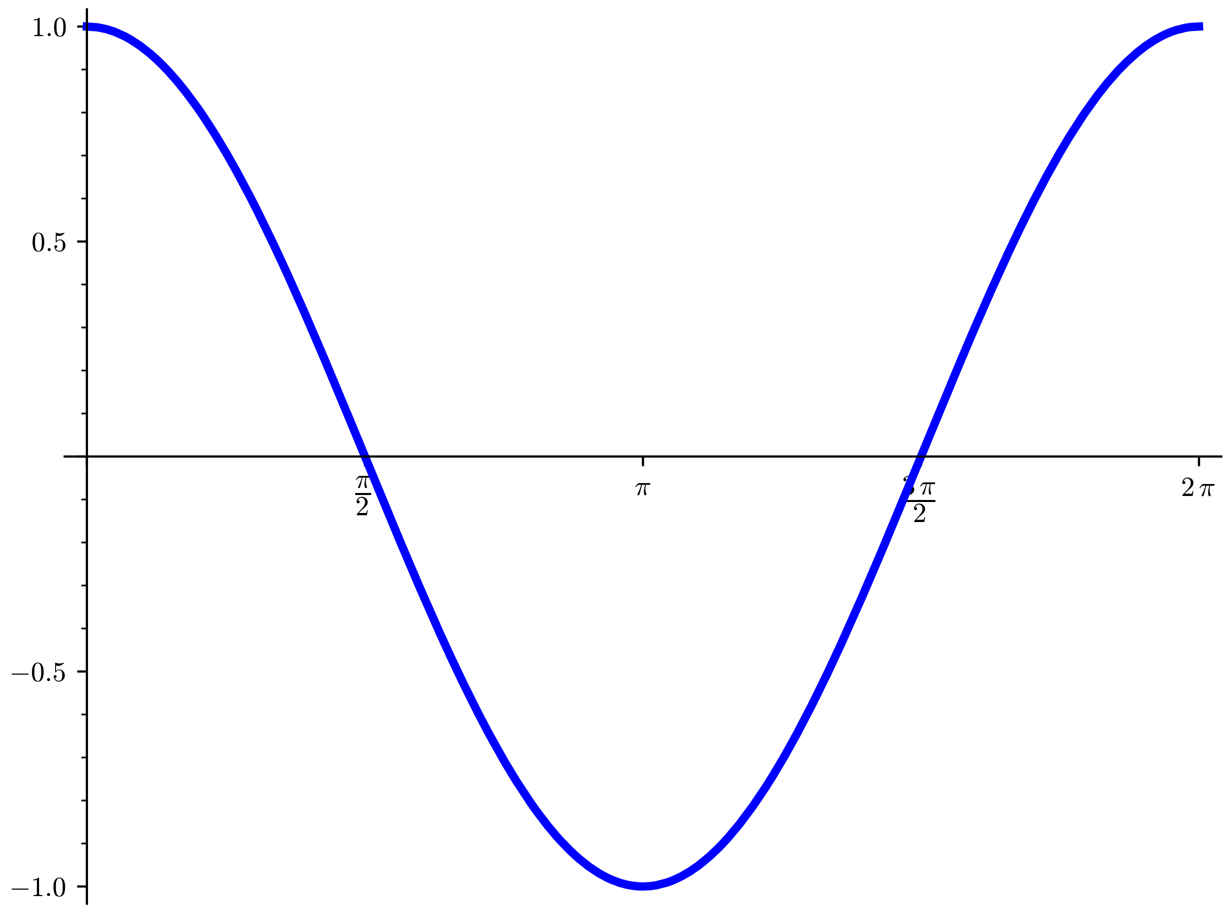
Observation 7.1.14 Now that we can graph both the standard sine and cosine curves, we can add them to our list of parent functions in Section A.1. We also show them graphed below on the interval $[0, 2\pi]$.

$f(x) = \sin x$



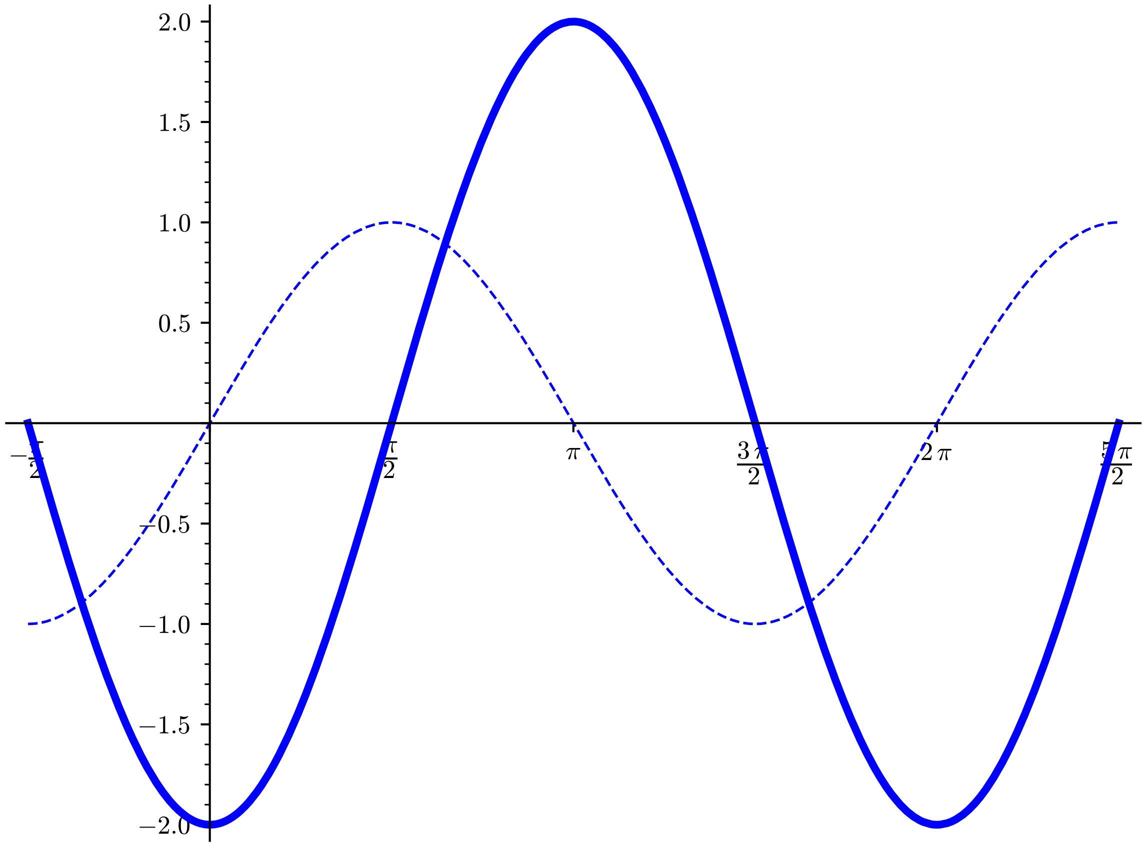
$f(x) = \cos x$

Properties of Sine and Cosine Graphs (PF1)



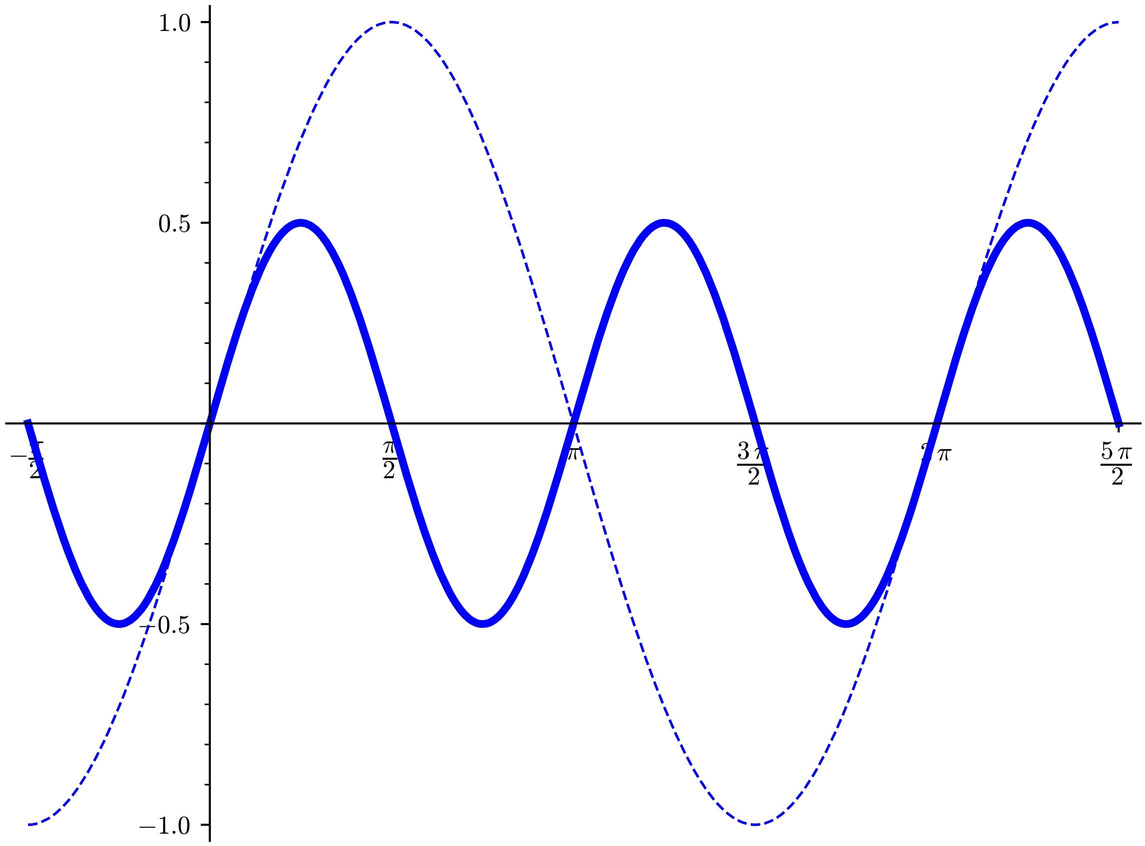
Properties of Sine and Cosine Graphs (PF1)

Activity 7.1.15 Find the amplitude, period, and phase shift of each of the following sine functions shown as a solid line. To help, $f(x) = \sin x$ is shown as a dotted line.



(a)

Properties of Sine and Cosine Graphs (PF1)



(b)

Properties of Sine and Cosine Graphs (PF1)

Activity 7.1.16 Find the amplitude, period, and phase shift of each of the following functions.

(a) $f(x) = \dfrac{3}{2}\cos(x + \pi)$

(b) $f(x) = 3\sin(2x - \pi)$

(c) $f(x) = 4\sin\left(3x + \dfrac{\pi}{2}\right)$

(d) $f(x) = \dfrac{1}{2}\cos\left(2x + \dfrac{\pi}{4}\right)$

Properties of Sine and Cosine Graphs (PF1)

Activity 7.1.17 Find an expression for each of the following transformations of the sine and cosine functions.

(a) $\sin(x)$ with a phase shift left by π and an amplitude of 3.

(b) $\cos(x)$ with a phase shift right by $\frac{\pi}{2}$ and a period of π.

(c) $\sin(x)$ with a phase shift right by $\frac{\pi}{6}$, a period of $\frac{\pi}{3}$, and an amplitude of 5.

Properties of Sine and Cosine Graphs (PF1)

Activity 7.1.18 To graph $f(x) = 3\sin(2x - \pi) - 3$, lets apply one transformation at a time.

(a) Which of the following is equivalent to $f(x) = 3\sin(2x - \pi) - 3$?

A. $3\sin\left(2\left(x - \pi\right)\right) - 3$

B. $3\sin\left(2\left(x - \frac{\pi}{2}\right)\right) - 3$

C. $3\sin\left(2\left(x - \frac{\pi}{3}\right)\right) - 2$

D. $3\sin\left(2\left(x - 2\pi\right)\right) - 3$

(b) Graph the function $f_1(x) = \sin(2x)$.

(c) Graph the function $f_2(x) = \sin\left(2\left(x - \frac{\pi}{2}\right)\right)$.

(d) Graph the function $f_3(x) = 3\sin\left(2\left(x - \frac{\pi}{2}\right)\right)$.

(e) Graph the function $f_4(x) = 3\sin\left(2\left(x - \frac{\pi}{2}\right)\right) - 3$.

Properties of Sine and Cosine Graphs (PF1)

Activity 7.1.19 Graph each of the following functions.

(a) $2\sin(3x-\pi)+4$

(b) $4\cos(2x+2\pi)+1$

(c) $-3\sin\left(4x-\frac{\pi}{2}\right)+1$

7.2 Additional Trigonometric Functions (PF2)

Objectives

- Graph trigonometric functions including tangent, cotangent, secant, and cosecant functions and transformations of these functions, and determine the domain and range.

Additional Trigonometric Functions (PF2)

Remark 7.2.1 In the previous sections, we looked at graphs of the sine and cosine functions. We will now look at graphs of the other four trigonometric functions.

Activity 7.2.2 Consider the tangent function, $f(x) = \tan(x)$.

(a) Fill in the missing values in the table below for $f(x) = \tan(x)$. Find the exact values, then express as a decimal, approximated to two decimal places if needed. (Notice that the values in the table are all the standard angles found on the unit circle!)

x	$\tan(x)$ **(exact)**	$\tan(x)$ **(as a decimal)**
0		
$\frac{\pi}{6}$	$\frac{1}{\sqrt{3}}$	
$\frac{\pi}{4}$		
$\frac{\pi}{3}$		≈ 1.73
$\frac{\pi}{2}$		
$\frac{2\pi}{3}$		
$\frac{3\pi}{4}$		
$\frac{5\pi}{6}$		≈ -0.58
π		
$\frac{7\pi}{6}$		
$\frac{5\pi}{4}$		
$\frac{4\pi}{3}$	$\sqrt{3}$	
$\frac{3\pi}{2}$		
$\frac{5\pi}{3}$		
$\frac{7\pi}{4}$		
$\frac{11\pi}{6}$		
2π		

(b) What do you think is happening to the graph at $\frac{\pi}{2}$ and $\frac{3\pi}{2}$?

A. The graph has a hole.

B. The graph has a horizontal asymptote.

C. The graph has a vertical asymptote.

Hint. Recall that $\tan(x) = \dfrac{\sin(x)}{\cos(x)}$.

(c) Plot these values on a coordinate plane to approximate the graph of $f(x) = \tan(x)$. Then sketch in the graph of the tangent curve using the points as a guide.

(d) What is the domain of $\tan(x)$?

A. $\ldots \cup \left(-\frac{3\pi}{2}, -\frac{\pi}{2}\right) \cup \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \frac{3\pi}{2}\right) \cup \ldots$

B. $\ldots \cup (-\pi, 0) \cup (0, \pi) \cup (\pi, 2\pi) \cup \ldots$

C. $\left(0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \frac{3\pi}{2}\right) \cup \ldots$

D. $(-\infty, \infty)$

(e) What is the range of $\tan(x)$?

A. $[-1, 1]$

B. $(-\infty, -1] \cup [1, \infty)$

C. $(-\infty, 0) \cup (0, \infty)$

D. $(-\infty, \infty)$

(f) What is the period of $\tan(x)$?

A. $\frac{\pi}{2}$ B. π C. 2π D. 4π

Additional Trigonometric Functions (PF2)

Activity 7.2.3 Consider the secant function, $f(x) = \sec(x)$.

(a) Fill in the missing values in the table below for $f(x) = \sec(x)$. Find the exact values, then express as a decimal, approximated to two decimal places if needed. (Notice that the values in the table are all the standard angles found on the unit circle!)

x	$\sec(x)$ **(exact)**	$\sec(x)$ **(as a decimal)**
0		
$\frac{\pi}{6}$		
$\frac{\pi}{4}$		
$\frac{\pi}{3}$		
$\frac{\pi}{2}$		
$\frac{2\pi}{3}$		
$\frac{3\pi}{4}$		≈ -1.41
$\frac{5\pi}{6}$		
π		
$\frac{7\pi}{6}$	$-\frac{2}{\sqrt{3}}$	
$\frac{5\pi}{4}$		
$\frac{4\pi}{3}$		
$\frac{3\pi}{2}$		
$\frac{5\pi}{3}$	2	
$\frac{7\pi}{4}$		
$\frac{11\pi}{6}$		
2π		

(b) What do you think is happening to the graph at $\frac{\pi}{2}$ and $\frac{3\pi}{2}$?

A. The graph has a hole.

B. The graph has a horizontal asymptote.

C. The graph has a vertical asymptote.

(c) Plot these values on a coordinate plane to approximate the graph of $f(x) = \sec(x)$. Then sketch in the graph of the secant curve using the points as a guide.

(d) What is the domain of $\sec(x)$?

A. $\ldots \cup \left(-\frac{3\pi}{2}, -\frac{\pi}{2}\right) \cup \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \frac{3\pi}{2}\right) \cup \ldots$

B. $\ldots \cup (-\pi, 0) \cup (0, \pi) \cup (\pi, 2\pi) \cup \ldots$

C. $\left(0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \frac{3\pi}{2}\right) \cup \ldots$

D. $(-\infty, \infty)$

(e) What is the range of $\sec(x)$?

A. $[-1, 1]$

B. $(-\infty, -1] \cup [1, \infty)$

C. $(-\infty, 0) \cup (0, \infty)$

D. $(-\infty, \infty)$

(f) What is the period of $\sec(x)$?

A. $\frac{\pi}{2}$ B. π C. 2π D. 4π

Observation 7.2.4 Since $\sec(x) = \dfrac{1}{\cos(x)}$, we can see that their graphs are related: $\sec(x)$ (solid blue curve) has a vertical asymptote everywhere $\cos(x)$ (dotted green curve) has a zero, and for every point (a, b) on the graph of $\cos(x)$, the point $(a, \frac{1}{b})$ is on the graph of $\sec(x)$.

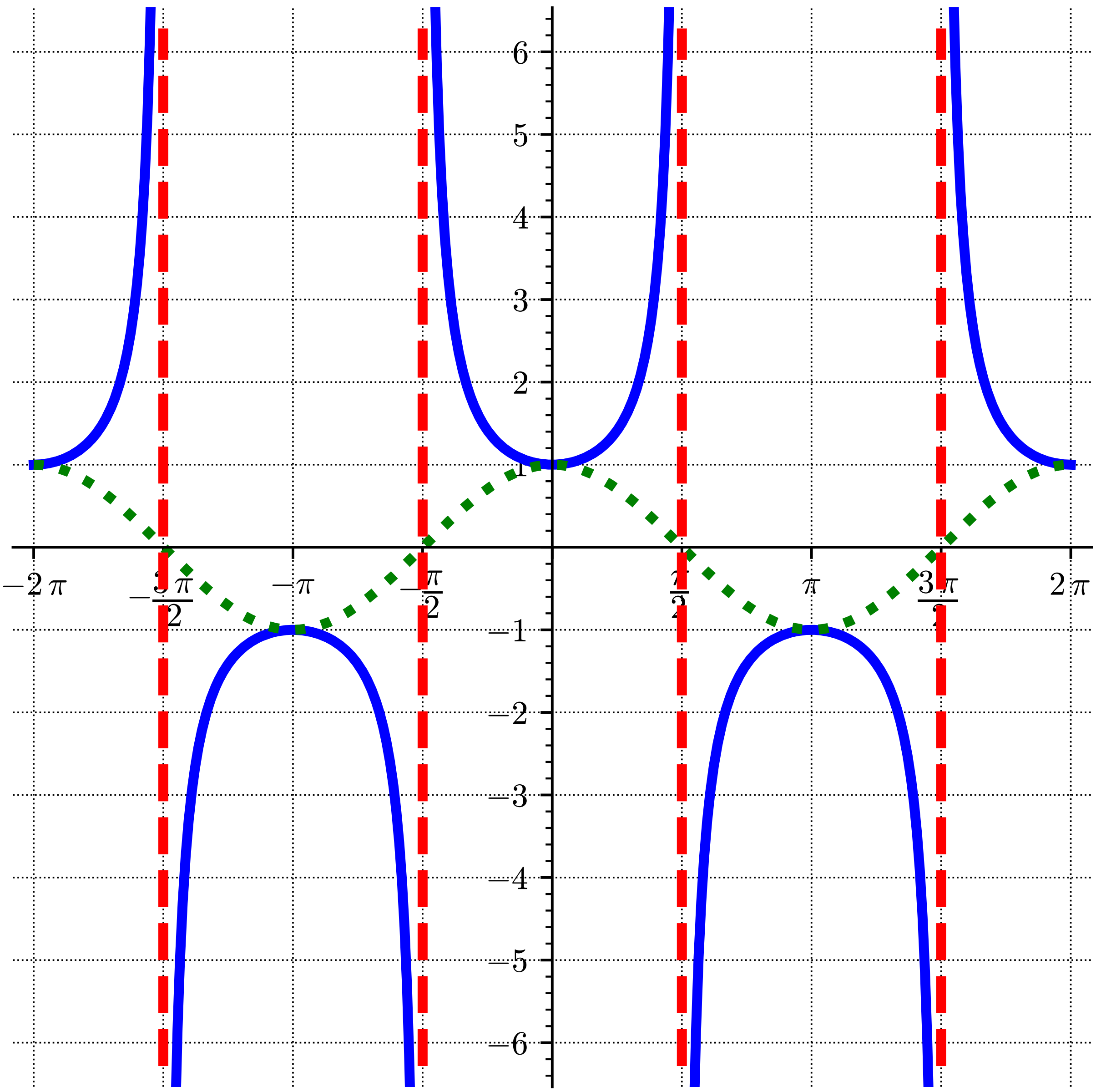


Figure 7.2.5 $y = \sec(x)$

Activity 7.2.6 Consider the cosecant function, $f(x) = \csc(x)$. While we could make a table as in Activity 7.2.3, let's instead take advantage of the fact that the graphs of $\csc(x)$ and its reciprocal $\sin(x)$ will be related in the same way as the graphs of $\sec(x)$ and its reciprocal $\cos(x)$.

(a) Where does $\sin(x)$ have zeros?

Hint. Recall the graph of $\sin(x)$

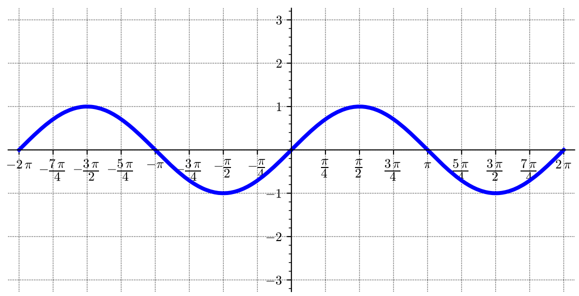


Figure 7.2.7

(b) Where does $\csc(x)$ have vertical asymptotes?

(c) Where does $\sin(x)$ have local maximum and minimum values?

(d) Where does $\csc(x)$ have local maximum and minimum values?

(e) On what intervals is $\sin(x)$ increasing and decreasing?

(f) On what intervals is $\csc(x)$ increasing and decreasing?

(g) Use your answers to the previous tasks to plot the graph of $y = \csc(x)$.

Hint. It may be helpful to first sketch the graph of $\sin(x)$.

(h) What is the domain of $\csc(x)$?

A. $\ldots \cup \left(-\frac{3\pi}{2}, -\frac{\pi}{2}\right) \cup \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \frac{3\pi}{2}\right) \cup \ldots$

B. $\ldots \cup (-\pi, 0) \cup (0, \pi) \cup (\pi, 2\pi) \cup \ldots$

C. $(0, \pi) \cup (\pi, 2\pi) \cup \ldots$

D. $(-\infty, \infty)$

(i) What is the range of $\csc(x)$?

A. $[-1, 1]$

B. $(-\infty, -1] \cup [1, \infty)$

C. $(-\infty, 0) \cup (0, \infty)$

D. $(-\infty, \infty)$

Additional Trigonometric Functions (PF2)

(j) What is the period of $\csc(x)$?

A. $\frac{\pi}{2}$

B. π

C. 2π

D. 4π

Observation 7.2.8 Since $\csc(x) = \dfrac{1}{\sin(x)}$, we can see that their graphs are related: $\csc(x)$ (solid blue) has a vertical asymptote everywhere $\sin(x)$ (dotted green) has a zero, and for every point (a, b) on the graph of $\sin(x)$, the point $(a, \frac{1}{b})$ is on the graph of $\csc(x)$.

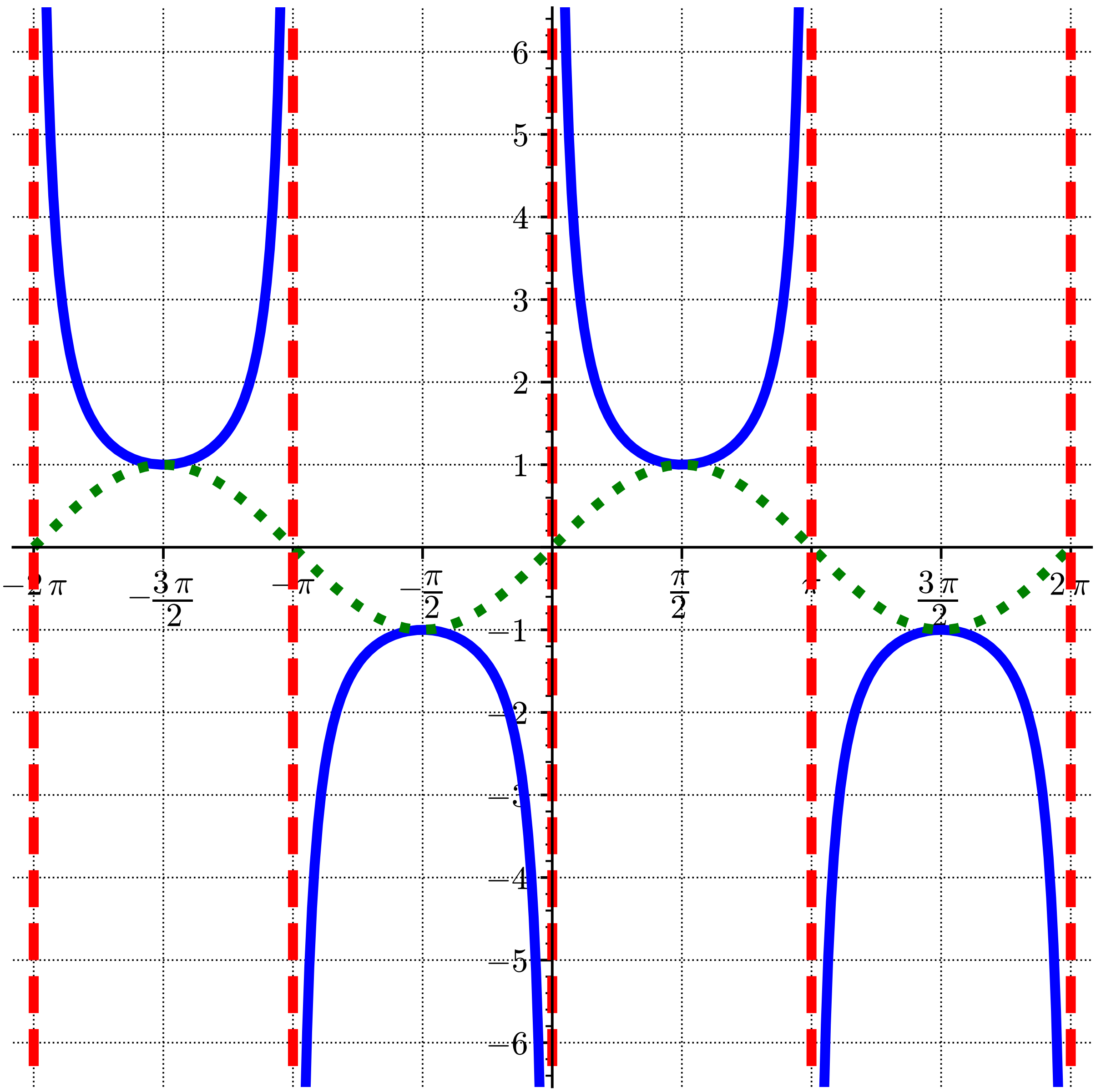


Figure 7.2.9 $y = \csc(x)$

Activity 7.2.10 Consider the cotangent function, $f(x) = \cot(x)$.

(a) Where does $\tan(x)$ have zeros?

Hint. Recall the graph of $\tan(x)$

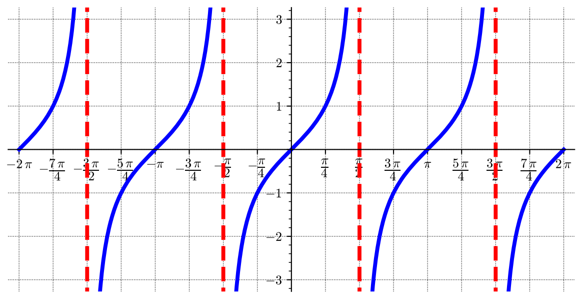


Figure 7.2.11

(b) Where does $\cot(x)$ have vertical asymptotes?

(c) Where does $\tan(x) = 1$ and where does $\tan(x) = -1$?

(d) Where does $\cot(x) = 1$ and where does $\cot(x) = -1$?

(e) On what intervals is $\tan(x)$ increasing and decreasing?

(f) On what intervals is $\cot(x)$ increasing and decreasing?

(g) Use your answers to the previous tasks to plot the graph of $y = \cot(x)$.

Hint. It may be helpful to first sketch the graph of $\tan(x)$.

(h) What is the domain of $\cot(x)$?

A. $\ldots \cup \left(-\frac{3\pi}{2}, -\frac{\pi}{2}\right) \cup \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \frac{3\pi}{2}\right) \cup \ldots$

B. $\ldots \cup (-\pi, 0) \cup (0, \pi) \cup (\pi, 2\pi) \cup \ldots$

C. $(0, \pi) \cup (\pi, 2\pi) \cup \ldots$

D. $(-\infty, \infty)$

(i) What is the range of $\cot(x)$?

A. $[-1, 1]$

B. $(-\infty, -1] \cup [1, \infty)$

C. $(-\infty, 0) \cup (0, \infty)$

D. $(-\infty, \infty)$

(j) What is the period of $\cot(x)$?

A. $\frac{\pi}{2}$ B. π C. 2π D. 4π

Observation 7.2.12 Since $\cot(x) = \dfrac{1}{\tan(x)}$, we can see that their graphs are related: $\cot(x)$ has a vertical asymptote everywhere $\tan(x)$ has a zero (and vice versa), and for every point (a, b) on the graph of $\tan(x)$, the point $(a, \frac{1}{b})$ is on the graph of $\cot(x)$.

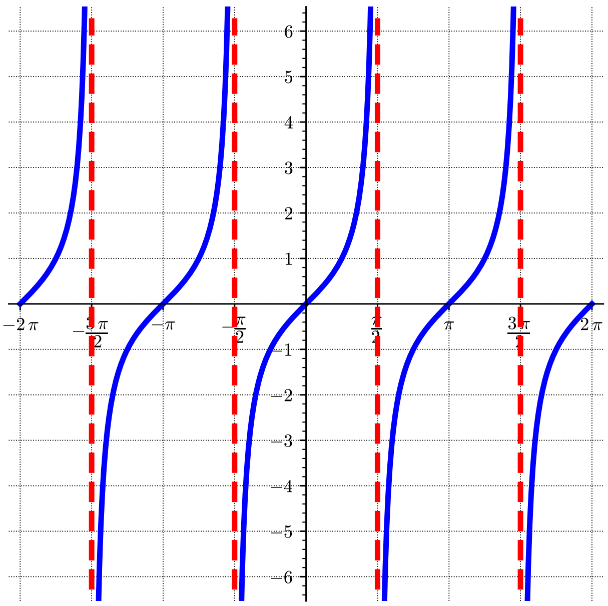


Figure 7.2.13 $y = \tan(x)$

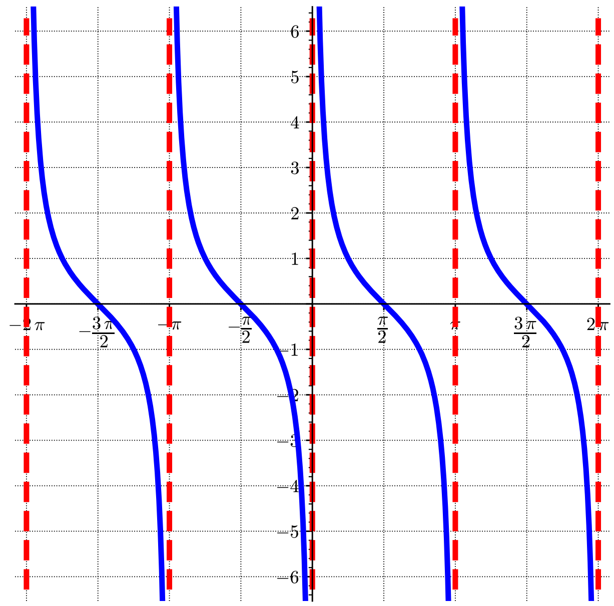


Figure 7.2.14 $y = \cot(x)$

Additional Trigonometric Functions (PF2)

Remark 7.2.15 The graphs of all six Trigonometric Functions are listed in Section A.1.

Additional Trigonometric Functions (PF2)

Observation 7.2.16 Everything we learned about transformations of functions in Section 2.4 applies equally well to trigonometric functions.

Activity 7.2.17 Consider the function $g(x) = \sec\left(x + \frac{\pi}{2}\right)$.

(a) How is this graph related to the graph of $\sec(x)$?

A. It is shifted left $\frac{\pi}{2}$.

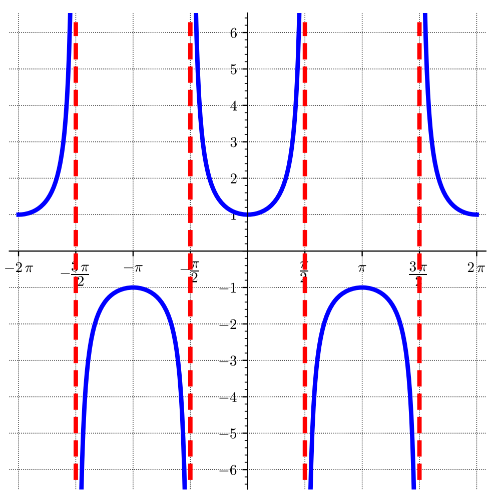
B. It is shifted right $\frac{\pi}{2}$.

C. It is shifted up $\frac{\pi}{2}$.

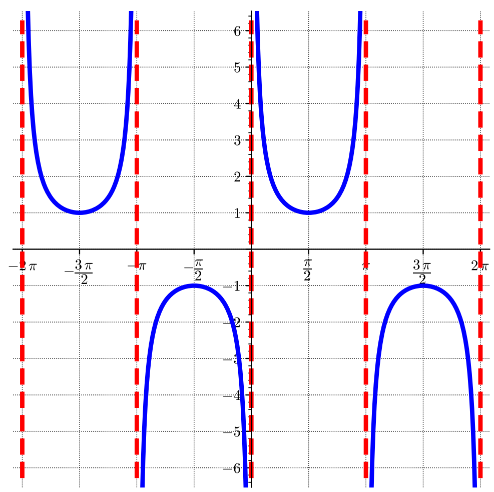
D. It is shifted down $\frac{\pi}{2}$.

(b) Which of the following is the graph of $g(x) = \sec\left(x + \frac{\pi}{2}\right)$?

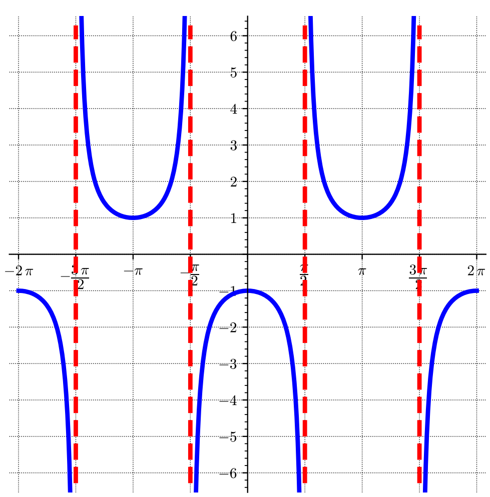
A.



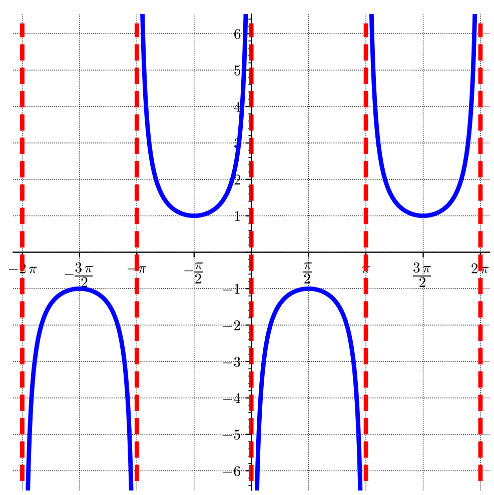
C.



B.



D.



(c) What is the domain of $g(x) = \sec\left(x + \frac{\pi}{2}\right)$?

(d) What is the range of $g(x) = \sec\left(x + \frac{\pi}{2}\right)$?

Additional Trigonometric Functions (PF2)

(e) What is the period of $g(x) = \sec\left(x + \frac{\pi}{2}\right)$?

Activity 7.2.18 Consider the function $h(x) = \tan(2x)$

(a) What sort of transformation of $\tan(x)$ is $\tan(2x)$?

A. Horizontal stretch
B. Horizontal compression
C. Vertical stretch
D. Vertical compression

(b) Which of the following are different for $\tan(2x)$ than for $\tan(x)$? Select all that apply.

A. Location of vertical asymptotes
B. Period
C. Domain
D. Range

(c) Where are the vertical asymptotes of $h(x) = \tan(2x)$?

A. $\ldots, -2\pi, -\pi, 0, \pi, 2\pi, \ldots$
B. $\ldots, -3\pi, -\pi, \pi, 3\pi, \ldots$
C. $\ldots, -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$
D. $\ldots, -\frac{3\pi}{4}, -\frac{\pi}{4}, \frac{\pi}{4}, \frac{3\pi}{4}, \ldots$

Hint. Recall that $\tan(x)$ has vertical asymptotes at $\frac{\pi}{2} + \pi k$ for each integer k.

(d) Where are the zeros of $h(x) = \tan(2x)$?

A. $\ldots, -2\pi, -\pi, 0, \pi, 2\pi, \ldots$
B. $\ldots, -3\pi, -\pi, \pi, 3\pi, \ldots$
C. $\ldots,, -\frac{\pi}{2}, 0, \frac{\pi}{2}, \ldots$
D. $\ldots, -\frac{\pi}{4}, 0, \frac{\pi}{4},, \ldots$

Hint. Recall that $\tan(x)$ has zeroes asymptotes at πk for each integer k.

(e) Graph $h(x) = \tan(2x)$.

(f) What is the domain of $h(x) = \tan(2x)$?

(g) What is the range of $h(x) = \tan(2x)$?

(h) What is the period of $h(x) = \tan(2x)$?

Activity 7.2.19 Consider the function $k(x) = 3\csc\left(\frac{x}{2}\right)$.

(a) Where are the vertical asymptotes for $k(x) = 3\csc\left(\frac{x}{2}\right)$ located?

(b) Where are the local minima and local maxima for $k(x) = 3\csc\left(\frac{x}{2}\right)$ located?

(c) What are the local minimum and local maximum values for $k(x) = 3\csc\left(\frac{x}{2}\right)$?

(d) Graph $k(x) = 3\csc\left(\frac{x}{2}\right)$.

(e) What is the domain of $k(x) = 3\csc\left(\frac{x}{2}\right)$?

(f) What is the range of $k(x) = 3\csc\left(\frac{x}{2}\right)$?

(g) What is the period of $k(x) = 3\csc\left(\frac{x}{2}\right)$?

7.3 Inverse Trig Functions (PF3)

Objectives

- Determine the inverse sine, cosine, and tangent values; graph inverse trig functions and determine the limitations on the domain and range.

Inverse Trig Functions (PF3)

Activity 7.3.1 Which of the following angles satisfy $\cos(\theta) = \frac{1}{2}$?

A. $\dfrac{\pi}{6}$

B. $\dfrac{\pi}{3}$

C. $\dfrac{2\pi}{3}$

D. $\dfrac{5\pi}{6}$

E. $\dfrac{7\pi}{6}$

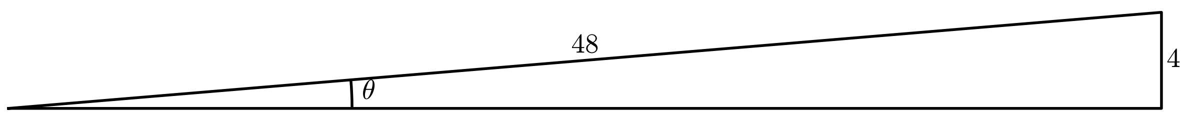
F. $\dfrac{4\pi}{3}$

G. $\dfrac{5\pi}{3}$

H. $\dfrac{11\pi}{6}$

Inverse Trig Functions (PF3)

Activity 7.3.2 A carpenter is cutting a hand rail for a ramp on his mitre saw. The ramp goes up 4 feet, and the length of the hand rail is 48 feet long. Which of the following equations determines the angle of the ramp, which the carpenter will use to set his saw?



A. $\sin(\theta) = \frac{1}{12}$

B. $\cos(\theta) = \frac{1}{12}$

C. $\tan(\theta) = \frac{1}{12}$

D. $\cot(\theta) = \frac{1}{12}$

Inverse Trig Functions (PF3)

Observation 7.3.3 Often in applications, in addition to finding the sine or cosine of an angle, we need to find an angle with a given sine or cosine or tangent. In other words, we need to have inverse functions of our six trigonometric functions.

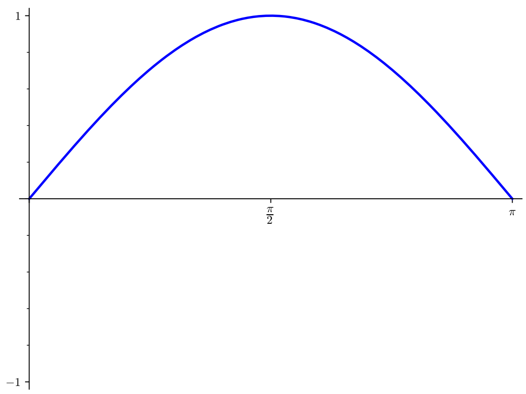
Remark 7.3.4 In Activity 7.3.1, we saw that there are many angles with a given sine or cosine. We must systematically choose one of these to define an inverse function.

Inverse Trig Functions (PF3)

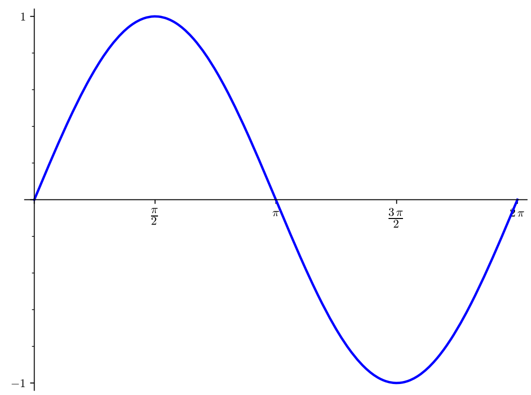
Activity 7.3.5 By restricting the domain, we can find a part of the sine function which is one-to-one, and thus allows us to define an inverse function.

Which of the following domain restrictions is one-to-one?

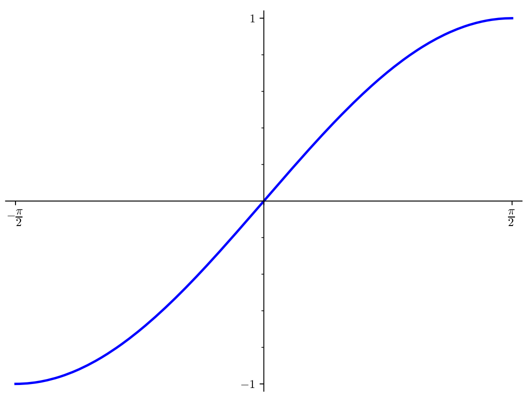
A. $0 \le x \le \pi$



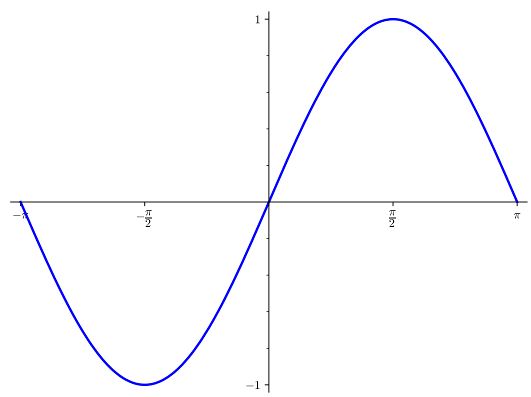
C. $0 \le x \le 2\pi$



B. $-\dfrac{\pi}{2} \le x \le \dfrac{\pi}{2}$



D. $-\pi \le x \le \pi$



Hint. Use the horizontal line test.

Inverse Trig Functions (PF3)

Definition 7.3.6 The **arcsine** function, denoted $\arcsin(x)$, is the inverse of the restriction of $\sin(x)$ to the domain $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

In other words, $\arcsin(x)$ is the unique angle θ with $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ such that $\sin(\theta) = x$. ◊

Inverse Trig Functions (PF3)

Activity 7.3.7 Compute each of the following, without the use of technology.

(a) $\arcsin\left(\frac{1}{2}\right)$

(b) $\arcsin(-1)$

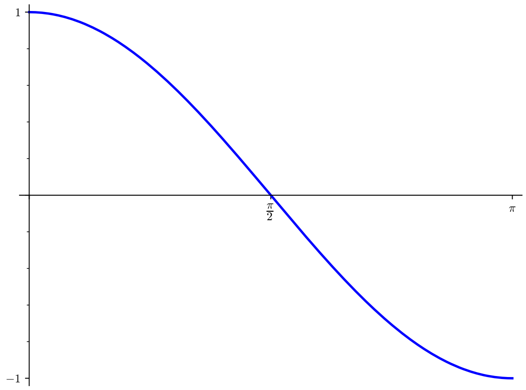
(c) $\arcsin\left(\frac{\sqrt{2}}{2}\right)$

(d) $\arcsin\left(-\frac{\sqrt{3}}{2}\right)$

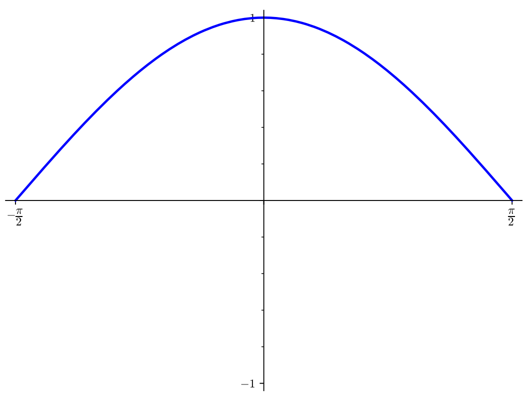
Inverse Trig Functions (PF3)

Activity 7.3.8 Which of the following domain restrictions of $\cos(x)$ is one-to-one?

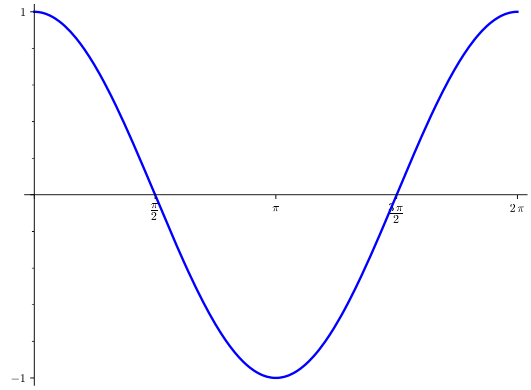
A. $0 \leq x \leq \pi$



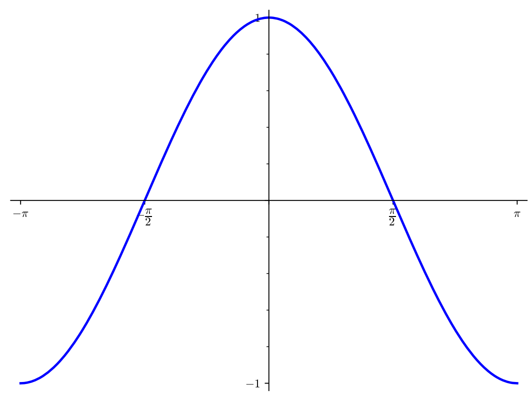
B. $-\dfrac{\pi}{2} \leq x \leq \dfrac{\pi}{2}$



C. $0 \leq x \leq 2\pi$



D. $-\pi \leq x \leq \pi$



Hint. Use the horizontal line test.

Inverse Trig Functions (PF3)

Definition 7.3.9 The **arccosine** function, denoted $\arccos(x)$, is the inverse of the restriction of $\cos(x)$ to the domain $[0, \pi]$.

In other words, $\arccos(x)$ is the unique angle θ with $0 \leq \theta \leq \pi$ such that $\cos(\theta) = x$. $\Diamond$

Inverse Trig Functions (PF3)

Activity 7.3.10 Compute each of the following, without the use of technology.

(a) $\arccos\left(\frac{1}{2}\right)$

(b) $\arccos(-1)$

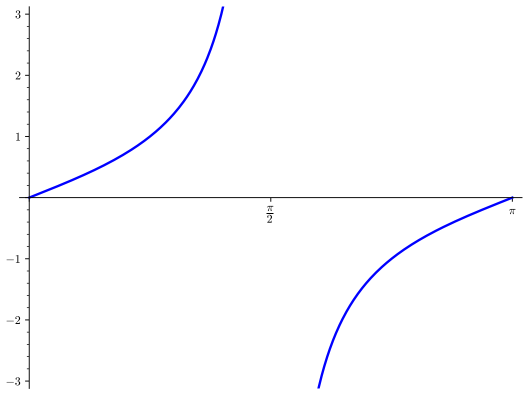
(c) $\arccos\left(\frac{\sqrt{2}}{2}\right)$

(d) $\arccos\left(-\frac{\sqrt{3}}{2}\right)$

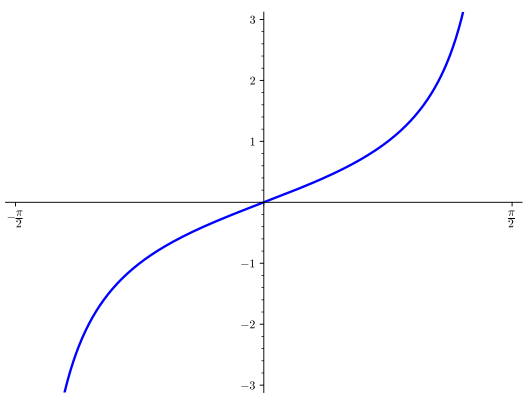
Inverse Trig Functions (PF3)

Activity 7.3.11 Which of the following domain restrictions of $\tan(x)$ is one-to-one?

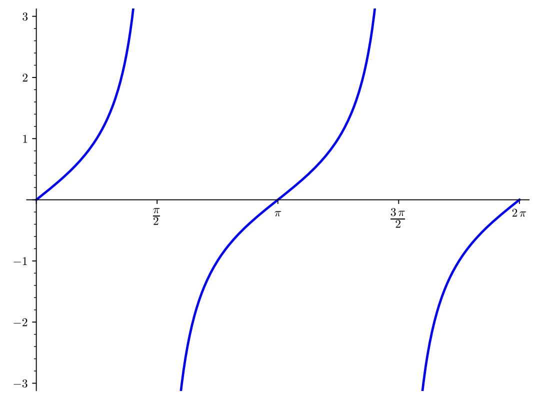
A. $0 \leq x \leq \pi$



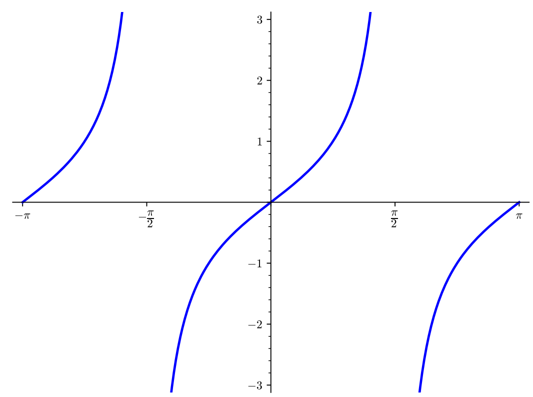
B. $-\dfrac{\pi}{2} \leq x \leq \dfrac{\pi}{2}$



C. $0 \leq x \leq 2\pi$



D. $-\pi \leq x \leq \pi$



Hint. Use the horizontal line test.

Inverse Trig Functions (PF3)

Definition 7.3.12 The **arctangent** function, denoted $\arctan(x)$, is the inverse of the restriction of $\tan(x)$ to the domain $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

In other words, $\arctan(x)$ is the unique angle θ with $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ such that $\tan(\theta) = x$.

$\Diamond$

Observation 7.3.13 Note that while $\arcsin(x)$ and $\arccos(x)$ were defined by restricting the domain to a closed interval, since $\tan(x)$ is not defined at $-\frac{\pi}{2}$ or $\frac{\pi}{2}$, we define $\arctan(x)$ by restricting the domain to the open interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

Inverse Trig Functions (PF3)

Activity 7.3.14 Compute each of the following, without the use of technology.

(a) $\arctan(1)$

(b) $\arctan\left(-\sqrt{3}\right)$

(c) $\arctan(0)$

(d) $\arctan\left(\frac{\sqrt{3}}{3}\right)$

Activity 7.3.15 Sometimes, as in Activity 7.3.2, we need to find an inverse trigonometric function that does not produce one of our special angles.

Compute each of the following using technology (e.g. a calculator).

Note that most calculators use the shorter notation, e.g. $\sin^{-1}$, for this operation.

(a) $\arcsin\left(\frac{1}{12}\right)$

(b) $\arccos\left(-\dfrac{3}{5}\right)$

(c) $\arctan(2)$

Inverse Trig Functions (PF3)

Observation 7.3.16 Next, we look at the graphs of $\arcsin(x)$, $\arccos(x)$, and $\arctan(x)$.

Inverse Trig Functions (PF3)

Activity 7.3.17 Consider the function $f(x) = \arcsin(x)$.

(a) Complete the table of values.

x	$\arcsin(x)$
-1	
$-\frac{\sqrt{3}}{2}$	
$-\frac{\sqrt{2}}{2}$	
$-\frac{1}{2}$	
0	
$\frac{1}{2}$	
$\frac{\sqrt{2}}{2}$	
$\frac{\sqrt{3}}{2}$	
1	

Hint. Recall that $\theta = \arcsin(x)$ means $\sin(\theta) = x$.

(b) Plot these values on a coordinate plane to approximate the graph of $f(x) = \arcsin(x)$. Then sketch the graph of the arcsine curve using the points as a guide.

(c) What is the domain of $f(x) = \arcsin(x)$?

(d) What is the range of $f(x) = \arcsin(x)$?

Activity 7.3.18 Consider the function $f(x) = \arccos(x)$.

(a) Complete the table of values.

x	$\arccos(x)$
-1	
$-\frac{\sqrt{3}}{2}$	
$-\frac{\sqrt{2}}{2}$	
$-\frac{1}{2}$	
0	
$\frac{1}{2}$	
$\frac{\sqrt{2}}{2}$	
$\frac{\sqrt{3}}{2}$	
1	

Hint. Recall that $\theta = \arccos(x)$ means $\cos(\theta) = x$.

(b) Plot these values on a coordinate plane to approximate the graph of $f(x) = \arccos(x)$. Then sketch the graph of the arccosine curve using the points as a guide.

(c) What is the domain of $f(x) = \arccos(x)$?

(d) What is the range of $f(x) = \arccos(x)$?

Inverse Trig Functions (PF3)

Activity 7.3.19 Consider the function $f(x) = \arctan(x)$.

(a) Complete the table of values.

x	$\arctan(x)$
$-\sqrt{3}$	
-1	
$-\frac{\sqrt{3}}{3}$	
0	
$\frac{\sqrt{3}}{3}$	
1	
$\sqrt{3}$	

Hint. Recall that $\theta = \arctan(x)$ means $\tan(\theta) = x$.

(b) Plot these values on a coordinate plane to approximate the graph of $f(x) = \arctan(x)$. Then sketch the graph of the arctangent curve using the points as a guide.

(c) What is the domain of $f(x) = \arctan(x)$?

(d) What is the range of $f(x) = \arctan(x)$?

Inverse Trig Functions (PF3)

Activity 7.3.20 Sometimes when solving applied problems, we need to exactly (not approximately) evaluate expressions like $\sin\left(\arccos\left(\frac{5}{13}\right)\right)$.

(a) Which of the following sentences describe the expression $\sin\left(\arccos\left(\frac{5}{13}\right)\right)$?

A. The angle whose cosine is the same as the sine of $\frac{5}{13}$.

B. The angle whose sine is the same as the cosine of $\frac{5}{13}$.

C. The cosine of the angle whose sine is $\frac{5}{13}$.

D. The sine of the angle whose cosine is $\frac{5}{13}$.

(b) Let $\theta = \arccos(\frac{5}{13})$. Draw a right triangle with an angle of θ, and find the lengths of its three sides.

(c) Find $\sin(\theta)$. Since we defined $\theta = \arccos(\frac{5}{13})$, this gives us $\sin(\arccos(\frac{5}{13}))$.

A. $\dfrac{5}{13}$

B. $\dfrac{12}{13}$

C. $\dfrac{5}{12}$

D. $\dfrac{13}{12}$

Inverse Trig Functions (PF3)

Activity 7.3.21 Compute each of the following.

(a) $\tan(\arcsin(\frac{8}{17}))$

(b) $\sec(\arctan(\frac{24}{7}))$

(c) $\tan(\arcsin(\frac{3}{4}))$

(d) $\cos(\arcsin(x))$

Hint. Draw an appropriate right triangle with two sides as x and 1.

Chapter 8

Trigonometric Equations (TE)

Objectives

How can we use trigonometry to solve equations and understand arbitrary triangles?
By the end of this chapter, you should be able to...

1. Use the sum and difference, the double-angle, and power-reducing formulas for cosine, sine, and tangent and the Pythagorean identities.
2. Symbolically verify identities by using various trig identities.
3. Find all solutions to a trigonometric equation.
4. Use trigonometric functions and the Pythagorean Theorem to solve right triangles.
5. Use the law of sines and/or the law of cosines to solve non-right triangles.
6. Solve applications that require the use of trigonometry to find a side or angle.

8.1 Trigonometric Identities (TE1)

Objectives

- Use the sum and difference, the double-angle, and power-reducing formulas for cosine, sine, and tangent and the Pythagorean identities.

Remark 8.1.1 Pythagorean Identities, as we have seen in a previous section, are derived from the Pythagorean Theorem ($a^2 + b^2 = c^2$).

For example, consider a point P on the unit circle, with coordinates (x, y). If we draw a right triangle (as shown in the figure below), the Pythagorean Theorem says that $x^2+y^2 = 1$.

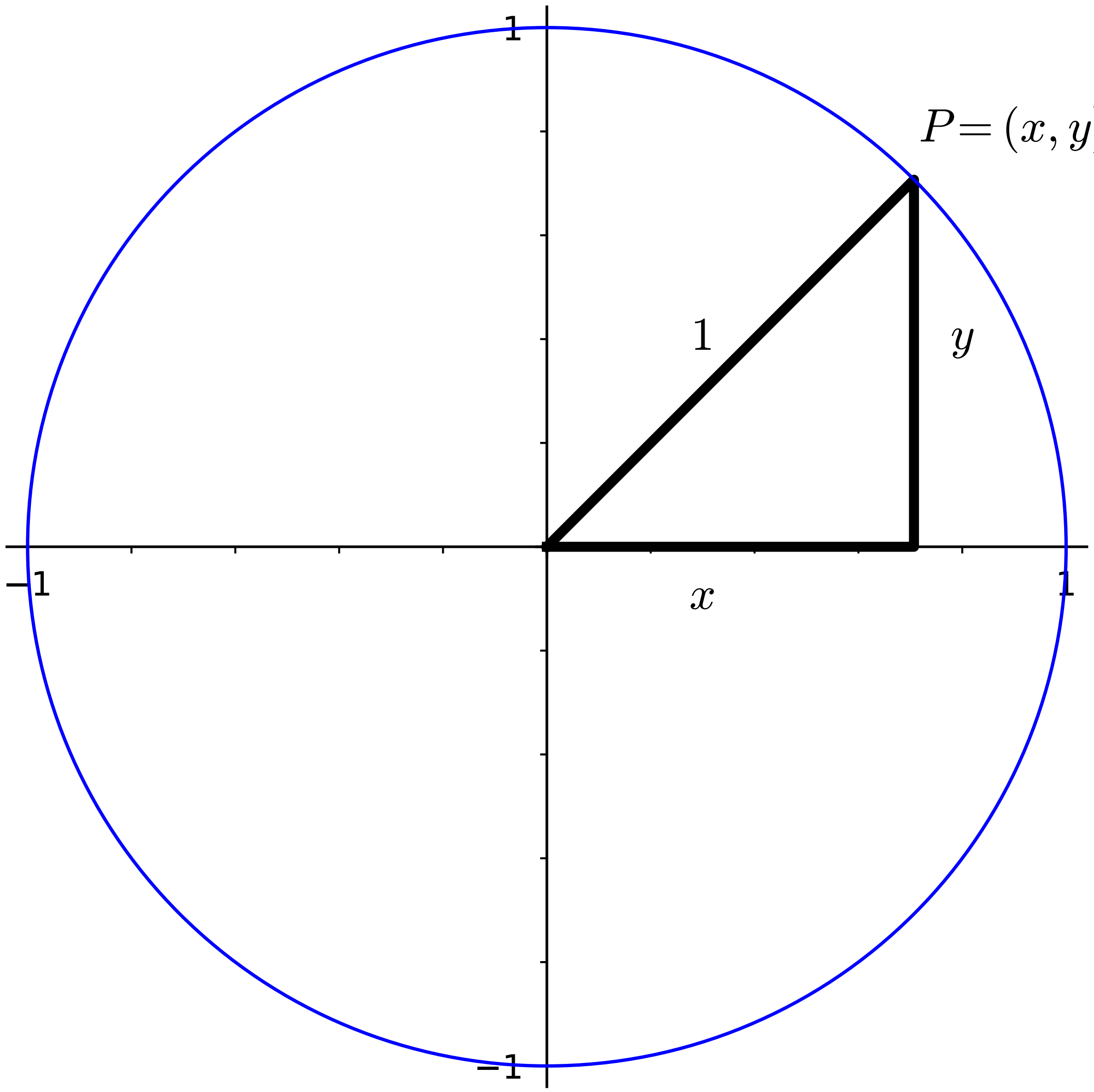


Figure 8.1.2

But, remember, that the x-coordinate of the point corresponds to $\cos\theta$ and the y-coordinate corresponds to $\sin\theta$. Thus, we get:

$$\sin^2\theta + \cos^2\theta = 1.$$

Pythagorean Identities are used in solving many trigonometric problems where one trigonometric ratio is given and we are expected to find the other trigonometric ratios. The next two activities will lead us to find the other two Pythagorean Identities.

Trigonometric Identities (TE1)

Activity 8.1.3 Let's begin with the Pythagorean Identity, $\cos^2\theta + \sin^2\theta = 1$.

(a) If we divide each term by $\cos^2\theta$, what would the resulting equation be?

(b) Recall the following trigonometric functions:

- $\sin\theta$
- $\csc\theta = \dfrac{1}{\sin\theta}$
- $\cos\theta$
- $\sec\theta = \dfrac{1}{\cos\theta}$
- $\tan\theta$
- $\cot\theta = \dfrac{\sin\theta}{\cos\theta}$

Using these trigonometric relationships, rewrite each fractional term in the equation you got in part (a) as a single trigonometric function.

Remark 8.1.4 In Activity 8.1.3, we derived the second Pythagorean Identity:

$$1 + \tan^2 \theta = \sec^2 \theta$$

Trigonometric Identities (TE1)

Activity 8.1.5 We can also derive the third Pythagorean identity from $\sin^2\theta + \cos^2\theta = 1$.

(a) What do we get if we divide each term by $\sin^2\theta$?

(b) Rewrite each fractional term in the equation you got in part (a) as a single trigonometric function.

Remark 8.1.6 In Activity 8.1.5, we derived the third Pythagorean Identity:

$$1 + \cot^2\theta = \csc^2\theta$$

Remark 8.1.7 Other identities and formulas (in addition to the Pythagorean Identities) can be used to solve various mathematical problems. The next few activities will lead us through an exploration of other types of identities and formulas.

Trigonometric Identities (TE1)

Activity 8.1.8 Refer back to the unit circle to determine the exact value of $\sin\left(\pi + \frac{\pi}{2}\right)$.

(a) What is the value of $\sin \pi$?

A. -1 B. 0 C. 1

(b) What is the value of $\sin\left(\frac{\pi}{2}\right)$?

A. -1 B. 0 C. 1

(c) Based on your answers from parts (a) and (b), what do you think the value of $\sin\left(\pi + \frac{\pi}{2}\right)$ is?

(d) Let's test your conjecture from part (c). What is the value of $\pi + \frac{\pi}{2}$?

A. $\frac{2\pi}{2}$

B. π

C. $\frac{3\pi}{2}$

D. $\frac{2\pi}{3}$

(e) What is the value of $\sin\left(\frac{3\pi}{2}\right)$?

A. -1 B. 0 C. 1

(f) Is it safe to assume that $\sin\left(\pi + \frac{\pi}{2}\right) = \sin \pi + \sin\left(\frac{\pi}{2}\right)$?

Remark 8.1.9 Notice that in Activity 8.1.9, we saw that $\sin\left(\pi+\frac{\pi}{2}\right) \neq \sin\pi + \sin\frac{\pi}{2}$. This is also true for both cosine and tangent as well - that is, if we want to find the cosine or tangent of the sum of two angles, we cannot assume that it is equal to the sum of the two trigonometric functions of each angle. For example, $\cos\left(\pi+\frac{\pi}{2}\right) \neq \cos\pi + \cos\frac{\pi}{2}$ and $\tan\left(\pi+\frac{\pi}{2}\right) \neq \tan\pi + \tan\frac{\pi}{2}$. The same is true for finding the difference of two angles.

Activity 8.1.10 Recall that the coordinates of points on the unit circle are given by $(\cos\theta, \sin\theta)$. In the first unit circle shown below, point P makes an angle α with the positive x-axis and has coordinates $(\cos\alpha, \sin\alpha)$ and point Q makes an angle β with the positive x-axis and has coordinates $(\cos\beta, \sin\beta)$. In the second figure, the triangle is rotated so that point B has coordinates $(1, 0)$.

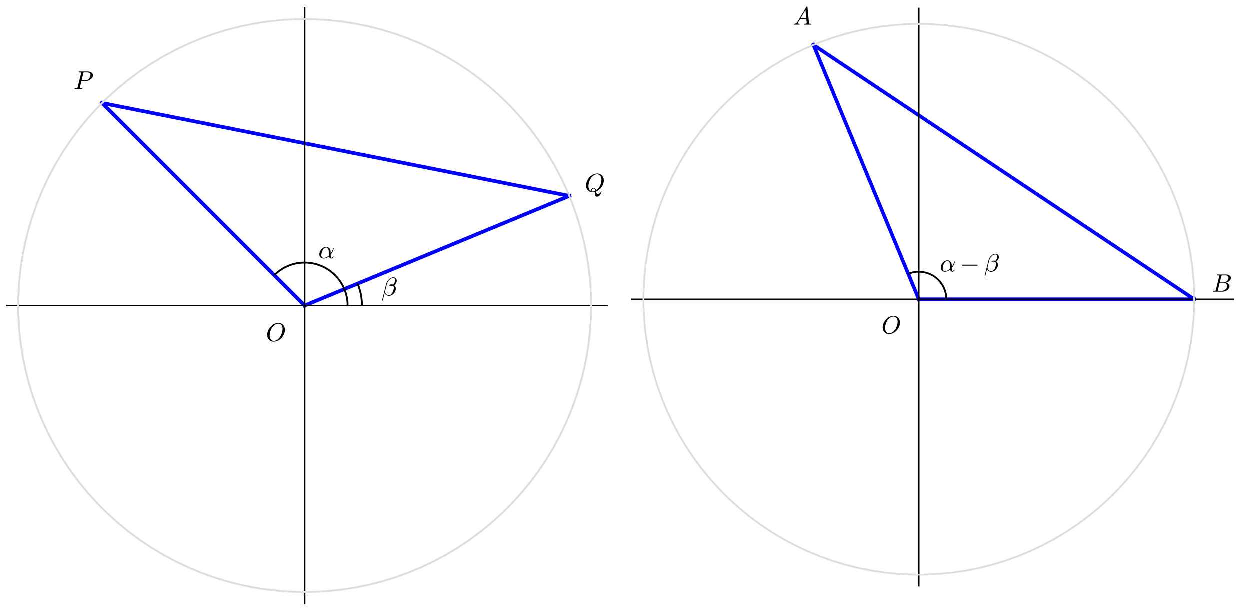


Figure 8.1.11 Triangle POQ and its rotation clockwise by β, Triangle AOB

(a) Note that angle AOB is equal to $(\alpha - \beta)$. What are the coordinates of point A?

(b) Triangles POQ and AOB are rotations of one another. What can we say about the lengths of PQ and AB?

(c) Let's use the distance formula, $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$, to find the length of PQ. What do you get when you plug in the coordinates of point P and point Q?

A. $d = \sqrt{(\cos\beta - \cos\alpha)^2 + (\sin\beta - \sin\alpha)^2}$

B. $d = \sqrt{(\cos\alpha - \sin\beta)^2 + (\sin\alpha - \cos\beta)^2}$

C. $d = \sqrt{(\cos\alpha - \cos\beta)^2 + (\sin\alpha - \sin\beta)^2}$

D. $d = \sqrt{(\sin\alpha - \cos\beta)^2 + (\cos\alpha - \sin\beta)^2}$

Hint. Remember that point P has coordinates $(\cos\alpha, \sin\alpha)$ and point Q has coordinates $(\cos\beta, \sin\beta)$.

(d) Begin simplifying your answer from part (c) by applying the algebraic identity $(a - b)^2 = a^2 - 2ab + b^2$. What do you get when squaring the two binomials under the radical?

A. $d = \sqrt{\sin^2\alpha - 2\sin\alpha\sin\beta + \sin^2\beta + \cos^2\alpha - 2\cos\alpha\cos\beta + \cos^2\beta}$

B. $d = \sqrt{\cos^2\alpha + 2\cos\alpha\cos\beta + \cos^2\beta + \sin^2\alpha + 2\sin\alpha\sin\beta + \sin^2\beta}$

C. $d = \sqrt{\cos^2\alpha - \cos\alpha\cos\beta + \cos^2\beta + \sin^2\alpha - \sin\alpha\sin\beta + \sin^2\beta}$

D. $d = \sqrt{\cos^2\alpha - 2\cos\alpha\cos\beta + \cos^2\beta + \sin^2\alpha - 2\sin\alpha\sin\beta + \sin^2\beta}$

(e) Simplify your answer from part (d) even further by applying the Pythagorean Identity, $\cos^2\theta + \sin^2\theta = 1$. What does your answer from part (d) simplify to?

A. $d = \sqrt{1 - \sin\alpha\sin\beta - \cos\alpha\cos\beta}$

B. $d = \sqrt{2 - 2\sin\alpha\sin\beta - 2\cos\alpha\cos\beta}$

C. $d = \sqrt{2 - 2\cos\alpha\cos\beta - 2\sin\alpha\sin\beta}$

D. $d = \sqrt{2 - \sin\alpha\sin\beta - \cos\alpha\cos\beta}$

Hint. You may need to rearrange some of your terms before applying the Pythagorean Identity.

(f) Using the same steps as in parts (c-e), use the distance formula to find the distance between points A and B.

A. $d = \sqrt{1 - 2\cos(\alpha - \beta)}$

B. $d = \sqrt{2 - 2\cos(\alpha - \beta)}$

C. $d = \sqrt{1 - \cos(\alpha - \beta)}$

D. $d = \sqrt{-2\cos(\alpha - \beta) + 2}$

Hint. Point A has coordinates $(\cos(\alpha - \beta), \sin(\alpha - \beta))$ and point B has coordinates $(1, 0)$.

(g) From part (b), we know that the length of PQ is equal to the length of AB. Therefore,

$$\sqrt{2 - 2\cos\alpha\cos\beta - 2\sin\alpha\sin\beta} = \sqrt{2 - 2\cos(\alpha - \beta)}.$$

Square both sides of this equation and solve for $\cos(\alpha - \beta)$.

Remark 8.1.12 In Activity 8.1.11, we derived the difference formula for cosine. The development of the sum/difference formulas are similar for sine and cosine and same for $\tan(\alpha - \beta)$ compared to $\tan(\alpha + \beta)$.

Trigonometric Identities (TE1)

Activity 8.1.13 Use the unit circle and Theorem 8.1.14 to find the exact value of $\cos 75°$.

(a) Split $75°$ into the sum of two angles which can be found on the unit circle (use unique angles such as $0°$, $30°$, $45°$, $60°$, $90°$, and $180°$, etc.).

(b) Rewrite $\cos 75°$ as $\cos(\alpha + \beta)$ where α and β are the two angles you found in part (a).

(c) Apply Theorem 8.1.14 to rewrite $\cos 75°$ with four trigonometric functions.

(d) Use the unit circle to find the value of each trigonometric function in the formula you wrote in part (c).

(e) Simplify your answer from part (d).

Trigonometric Identities (TE1)

Activity 8.1.14 Determine whether each statement is true or false by referring to Theorem 8.1.14.

(a) $\sin(45° - 30°) = \sin 45° - \sin 30°$

(b) $\cos 15° = \cos 60° \cos 45° + \sin 60° \sin 45°$

(c) $\tan 75° = \dfrac{\tan 120° - \tan 45°}{1 + \tan 120° \tan 45°}$

Activity 8.1.15 Apply Theorem 8.1.14 to find the exact value for each of the following trigonometric functions.

(a) $\sin 165°$

(b) $\tan 195°$

(c) $\cos\left(\frac{7\pi}{12}\right)$

(d) $\tan\left(\frac{11\pi}{12}\right)$

Trigonometric Identities (TE1)

Activity 8.1.16 Apply Theorem 8.1.14 to find the exact value for each of the following trigonometric functions.

(a) $\cos 87^\circ \cos 33^\circ - \sin 87^\circ \sin 33^\circ$

(b) $\sin 20^\circ \cos 80^\circ - \cos 20^\circ \sin 80^\circ$

(c) $\sin \frac{\pi}{12} \cos \frac{5\pi}{12} + \sin \frac{5\pi}{12} \cos \frac{\pi}{12}$

(d) $\dfrac{\tan 40^\circ - \tan 10^\circ}{1 + \tan 40^\circ \tan 10^\circ}$

Trigonometric Identities (TE1)

Activity 8.1.17 Recall from Theorem 8.1.14 the sum formulas for sine, cosine, and tangent.

(a) Suppose $\alpha = \beta$. Rewrite the left-hand and right-hand sides of the sum formula for sine $(\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta)$ in terms of α.

(b) Suppose $\alpha = \beta$. Rewrite the left-hand and right-hand sides of the sum formula for cosine $(\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta)$ in terms of α.

(c) Suppose $\alpha = \beta$. Rewrite the left-hand and right-hand sides of the sum formula for tangent $\left(\tan(\alpha+\beta) = \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}\right)$ in terms of α.

Trigonometric Identities (TE1)

Observation 8.1.18 You may have noticed that $\cos(2\theta)$ has three different forms. This is because we can use Pythagorean Identities to obtain the other forms.

For example, suppose we start with $\cos(2\theta) = \cos^2\theta - \sin^2\theta$. We can substitute the first $\cos^2\theta$ with $1 - \sin^2\theta$ (think about how you can rewrite the first Pythagorean Identity, $\cos^2\theta + \sin^2\theta = 1$). We will then have $1 - \sin^2\theta - \sin^2\theta$, which simplifies to $1 - 2\sin^2\theta$. Thus, $\cos(2\theta) = 1 - 2\sin^2\theta$. Using the same method, you can get $2\cos^2\theta - 1$.

Trigonometric Identities (TE1)

Activity 8.1.19 Suppose $\sin\alpha = \dfrac{2}{3}$ and α lies in Quadrant I.

(a) Use the Pythagorean Identities and Theorem 8.1.20 to find $\sin 2\alpha$.

(b) Use the Pythagorean Identities and Theorem 8.1.20 to find $\cos 2\alpha$.

(c) Use the Pythagorean Identities and Theorem 8.1.20 to find $\tan 2\alpha$.

Trigonometric Identities (TE1)

Activity 8.1.20 The double-angle formulas can be used to derive the reduction formulas, which are formulas we can use to reduce the power of a given expression involving even powers of sine or cosine.

(a) Let's start with the double angle formula for cosine to find our first power-reduction formula: $\cos 2\theta = 1 - 2\sin^2\theta$. Use your algebra skills to solve for $\sin^2\theta$.

(b) Given the second double angle formula,$\cos 2\theta = 2\cos^2\theta - 1$, solve for $\cos^2\theta$.

(c) To generate the power reduction formula for tangent, let's begin with its definition: $\tan^2\theta = \dfrac{\sin^2\theta}{\cos^2\theta}$. Use the formulas you created in parts (a) and (b) to rewrite $\tan^2\theta$.

Trigonometric Identities (TE1)

Activity 8.1.21 Rewrite $\sin^4 \beta$ as an expression without powers greater than one.

(a) How can we rewrite $\sin^4 \beta$ as a function squared?

(b) Rewrite $\sin^2 \beta$ using Theorem 8.1.24.

(c) Square the right-hand side of the equation you found in part (b).

(d) Notice that you still have a cosine function being squared in your equation in part (c). Substitute the $\cos^2 \beta$ using the power reduction formula in Theorem 8.1.24 and then simplify

Trigonometric Identities (TE1)

Activity 8.1.22 The next set of identities is the set of half-angle formulas, which can be derived from the reduction formulas. Let's derive the half-angle formula for $\sin \dfrac{\theta}{2}$.

(a) Recall the power reduction formula: $\sin^2 \theta = \dfrac{1 - \cos 2\theta}{2}$. Replace θ with $\dfrac{\theta}{2}$.

(b) Simplify the right-hand side of your equation in part (a).

(c) Now take the square root of both sides to solve for $\sin \theta$.

Remark 8.1.23 The derivation for $\cos\left(\frac{\theta}{2}\right)$ and $\tan\left(\frac{\theta}{2}\right)$ is similar to that in Activity 8.1.26 when starting with the power-reduction formulas for each trigonometric function.

Trigonometric Identities (TE1)

Activity 8.1.24 Find $\sin 15°$ using a half-angle formula in Theorem 8.1.28.

(a) Which quadrant is the angle $15°$ in?

(b) What angle is $15°$ half of?

(c) Which half angle formula should we use to determine $\sin 15°$?

(d) Substitute θ with $30°$ in the half-angle formula for sine.

(e) Use the unit circle to find the exact value of $\cos 30°$ and substitute that into the equation you got in part (d).

(f) Simplify your equation in part (e) to find the exact value of $\sin 15°$.

Trigonometric Identities (TE1)

Activity 8.1.25 Use the appropriate half-angle formula in Theorem 8.1.28 to find the exact value of each of the following.

(a) $\cos 75°$

(b) $\sin 165°$

8.2 Verifying Identities (TE2)

Objectives

- Symbolically verify identities by using various trig identities.

Verifying Identities (TE2)

Remark 8.2.1 Occasionally a question may ask you to "prove the identity" or "establish the identity." In these situations, you must show the algebraic manipulations that demonstrate that the left and right side of the equation are equal. You can think of a "prove the identity" problem as a simplification problem where you know the answer: you know what the end goal of the simplification should be, and just need to show the steps to get there.

To prove an identity, start with the expression on one side of the identity and manipulate it using algebra and trigonometric identities until you have simplified it to the expression on the other side of the equation. Do not treat the identity like an equation! The proof is establishing the two expressions are equal, so work with one side at a time rather than applying an operation simultaneously to both sides of the equation.

Verifying Identities (TE2)

Activity 8.2.2 Follow the steps to verify the identity

$$\tan\theta \cdot \cos\theta = \sin\theta.$$

(a) Rewrite the left-hand side so that each trigonometric function is written in terms of sine and cosine.

(b) Now simplify the left-hand side.

(c) Compare the simplified version of the left hand side to the original right hand side of the identity. What do you notice?

Remark 8.2.3 As we saw in Activity 8.2.2, one method that often helps in verifying identities is to rewrite everything in terms of sine and cosine to see if one side of the equation simplifies.

Verifying Identities (TE2)

Activity 8.2.4 Verify the identity:

$$\cos\theta + \sin\theta\tan\theta = \sec\theta.$$

(a) Rewrite the left-hand side in terms of sine and cosine.

(b) Now find a common denominator to add the two fractions. Add the fractions so that you have one fraction on the left-hand side.

(c) Simplify the numerator of your fraction by using one of the Pythagorean Identities.

Verifying Identities (TE2)

Activity 8.2.5 Verify the following identity:

$$\frac{\tan\theta - \cot\theta}{\sin\theta\cos\theta} = \sec^2\theta - \csc^2\theta.$$

(a) In some cases, the more complex side involves a fraction that can be split up. How can we rewrite the left side of the equation so that we end up with two fractions?

(b) Rewrite $\tan\theta$ and $\cot\theta$ in terms of sine and cosine.

(c) Simplify each complex fraction to verify that the left-hand side is equal to $\sec^2\theta - \csc^2\theta$.

Verifying Identities (TE2)

Remark 8.2.6 Starting with the more complex side can sometimes make the simplification easier.

Verifying Identities (TE2)

Activity 8.2.7 Refer back to Activity 8.2.5 to help you verify the identity:

$$\frac{\csc\theta - \sin\theta}{\sin\theta\csc\theta} = \csc\theta - \sin\theta.$$

Justify each step by listing what you used, such as algebra, Pythagorean identity, reciprocal identities, etc.

Verifying Identities (TE2)

Activity 8.2.8 Verify the following identity:

$$\frac{\cos^2\theta}{1+\sin\theta} = 1-\sin\theta$$

(a) Since the left side of the identity is more complicated, we should probably start there. We notice that the right side only involves sine. We will start by converting the cosine into something involving sine. Which identity could help us rewrite $\cos^2\theta$ into sine?

(b) Rewrite the numerator of the left-hand side into a function of sine (use the Pythagorean Identity you found in part (a)).

(c) Take a look at the numerator you now have. How can we factor the numerator?

(d) Simplify by canceling out terms.

Remark 8.2.9 As we saw in Activity 8.2.8, knowing how to factor can be very useful when simplifying trigonometric identities.

Remark 8.2.10 Using the property of conjugates is sometimes helpful in simplifying trigonometric identities. For an expression like $a + b$, the conjugate would be $a - b$. When you multiply conjugates, you often get a more useful expression. Sometimes multiplying by the conjugate will simplify an expression and help in verifying the given identity. Let's try this method in the next activity.

Verifying Identities (TE2)

Activity 8.2.11 Verify the following identity:

$$\frac{\cos\theta}{1-\sin\theta} = \frac{1+\sin\theta}{\cos\theta}.$$

(a) Let's start with the left-hand side. Multiply the left-hand side by the conjugate of $1-\sin\theta$.

Hint. When multiplying by the conjugate, do not distribute the $\cos\theta$ in the numerator.

(b) Rewrite the denominator of your fraction in part (a) so that the denominator is written in terms of cosine.

Hint. Use a Pythagorean Identity.

(c) Simplify your expression from part (b) to get the right-hand side.

Verifying Identities (TE2)

Remark 8.2.12 As we've seen from the activities from this section, there are some basic tools that can be helpful when verifying trigonometric identities. Here are some suggestions as you continue to work through these types of problems.

- Work on one side of the identity. It is usually better to start with the more complex side, as it is easier to simplify than to build.
- Look for opportunities to
  - Multiply expressions out and combine like terms.
  - Factor expressions in a fraction and to cancel common factors.
  - Split apart fractions.
  - Rewrite multiple fractions using a common denominator, and combine the fractions into a single fraction.
  - Simplify two term denominators by substituting with a Pythagorean identity.
  - Simplify two term denominators by multiplying numerator and denominator by the conjugate of the binomial denominator.
- Observe which functions are in the final expression, and look for opportunities to use identities that would lead to those functions.
- If all else fails, try rewriting all terms to sines and cosines.

Verifying Identities (TE2)

Activity 8.2.13 Use the tools you have learned in this section to verify each of the identities given below.

(a) $2\tan\theta\sec\theta = \dfrac{2\sin\theta}{1-\sin^2\theta}$

(b) $\dfrac{\sec^2\theta - 1}{\sec^2\theta} = \sin^2\theta$

(c) $\tan\theta + \dfrac{\cos\theta}{1+\sin\theta} = \sec\theta$

8.3 Trigonometric Equations (TE3)

Objectives

- Find all solutions to a trigonometric equation.

Trigonometric Equations (TE3)

Remark 8.3.1 Now that we've learned about properties of and identities involving trigonometric functions, we will now turn to solving equations that include trigonometric functions.

Trigonometric Equations (TE3)

Activity 8.3.2 Consider the equation $2\sin\theta - 1 = 0$.

(a) What first step would you take to isolate $\sin\theta$?

A. Divide both sides by 2.

B. Add 1 to both sides.

C. Subtract $2\sin\theta$ from both sides.

Hint. If given the equation $2x - 1 = 0$, what would your first step be?

(b) Continue using algebra to isolate $\sin\theta$. What equation would you have left?

A. $\sin\theta = \frac{1}{2}$

B. $\sin\theta = -\frac{1}{2}$

C. $\sin\theta = 1$

D. $\sin\theta = -1$

(c) For what values of θ, on the interval $[0, 2\pi)$, is the equation true?

A. $\frac{\pi}{6}$

B. $\frac{\pi}{3}$

C. $\frac{2\pi}{3}$

D. $\frac{5\pi}{6}$

E. $\frac{7\pi}{6}$

F. $\frac{4\pi}{3}$

Hint. Use the unit circle to help you find all the values of θ.

Trigonometric Equations (TE3)

Remark 8.3.3 One way to solve a trigonometric equation is to isolate the trigonometric term on one side. Once it has been isolated, you can then determine values of θ that satisfies the equation.

Trigonometric Equations (TE3)

Definition 8.3.4 In many cases, you will be asked to find the solution to a trigonometric equation within a given interval (such as $[0, 2\pi)$). If, however, you are asked to find the **general solution** of a trigonometric equation (like in Activity 8.3.2), you need to include all solutions. This requires the notation below, which includes all angles coterminal with the solutions in the interval $[0, 2\pi)$. If α is a solution to a trigonometric equation, then the general solution would be

$$\theta = \alpha + 2\pi n, n \in \mathbb{Z}$$

where $\mathbb{Z}$ represents the set of integers. ◊

Activity 8.3.5 From Activity 8.3.2, we were able to determine two values ($\frac{\pi}{6}$ and $\frac{5\pi}{6}$) that satisfy the equation $2\sin\theta - 1 = 0$ within the interval $[0, 2\pi)$. In this activity, we will explore other values of θ that also satisfy the equation.

(a) If we know that $\frac{\pi}{6}$ is a solution to $2\sin\theta - 1 = 0$, determine whether $\frac{\pi}{6} + 2\pi$ is a solution.

Hint. Recall that one complete revolution is 2π, so the positive x-axis can correspond to either an angle of 0 or 2π radians (or 4π, or 6π, or -2π, or -4π, etc. depending on the direction of rotation).

(b) Determine if $\frac{\pi}{6} - 2\pi$ is a solution to the equation $2\sin\theta - 1 = 0$.

(c) Determine if $\frac{\pi}{6} + \frac{\pi}{2}$ is a solution to the equation $2\sin\theta - 1 = 0$.

(d) What other solutions can you find to the equation $2\sin\theta - 1 = 0$?

(e) Refer back to parts (a) and (b) and use Definition 8.3.4 to find the general solution to $2\sin\theta - 1 = 0$.

Hint. As we see in parts (a) and (b), there are more angles that solve the equation. In fact, any angle coterminal with the either of the angles we found will also be solutions. That means, we can add or subtract any multiple of 2π to the solutions to find another solution.

Trigonometric Equations (TE3)

Activity 8.3.6 Consider the equation $(2\sin\theta + \sqrt{3})(\cos\theta - 1) = 0$.

(a) Refer back to the Zero Product Property Definition 1.5.3. What equations do we need to solve?

(b) Solve $2\sin\theta + \sqrt{3} = 0$ on the interval $[0, 2\pi)$. What values of θ satisfies that equation?

A. $\frac{\pi}{6}$

B. $\frac{5\pi}{3}$

C. $\frac{5\pi}{6}$

D. $\frac{7\pi}{6}$

E. $\frac{4\pi}{3}$

F. $\frac{11\pi}{6}$

(c) Solve $\cos\theta - 1 = 0$ on the interval $[0, 2\pi)$. What values of θ satisfies that equation?

A. 0

B. $\frac{\pi}{2}$

C. π

D. $\frac{3\pi}{2}$

E. 2π

(d) Refer back to parts (b) and (c). What are all the solutions that satisfy the equation $(2\sin\theta + \sqrt{3})(\cos\theta - 1) = 0$ on the interval $[0, 2\pi)$?

(e) Find the general solution of $(2\sin\theta + \sqrt{3})(\cos\theta - 1) = 0$.

Trigonometric Equations (TE3)

Activity 8.3.7 Trigonometric equations can also include other trig functions. Consider the equation $4 = 5 + \tan\theta$.

(a) Use algebra to isolate $\tan\theta$.

(b) For what values of θ, on the interval $[0, 2\pi)$, is the equation true?

A. $\dfrac{\pi}{4}$

B. $\dfrac{3\pi}{4}$

C. π

D. $\dfrac{5\pi}{4}$

E. $\dfrac{3\pi}{2}$

F. $\dfrac{7\pi}{4}$

Hint. Recall that $\tan\theta = \frac{\sin\theta}{\cos\theta}$.

(c) What is the general solution of $4 = 5 + \tan\theta$?

Trigonometric Equations (TE3)

Activity 8.3.8 Consider the equation $(\sin\theta - 1)(\csc\theta + 2) = 0$.

(a) Refer back to the Zero Product Property. What two equations do we solve?

(b) Solve $\sin\theta - 1 = 0$ on the interval $[0, 2\pi)$. What value(s) of θ satisfy that equation?

A. 0 B. $\frac{\pi}{2}$ C. π D. $\frac{3\pi}{2}$

(c) To solve $\csc\theta + 2 = 0$, isolate the $\csc\theta$ term. What does $\csc\theta$ equal?

(d) Recall that $\csc\theta = \frac{1}{\sin\theta}$. If $\csc\theta = -2$, what does $\sin\theta$ equal?

A. $\sin\theta = 2$

B. $\sin\theta = -2$

C. $\sin\theta = \frac{1}{2}$

D. $\sin\theta = -\frac{1}{2}$

(e) Using what you know from part (d), what value(s) of θ satisfy that equation $\csc\theta = -2$ on the interval $[0, 2\pi)$?

A. $\frac{\pi}{6}$

B. $\frac{2\pi}{3}$

C. $\frac{5\pi}{6}$

D. $\frac{7\pi}{6}$

E. $\frac{4\pi}{3}$

F. $\frac{11\pi}{6}$

(f) Refer back to parts (b) and (e). What are all the solutions that satisfy the equation $(\sin\theta - 1)(\csc\theta + 2) = 0$ on the interval $[0, 2\pi)$?

(g) Find the general solution of $(\sin\theta - 1)(\csc\theta + 2) = 0$.

Trigonometric Equations (TE3)

Activity 8.3.9 Solve each equation on the interval $[0, 2\pi)$.

(a) $3 + \sin\theta = 2$

(b) $\sin\theta(\tan\theta - 1) = 0$

Hint. Recall that $\tan\theta = \frac{\sin\theta}{\cos\theta}$.

(c) $3\tan\theta = 3\sqrt{3}$

(d) $\cos\theta + \sqrt{3} = -\cos\theta$

(e) $(\sec\theta - 1)(2\cos\theta + 1) = 0$

Hint. Recall that $\sec\theta = \frac{1}{\cos\theta}$.

Trigonometric Equations (TE3)

Remark 8.3.10 In some cases, it is not easy to isolate the trigonometric term like we have been doing so far. The next few activities will help us learn how to apply methods we have used before when solving algebraic equations to solve trigonometric equations.

Activity 8.3.11 Consider the equation $4\sin^2\theta - 2 = 0$.

(a) Use algebra to isolate the $\sin^2\theta$ term.

(b) To undo a squared term, you will need to take the square root (recall Definition 1.5.9). Take the square root of $\sin^2\theta$ to get $\sin\theta$. What does $\sin\theta$ equal?

A. $\pm\sqrt{\frac{1}{2}}$ B. $\pm\frac{1}{\sqrt{2}}$ C. $\pm\frac{1}{2}$ D. $\pm\frac{\sqrt{2}}{2}$

(c) Although there are multiple ways to represent $\sin\theta$, which solution(s) from part (b) do you see on the unit circle?

A. $\pm\sqrt{\frac{1}{2}}$ B. $\pm\frac{1}{\sqrt{2}}$ C. $\pm\frac{1}{2}$ D. $\pm\frac{\sqrt{2}}{2}$

(d) Notice there are two values of $\sin\theta$ to consider, which corresponds to four values of θ that satisfy the equation $\sin\theta = \pm\frac{\sqrt{2}}{2}$ on the interval $[0, 2\pi)$. What are these values of θ?

A. $\frac{\pi}{4}$

B. $\frac{\pi}{3}$

C. $\frac{3\pi}{4}$

D. $\frac{5\pi}{6}$

E. $\frac{5\pi}{4}$

F. $\frac{7\pi}{4}$

Trigonometric Equations (TE3)

Activity 8.3.12 Consider the equation $3\sec^2\theta - 4 = 0$.

(a) Using algebra, isolate the $\sec^2\theta$ term.

(b) Use the square root property to solve for $\sec\theta$. What does $\sec\theta$ equal?

A. $\pm\frac{4}{3}$ B. $\pm\frac{2}{3}$ C. $\pm\frac{2}{\sqrt{3}}$ D. $\pm\frac{\sqrt{3}}{2}$

(c) Recall that $\sec\theta = \frac{1}{\cos\theta}$. Knowing this relationship, what does $\cos\theta$ equal?

A. $\pm\frac{3}{2}$ B. $\pm\frac{2}{3}$ C. $\pm\frac{2}{\sqrt{3}}$ D. $\pm\frac{\sqrt{3}}{2}$

(d) What values of θ satisfy this equation on the interval $[0, 2\pi)$?

A. $\frac{\pi}{6}$

B. $\frac{\pi}{3}$

C. $\frac{5\pi}{6}$

D. $\frac{7\pi}{6}$

E. $\frac{4\pi}{3}$

F. $\frac{11\pi}{6}$

Remark 8.3.13 Notice in Activity 8.3.11 and Activity 8.3.12, we had to use the square root property to isolate the trigonometric term. The next few activities will highlight another tool to solve trigonometric equations.

Trigonometric Equations (TE3)

Activity 8.3.14 In this activity, we will solve trigonometric equations by first factoring. Consider the equation

$$2\sin\theta\cos\theta - \sqrt{2}\cos\theta = 0.$$

(a) One of the first methods of factoring you should always start with is GCF (Greatest Common Factor). Look at each term in the equation. What is the GCF?

(b) Factor out the GCF. What equation do you now have?

A. $2\sin\theta - \sqrt{2} = 0$

B. $\cos\theta(2\sin\theta - \sqrt{2}\cos\theta) = 0$

C. $\sin\theta(2\cos\theta - \sqrt{2}\cos\theta) = 0$

D. $\cos\theta(2\sin\theta - \sqrt{2}) = 0$

(c) Now use the Zero Product Property to create two equations. What two equations do you now have?

A. $2\sin\theta - \sqrt{2} = 0$

B. $\sin\theta = 0$

C. $\cos\theta = 0$

D. $2\sin\theta - \sqrt{2}\cos\theta = 0$

(d) Solve each of the equations you got in part (c). What value(s) of θ satisfy both equations on the interval $[0, 2\pi)$?

A. 0

B. $\frac{\pi}{2}$

C. π

D. $\frac{3\pi}{2}$

E. $\frac{\pi}{4}$

F. $\frac{3\pi}{4}$

Trigonometric Equations (TE3)

Activity 8.3.15 Consider the equation $2\cos^2\theta - 5\cos\theta - 3 = 0$.

(a) Notice that this trigonometric equation looks similar to a quadratic equation. Let's we replace every $\cos\theta$ with x to get $2x^2 - 5x - 3 = 0$. What are the factors of the left hand side of this equation?

A. $(2x + 1)$

B. $(x + 3)$

C. $(2x - 1)$

D. $(x - 3)$

(b) Now use the Zero Product Property to create two equations in terms of $\cos\theta$. What two equations do you now have?

A. $(2\cos\theta + 1)$

B. $(\cos\theta + 3)$

C. $(2\cos\theta - 1)$

D. $(\cos\theta - 3)$

(c) Solve each of the equations you got in part (b). What values of θ satisfies both equations on the interval $[0, 2\pi)$?

A. 0

B. $\dfrac{\pi}{3}$

C. $\dfrac{2\pi}{3}$

D. $\dfrac{4\pi}{3}$

E. $\dfrac{5\pi}{3}$

F. 2π

Remark 8.3.16 Notice in Activity 8.3.15, one of the equations did not provide a value of θ that satisfies the equation. This can happen sometimes!

Trigonometric Equations (TE3)

Activity 8.3.17 Solve each of the following trigonometric equations on the interval $[0, 2\pi)$.

(a) $2\cos^2\theta + \cos\theta = 0$

(b) $\sin^2\theta - 1 = 0$

(c) $2\cos^2\theta - \cos\theta = 3$

Hint. Get everything over to one side before you factor!

(d) $2\sin^2\theta - 4 = 7\sin\theta$

(e) $2\sec^2\theta - 2\sec\theta - 4 = 0$

Trigonometric Equations (TE3)

Remark 8.3.18 In this last activity, let's look at trigonometric equations that involve multiple angles.

Trigonometric Equations (TE3)

Activity 8.3.19 Solve the equation $\sin(2\theta) = \frac{1}{2}$. Find the general solution, then find all solutions on the interval $[0, 2\pi)$.

(a) Notice that this equation involves $\sin(2\theta)$, a multiple angle, rather that $\sin(2\theta)$ like we have been dealing with. Let's replace 2θ with α so it looks more like the equations we've been solving so far. What values of α would satisfy the new equation $\sin\alpha = \frac{1}{2}$ on the interval $[0, 2\pi)$?

A. $\frac{\pi}{6}$

B. $\frac{\pi}{3}$

C. $\frac{5\pi}{6}$

D. $\frac{4\pi}{3}$

E. $\frac{5\pi}{3}$

F. $\frac{11\pi}{6}$

(b) What would the general solution of the equation $\sin\alpha = \frac{1}{2}$ be?

(c) For $\sin\alpha = \frac{1}{2}$, we know the general solution is

$$\alpha = \frac{\pi}{6} + 2\pi(n)$$

and

$$\alpha = \frac{5\pi}{6} + 2\pi(n).$$

And because $\alpha = 2\theta$, we can replace α with 2θ in those solutions. Therefore,

$$2\theta = \frac{\pi}{6} + 2\pi(n)$$

and

$$2\theta = \frac{5\pi}{6} + 2\pi(n).$$

Solve each general equation to find values of θ that satisfies the equation $\sin(2\theta) = \frac{1}{2}$.

Hint. Solve for θ.

(d) We now need to find all values of θ that satisfy our original equation on the interval $[0, 2\pi)$. We can find these angles by substituting values of n in the general solution. We will start with $n = 0$. What values of θ result when $n = 0$? Are these angles on the interval $[0, 2\pi)$?

(e) Now let $n = 1$. What values of θ would satisfy the equation $\sin(2\theta) = \frac{1}{2}$ on the interval $[0, 2\pi)$?

(f) Suppose $n = 2$. What values of θ would satisfy the equation $\sin(2\theta) = \frac{1}{2}$ on the interval $[0, 2\pi)$?

(g) Let's now try negative values of n. Suppose $n = -1$. What values of θ would satisfy the equation $\sin(2\theta) = \frac{1}{2}$ on the interval $[0, 2\pi)$?

(h) Look back at parts (d)-(g). Notice that that once we reach values outside of $[0, 2\pi)$ in either direction, we can stop substituting values of n in that direction. What are all the solutions to $\sin(2\theta) = \frac{1}{2}$ on the interval $[0, 2\pi)$?

Trigonometric Equations (TE3)

Observation 8.3.20 Recall that $y = \sin(2\theta)$ is a horizontal compression by a factor of 2 of the function $y = \sin\theta$. On an interval of 2π, we can graph two periods of $y = \sin(2\theta)$, as opposed to one cycle of $y = \sin\theta$. This compression of the graph suggests there may be twice as many x-intercepts or solutions to $\sin(2\theta) = \frac{1}{2}$ compared to $\sin\theta = \frac{1}{2}$.

8.4 Solving Right Triangles (TE4)

Objectives

- Use trigonometric functions and the Pythagorean Theorem to solve right triangles.

Solving Right Triangles (TE4)

Activity 8.4.1 Let's revisit the triangle ABC from Activity 6.3.12, where $a = 35$, $b = 12$, and $c = 37$, with c being the hypotenuse of the triangle.

(a) Find $\cos A$.

A. $\frac{12}{35}$

B. $\frac{12}{37}$

C. $\frac{37}{35}$

D. $\frac{37}{12}$

(b) Suppose we want to know the measure of angle A. We can find the measure of angle A in three different ways by using either sine, cosine, or tangent (since all side lengths are given). For each trigonometric function, write the trigonometric ratio that can be used to find the measure of angle A.

(c) Now that we have set up a trigonometric ratio to help find the measure of angle A, how can we use these ratios to determine how big A is?

Solving Right Triangles (TE4)

Remark 8.4.2 Sometimes you will need to use inverse trigonometric functions (e.g. $\arcsin(x)$) to find the measure of an angle. Recall from Section 7.3 that the inverse trig function keys on your calculator are usually denoted $\sin^{-1}$.

Solving Right Triangles (TE4)

Activity 8.4.3 Refer back to Activity 6.3.14, where you were given all the sides of a right triangle, but no angle measures. (In triangle ABC, $a = 35$, $b = 12$, and $c = 37$, with c being the hypotenuse of the triangle).

(a) What is the trigonometric ratio for $\cos A$?

A. $\frac{35}{12}$

B. $\frac{35}{37}$

C. $\frac{12}{35}$

D. $\frac{12}{37}$

(b) Use the inverse trig function, $\cos^{-1}$ to find the measure of angle A. (Make sure your calculator is in degree mode!)

(c) What is the trigonometric ratio for $\sin A$?

A. $\frac{35}{12}$

B. $\frac{35}{37}$

C. $\frac{12}{35}$

D. $\frac{12}{37}$

(d) Use the inverse trig function, $\sin^{-1}$ to find the measure of angle A. (Make sure your calculator is in degree mode!)

(e) What is the trigonometric ratio for $\tan A$?

A. $\frac{35}{12}$

B. $\frac{35}{37}$

C. $\frac{12}{35}$

D. $\frac{12}{37}$

(f) Use the inverse trig function, $\tan^{-1}$ to find the measure of angle A. (Make sure your calculator is in degree mode!)

(g) Refer back to parts (b), (d), and (f). What do you notice about your answers from those parts?

(h) Now that we know the measure of angle A, find the measure of angle B.

Solving Right Triangles (TE4)

Remark 8.4.4 Determining all of the side lengths and angle measures of a right triangle is known as **solving** a right triangle. In Activity 6.3.14 and Activity 8.4.3, we were given all the sides of the triangle and used trigonometric ratios to determine the measure of the angles.

Solving Right Triangles (TE4)

Activity 8.4.5 Solve the following triangles using your knowledge of right triangles, the Pythagorean Theorem and trigonometric functions. Be sure to draw a picture to help you determine the relationship between the angles and sides.

(a) In triangle ABC, $B = 53°$ and $c = 5$ meters (with c being the hypotenuse).

(b) In triangle ABC, $A = 28°$ and $b = 29.3$ miles (with c being the hypotenuse).

(c) In triangle ABC, $a = 8$ feet, $b = 17$ feet, and $c = 15$ feet (with b being the hypotenuse).

8.5 Trigonometric Laws (TE5)

Objectives

- Use the law of sines and/or the law of cosines to solve non-right triangles.

Trigonometric Laws (TE5)

Remark 8.5.1 In Section 6.4 and Section 8.4, we learned how to solve right triangles (and special right triangles) using trigonometric ratios. In this section, we will learn how to solve oblique (non-right triangles).

Activity 8.5.2 Suppose you are given the following triangle, where h is the altitude of triangle ABC. By drawing the altitude, we've now created two right triangles. If we label the point where the altitude intersects C as point D, then we can call the triangles ACD and BCD.

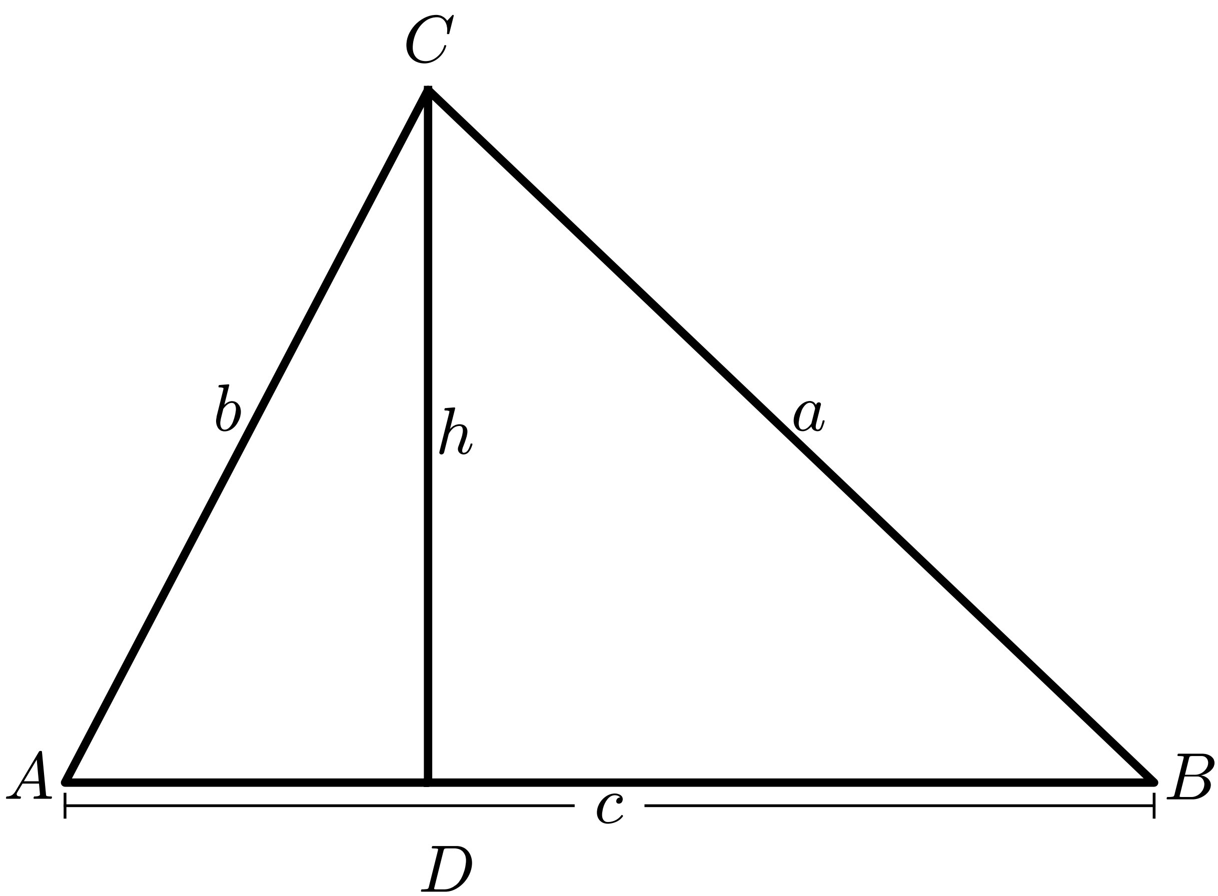


Figure 8.5.3

(a) Find $\sin A$.

A. $\sin A = \dfrac{a}{h}$

B. $\sin A = \dfrac{h}{b}$

C. $\sin A = \dfrac{h}{a}$

D. $\sin A = \dfrac{a}{b}$

(b) Find $\sin B$.

A. $\sin B = \dfrac{a}{h}$

B. $\sin B = \dfrac{h}{b}$

C. $\sin B = \dfrac{h}{a}$

D. $\sin B = \dfrac{a}{b}$

(c) Take your answers from parts (a) and (b) and solve each equation for h (the altitude). What does h equal in each case?

(d) Notice that we now have two ways to express h. Set the two expressions of h equal to one another and then rearrange this equation so that a and $\sin A$ are on the same side.

Remark 8.5.4 The Law of Sines is helpful when we have

1. two angles and one of their corresponding sides.
2. two sides and one of their corresponding angles.

Activity 8.5.5 Suppose you are given triangle ABC where $A = 35°$, $B = 25°$, and $a = 10$ as shown in the figure below.

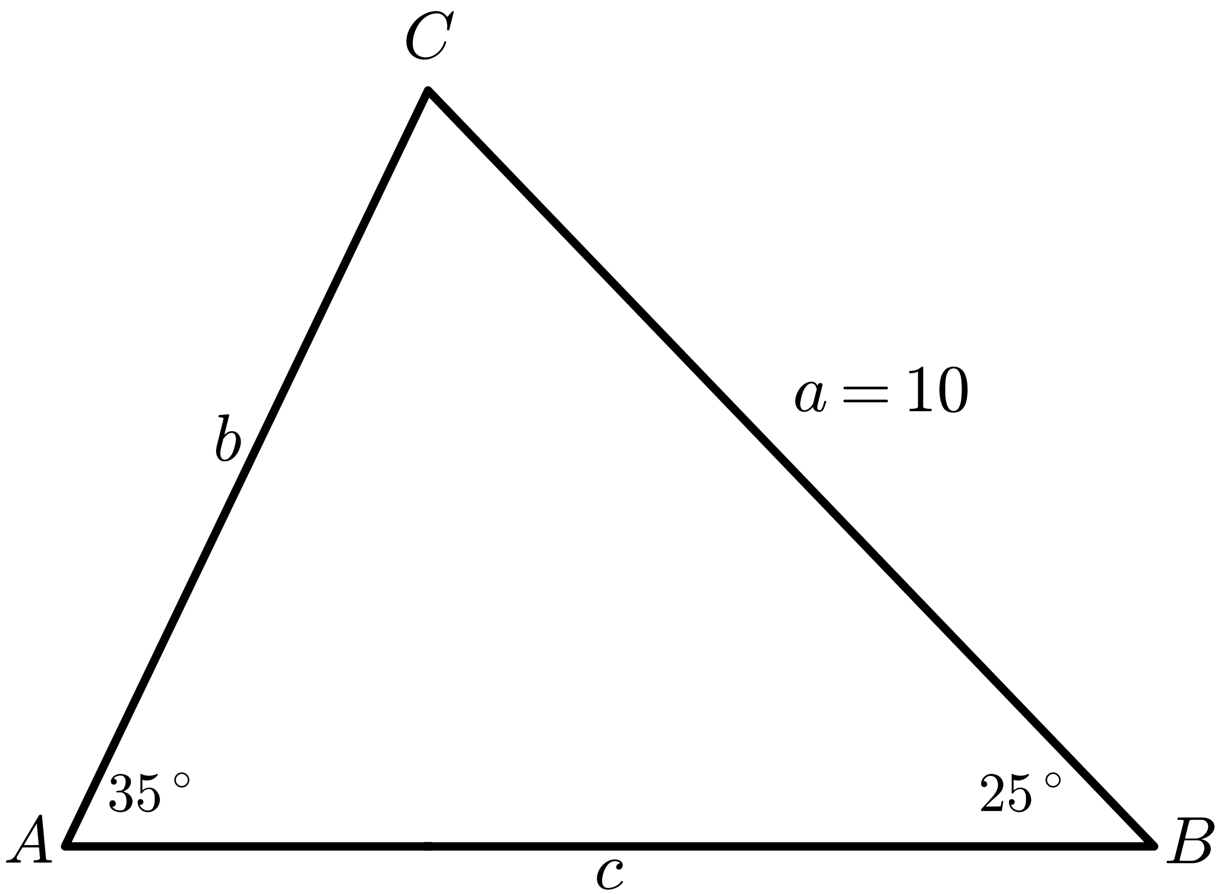


Figure 8.5.6

(a) What parts of the triangle are missing?

(b) What is the value of C?

A. 60°   B. 155°   C. 145°   D. 120°

Hint. 180°

(c) Use the Law of Sines to set up a proportion to find c. What does that proportion look like?

(d) Solve for c from the proportion you got in part (c). What is the value of c to the nearest tenth?

A. 8.7   B. 6.6   C. 15.1   D. 0.1

(e) We now have side b left to find. Which method would be the best method to find b?

A. Use the Law of Sines again with angle B, angle C, and side c.

B. Use the Pythagorean Theorem

C. Use the Law of Sines again with angle A, side a, and angle B.

(f) Use the method you identified in the previous part to find b. What is the value of b to the nearest tenth?

A. 5.7 B. 4.2 C. 13.6 D. 7.4

Remark 8.5.7 The triangle in Activity 8.5.6 is known as an AAS (Angle-Angle-Side) triangle, which means we know two angles and one side (which is NOT between the angles) in that order. This is just one type of triangle you might see when solving oblique triangles. The next activity will show another type of triangle you might encounter.

Activity 8.5.8 Suppose you are given triangle ABC where $A = 76°$, $B = 34°$, and $c = 9$ as shown in the figure below.

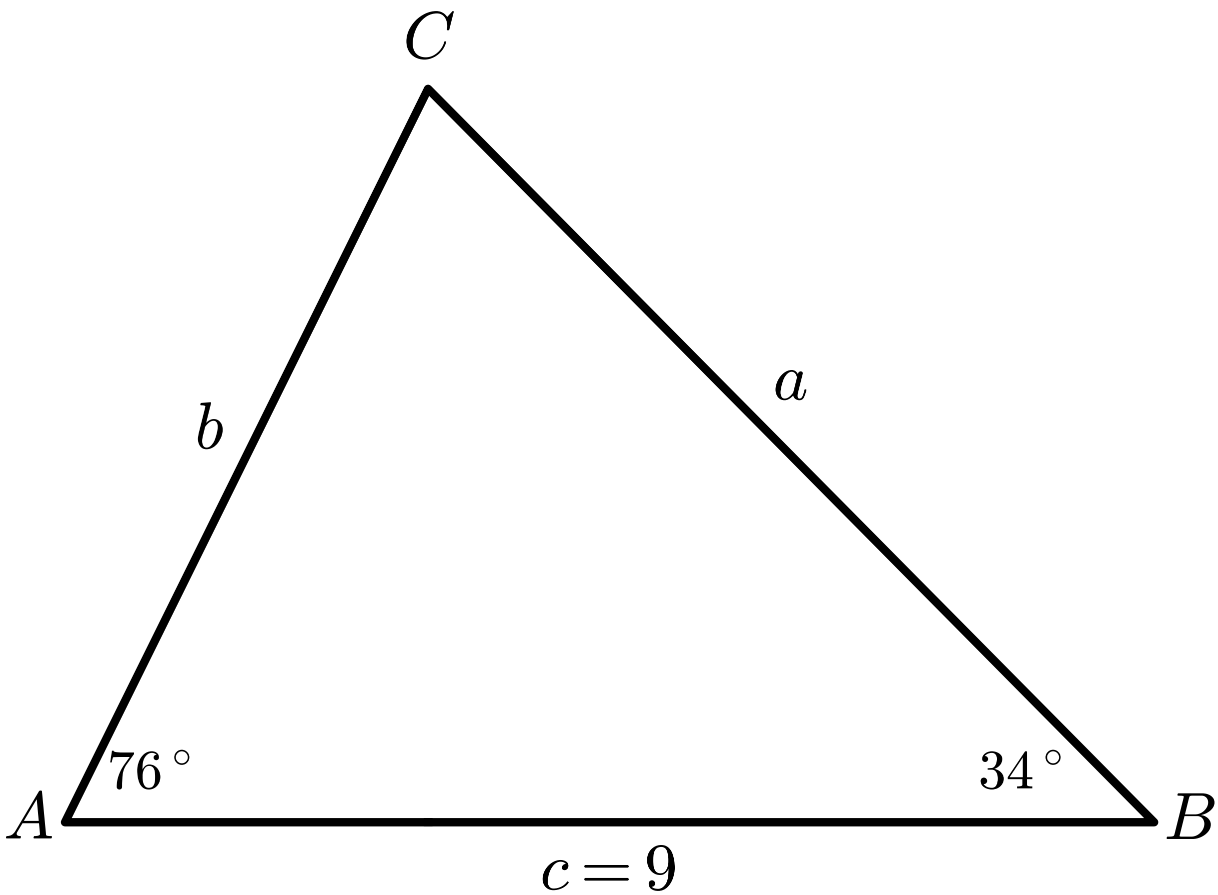


Figure 8.5.9

(a) What parts of the triangle are missing?

(b) What is the value of C?

A. 70° B. 110° C. 104° D. 146°

(c) Use the Law of Sines to set up a proportion to find a. What is the value of a to the nearest tenth?

A. 8.7 B. 9.3 C. 8.5 D. 9.1

(d) Use the Law of Sines to set up a proportion to find b. What is the value of b to the nearest tenth?

A. 5.0 B. 8.5 C. 15.1 D. 5.4

Remark 8.5.10 The triangle in Activity 8.5.9 is known as an ASA (Angle-Side-Angle) triangle, which means we know two angles and a side BETWEEN the angles.

Activity 8.5.11 Suppose you are given triangle ABC, where $A = 30°$, $a = 7$, and $b = 16$ as shown in the figure below.

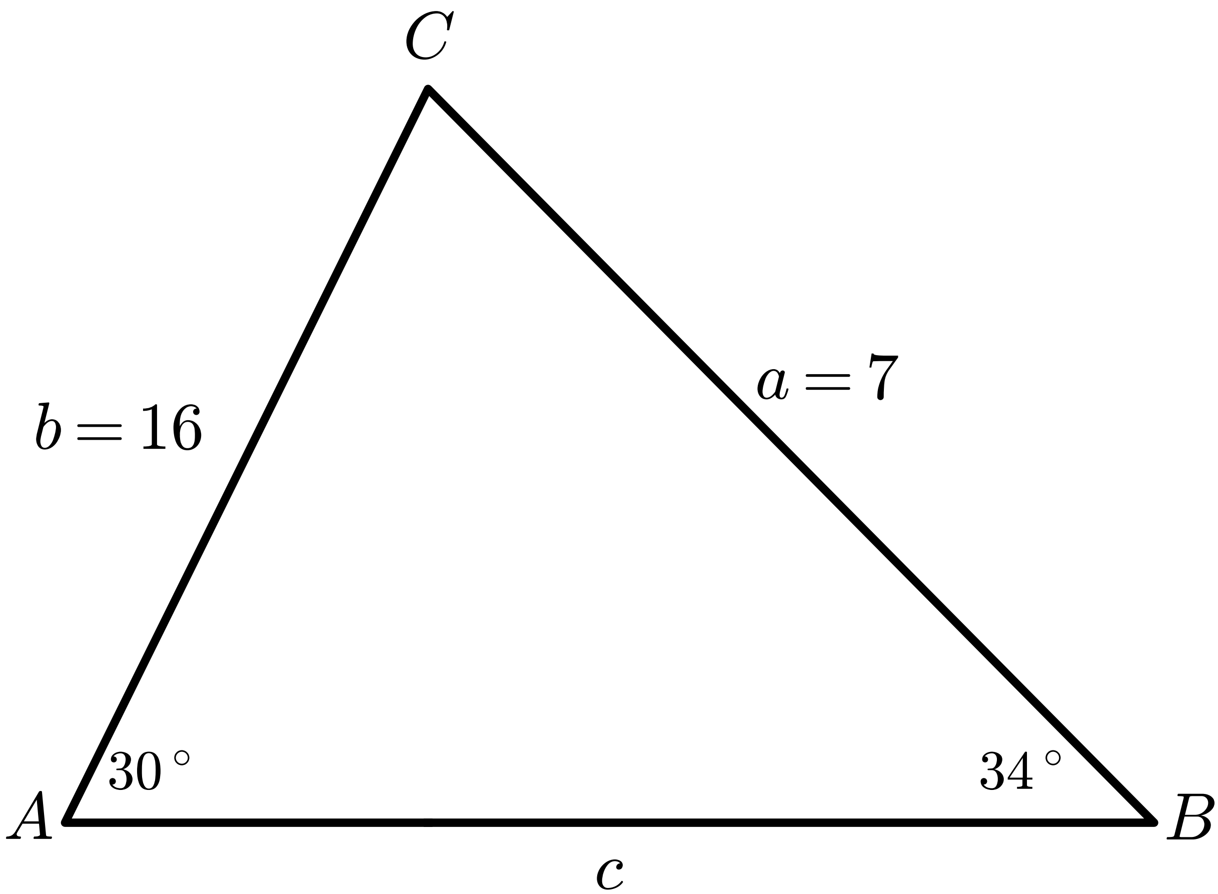


Figure 8.5.12

(a) What parts of the triangle are missing?

(b) Use the Law of Sines to set up a proportion to find $\sin B$. What is the value of $\sin B$?

A. 8.0 B. 1.14 C. 0.875 D. 0.031

(c) Use part (b) to determine the value of B.

Remark 8.5.13 In Activity 8.5.12, we can see that no triangle can be created because $\sin B$ was equal to a value greater than 1. This is because the largest value that the sine of an angle can have is 1 (refer back to Section 6.5).

Trigonometric Laws (TE5)

Activity 8.5.14 Suppose you are given triangle ABC, where $A = 30°$, $a = 10$, and $b = 16$.

(a) Draw triangle ABC and label what is known. What parts of the triangle are missing?

(b) Use the Law of Sines to determine the value of B (to the nearest degree).

A. 143° B. 53° C. 80° D. 37°

(c) Continue solving triangle ABC. What are the missing values (to the nearest degree/ whole number)?

(d) Let's go back to part (b) when we were asked to solve for B. We needed to solve $\sin B = \frac{4}{5}$. Using the inverse sine function on our calculator, we got that $B \approx 53°$. However, there is another angle between 0° and 180° whose sine is $\frac{4}{5}$. Which of the following values could also be B?

A. 143° B. 37° C. 80° D. 127°

(e) Why do you think there are two values of B for this triangle?

(f) Continue solving this other triangle ABC. What are the missing values (to the nearest degree/whole number)?

Remark 8.5.15 So far we have seen that when given two sides and an angle (also known as a SSA (Side-Side-Angle) triangle), we can have no solution (i.e., no triangle can be created) or two solutions (i.e., there are two possible triangles). There is still one more case we need to explore.

Trigonometric Laws (TE5)

Activity 8.5.16 Suppose you are given triangle ABC, where $A = 30°$, $a = 20$, and $b = 16$.

(a) Draw triangle ABC and label what is known. What parts of the triangle are missing?

(b) Use the Law of Sines to determine the value of B (to the nearest degree).

A. 24° B. 39° C. 141° D. 156°

(c) Choose the smaller value of B and solve triangle ABC. What are the missing values (to the nearest degree/whole number)?

(d) Choose the larger value of B and solve triangle ABC. What are the missing values (to the nearest degree/whole number)?

Observation 8.5.17 In the previous three activities, we saw that when we use the Law of Sines to find an angle, an ambiguity can arise due to the sine function being positive in Quadrant I and Quadrant II. In other words, if two sides and the non-included angle are given (SSA), three situations may occur.

1. NO triangle exists (no solution)
2. TWO different triangle exist (2 solutions)
3. Exactly ONE triangle exists (1 solution)

The **Ambiguous Case of the Law of Sines** states that when using the Law of Sines to find a missing side length, the possibility of two solutions for the measure of the same side may occur.

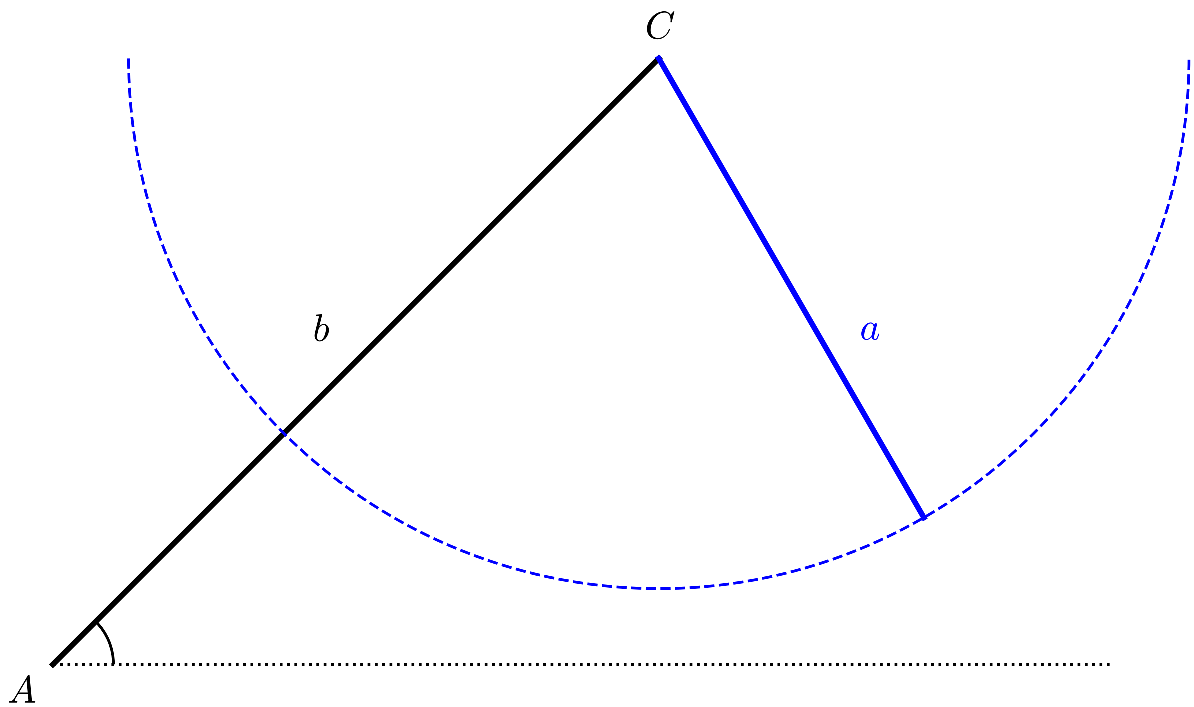


Figure 8.5.18 If a is too short for a given angle A and side b, no matter how that leg is swung around the dashed circle, it will not meet a third side of any length along the dotted line.

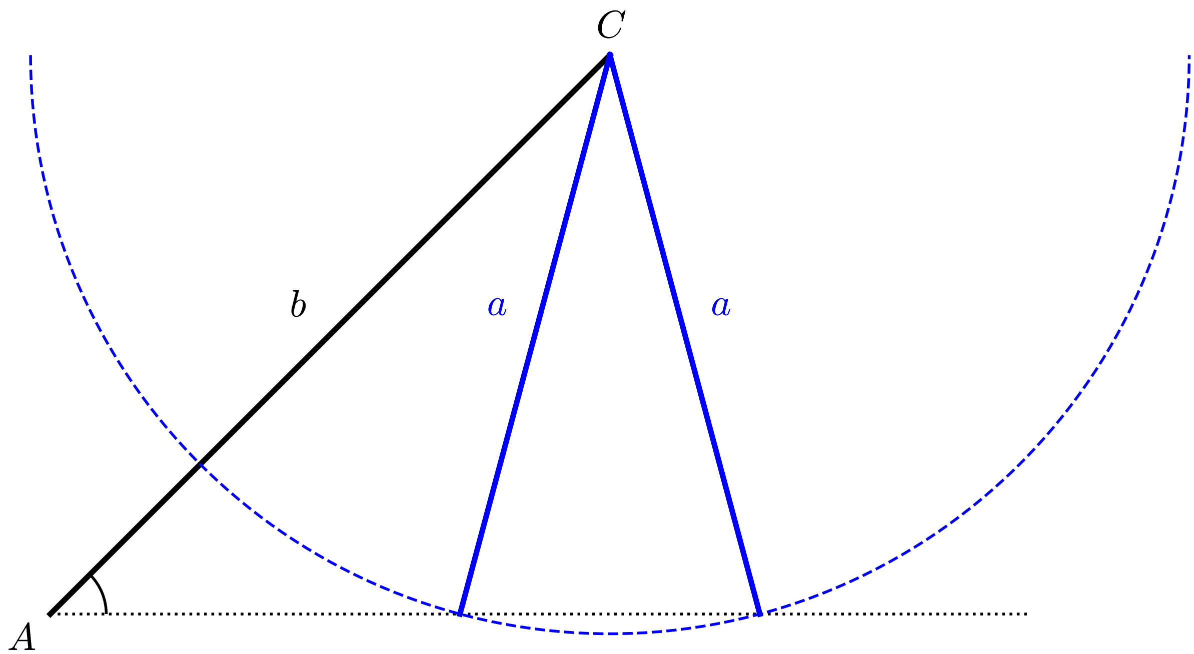


Figure 8.5.19 If a is larger, the circle of radius a centered at C intersects the horizontal dashed line in two places, giving two possible solutions.

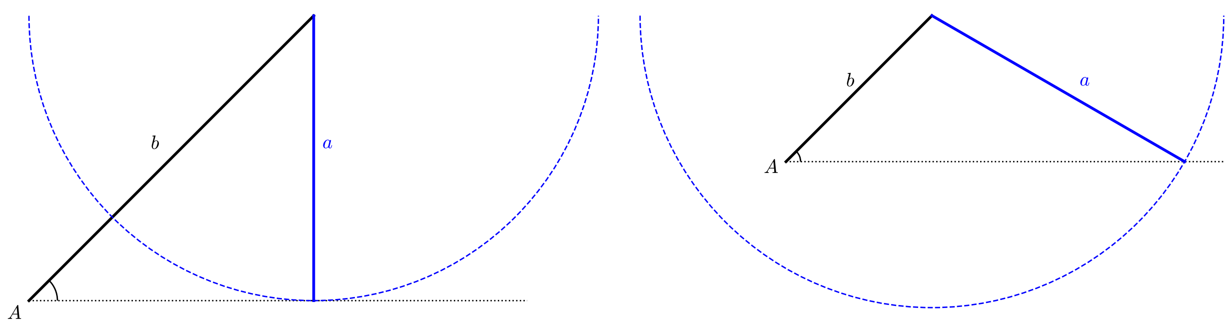


Figure 8.5.20 If the length of a is the same as the height of the triangle, the two possibilities converge and there is a single solution of a right triangle. On the other hand, if the length of a increases enough, the potential second solution swings past the angle A and thus does not exist.

Trigonometric Laws (TE5)

Activity 8.5.21 State the number of possible triangles that can be formed with the given measurements. Then, solve each triangle. Round your answers to the nearest tenth.

(a) $C = 145°$, $b = 7$, $c = 33$

(b) $B = 84°$, $a = 18$, $b = 9$

(c) $B = 45°$, $a = 28$, $b = 27$

Trigonometric Laws (TE5)

Activity 8.5.22 Suppose you are given triangle ABC, where $A = 70°$, $b = 14$, and $c = 7$.

(a) Draw triangle ABC and label what is known. What parts of the triangle are missing?

(b) Use the Law of Sines to solve triangle ABC.

Remark 8.5.23 Notice in Activity 8.5.23, we do not currently have enough information to be able to solve for triangle ABC since the Law of Sines cannot be used. In the next activity, we will explore another method that can be used to solve oblique triangles.

Activity 8.5.24 Suppose you are given the following triangle, where h is the altitude of triangle ABC.

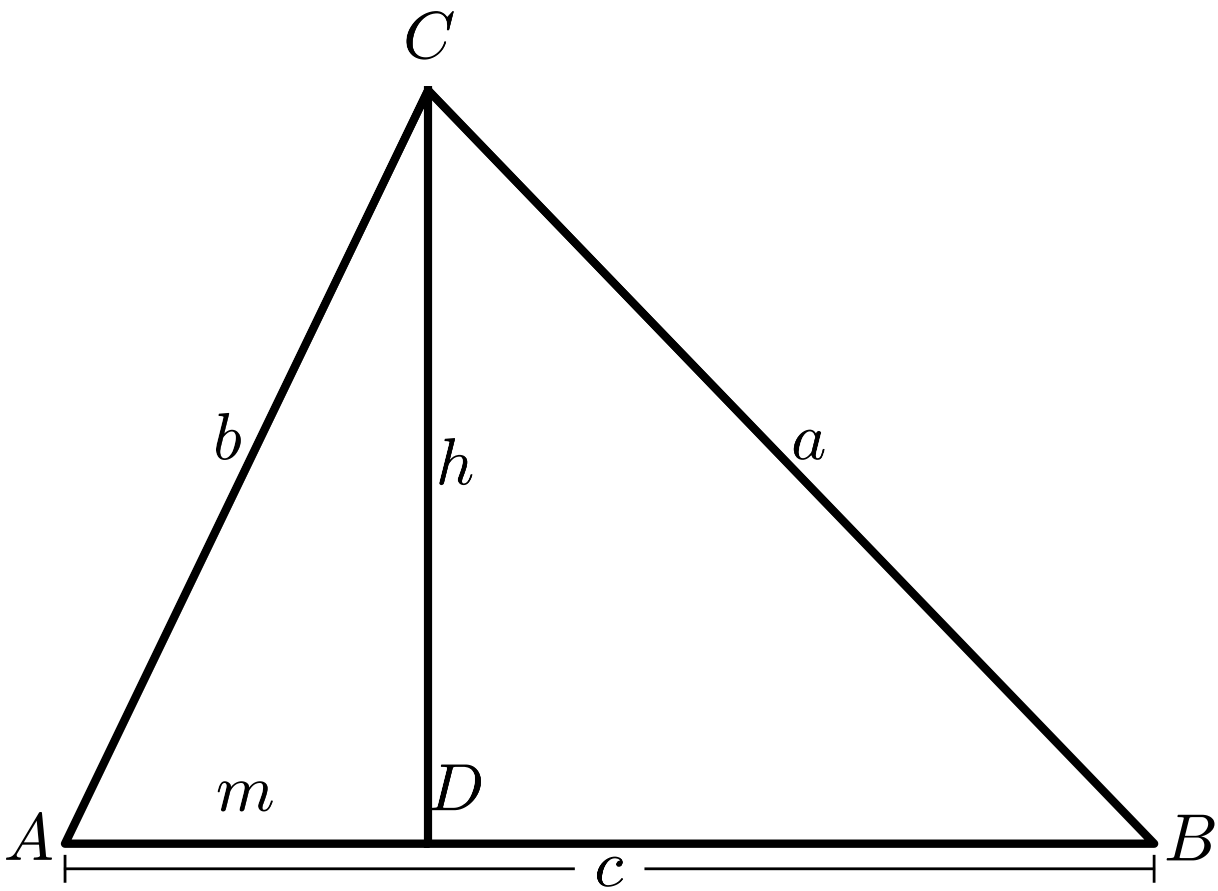


Figure 8.5.25

(a) Using right triangle trigonometry, find $\sin A$.

A. $\sin A = \dfrac{b}{h}$

B. $\sin A = \dfrac{h}{b}$

C. $\sin A = \dfrac{m}{b}$

D. $\sin A = \dfrac{b}{m}$

(b) Using right triangle trigonometry, find $\cos A$.

A. $\cos A = \dfrac{b}{h}$

B. $\cos A = \dfrac{h}{b}$

C. $\cos A = \dfrac{m}{b}$

D. $\cos A = \dfrac{b}{m}$

(c) Take your equation in part (a) and solve for h.

(d) Take your equation in part (b) and solve for m.

(e) Recall that we can use the Pythagorean Theorem to represent the relationship between the sides of a right triangle. For example, $h^2 + (c - m)^2 = a^2$ can be used to represent the relationship of the sides of triangle BCD. Take your equations from parts (c) and (d) to substitute m and h into that equation.

(f) Simplify your equation from part (e).

Hint. When simplifying the term $(c - b\cos A)^2$, don't forget that $(c - b\cos A)^2 = (c - b\cos A) * (c - b\cos A)$. You may also have to use a trig identity to simplify!

Trigonometric Laws (TE5)

Activity 8.5.26 Let's revisit Activity 8.5.23 to see how we can apply the Law of Cosines to solve for this triangle. Recall that we were given triangle ABC, where $A = 70°$, $b = 14$, and $c = 7$.

(a) Draw triangle ABC and label what is known. What parts of the triangle are missing?

(b) Use the Law of Cosines to find a (to the nearest tenth).

A. 178.0 B. 17.7 C. 13.3 D. 7.8

(c) Now that we have another side of triangle ABC, what would you use to find angle B?

A. Law of Sines

B. Law of Cosines

C. Pythagorean Theorem

(d) Find the measure of B and C (to the nearest degree).

Trigonometric Laws (TE5)

Activity 8.5.27 Suppose you are given triangle ABC, where $a = 30$, $b = 20$, and $c = 17$.

(a) Draw triangle ABC and label what is known. What parts of the triangle are missing?

(b) Which law should you start with to solve triangle ABC?

(c) What is the value of A (to the nearest degree)?

A. 109° B. 39° C. 108° D. 33°

(d) Now that you have found angle A, which law would be the best to use to find the measure of angle B?

(e) What is the value of B (to the nearest degree)?

A. 109° B. 39° C. 108° D. 33°

(f) What is the value of C (to the nearest degree)?

A. 109° B. 39° C. 108° D. 33°

Remark 8.5.28 Notice in Activity 8.5.29, you were given a triangle with three known sides. This type of triangle is known as a side-side-side (SSS) triangle.

Trigonometric Laws (TE5)

Activity 8.5.29 State whether the Law of Sines or the Law of Cosines is the best choice to use to determine the indicated angle/side.

(a) Given $B = 112°$, $a = 12$, and $b = 25$, find A.

Hint. It might help to draw a picture!

(b) Given $B = 87°$, $a = 15$, and $c = 16$, find b.

(c) Given $a = 37$, $b = 55$, and $c = 30$, find C.

(d) Given $A = 108°$, $B = 40°$, and $b = 20$, find c.

Activity 8.5.30 Now that we can determine which laws to use, let's go back to Activity 8.5.31 to solve each triangle (to the nearest degree/whole number).

(a) $B = 112°$, $a = 12$, and $b = 25$.

(b) $B = 87°$, $a = 15$, and $c = 16$.

(c) $a = 37$, $b = 55$, and $c = 30$.

(d) $A = 108°$, $B = 40°$, and $b = 20$.

8.6 Applications of Trigonometry (TE6)

Objectives

- Solve applications that require the use of trigonometry to find a side or angle.

Applications of Trigonometry (TE6)

Remark 8.6.1 Trigonometry is the branch of mathematics that focuses on the relationships between the angles and sides of triangles. It has a wide range of applications across various fields, including science, engineering, architecture, and more. In this section, we will look at some common ways trigonometry is used.

Applications of Trigonometry (TE6)

Activity 8.6.2 A pilot signals to an air traffic controller that she wants to land. She wants to know what angle of descent to take when she is currently at 40,000 feet. Her plane is a horizontal distance of 750,000 feet from the runway, as the air traffic controller can see on the radar.

(a) Draw a diagram to represent this situation.

(b) The pilot wants to know what angle of descent to take in order to reach the runway. The angle of descent is the angle between the flight path and the ground. In other words, it is an angle that is formed by the horizontal line and the pilot's line of sight to the runway. Where is this angle located in your diagram?

(c) Draw a line that represents the line of sight of the pilot. What do you notice about this line and the base of your triangle?

(d) If we know that the pilot's line of sight is parallel to the base of the triangle you created in part (a), then the hypotenuse of the triangle could also be considered a transversal that cuts the two parallel lines. What angle of the triangle is congruent to the "angle of descent" the pilot wants to take in order to descend the plane 750,000 feet from the runway?

(e) Now that we know which angle is congruent to the "angle of descent" the pilot needs, which of the 6 trig functions could we use to find the angle at which the pilot should descend?

(f) Find the measure of the angle the pilot needs to make her initial descent.

Remark 8.6.3 Notice that Activity 8.6.2, the angle that we needed to find was an angle that was not inside the right triangle. In these cases, it would be helpful to use prior knowledge of parallel lines and angle relationships to determine which other angle is congruent to that given angle.

Definition 8.6.4 Many applications of trigonometry will include the angle of elevation and the angle of depression which are formed by two parallel lines cut by a transversal.

- An **angle of elevation** is the angle formed by a horizontal line and a line of sight to a point above the line.
- An **angle of depression** is the angle formed by a horizontal line and a line of sight to a point below the line.

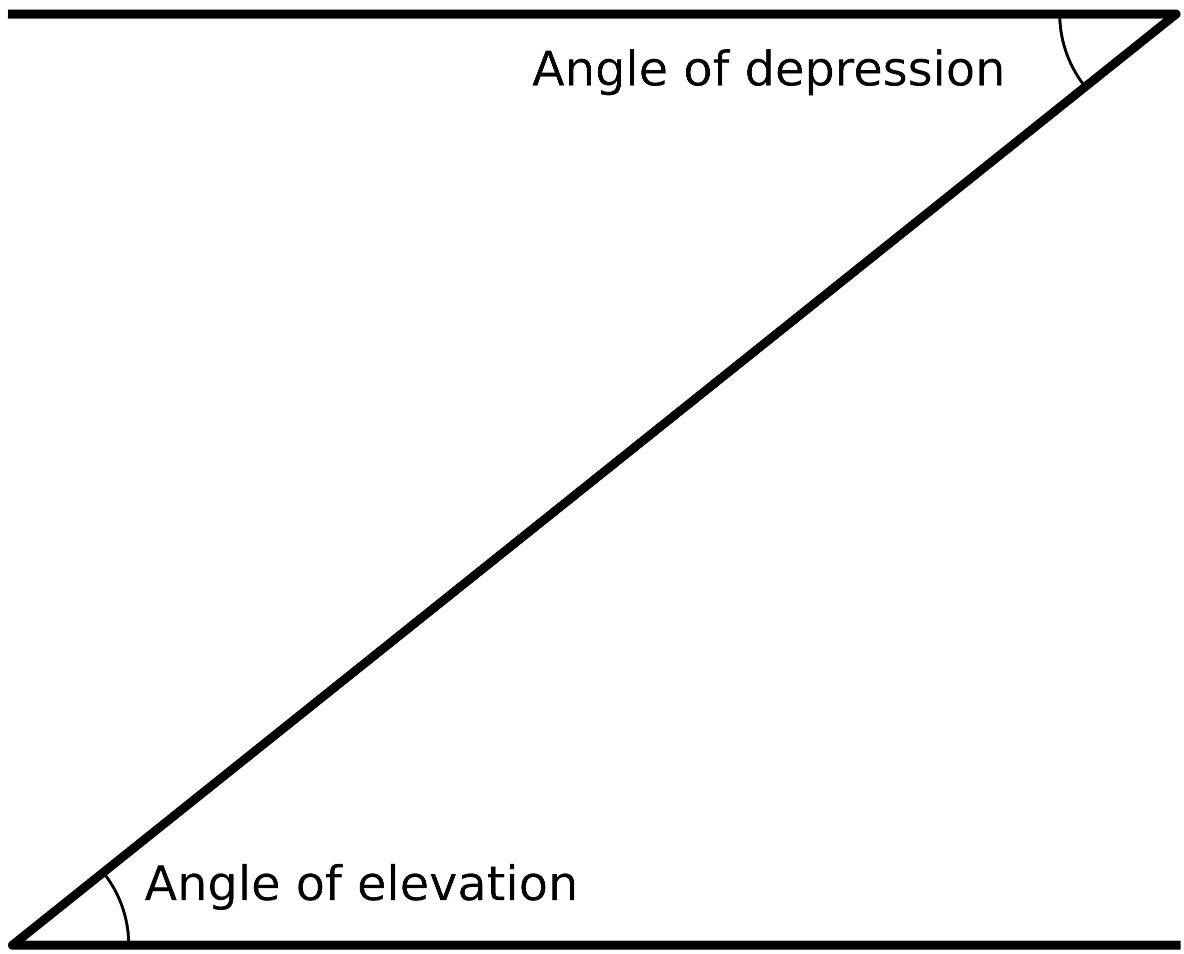


Figure 8.6.5 The angle of elevation and the angle of depression are congruent.

Notice that because both the angle of elevation and the angle of depression are formed by horizontal lines that are parallel, the angle of elevation is congruent to the angle of depression (by the alternate interior angles theorem). ◊

Applications of Trigonometry (TE6)

Activity 8.6.6 An observer who is 1.5 meters tall is standing 67 meters away from the Seattle Space Needle.

(a) Draw a diagram to represent this situation.

(b) If the angle of elevation from where the observers stands to the top of the Space Needle is 70°, which trig function could you use to find the height of the Space Needle?

(c) How tall is the Space Needle (to the nearest meter)?

Hint. Don't forget to account for the height of the observer!

Applications of Trigonometry (TE6)

Activity 8.6.7 Use Definition 8.6.4 and your knowledge of right triangles to solve each of the following. It might be helpful to draw a diagram to represent the situation before solving.

(a) Sarah's kite is flying above a field at the end of 65 meters of string. If the angle of elevation to the kite measures 70°, how high is the kite above Sarah's head?

(b) Standing on a cliff 380 meters above the sea, Sean sees an approaching ship and measures its angle of depression, obtaining 9 degrees. How far from shore is the ship (to the nearest meter)?

(c) A 14-foot ladder is used to scale a 13-foot wall. At what angle of elevation (to the nearest degree) must the ladder be situated in order to reach the top of the wall?

(d) A submarine starts on the surface, and dives at an angle of 13° to the surface. It goes diagonally a distance of 890 meters before reaching the bottom. How far is it along the ocean surface from the point where the submarine started to the point directly above where it reached the bottom?

Applications of Trigonometry (TE6)

Remark 8.6.8 All applications we have done so far have involved right triangles. Let's now look at some examples of non-right triangle applications.

Applications of Trigonometry (TE6)

Activity 8.6.9 Airplane A is flying directly towards the airport which is 20 miles away. From Airplane A's point of view, the angle between the airport and Airplane B is 45° and from Airplane B's point of view, the angle between the airport and Airplane A is 50°.

(a) Draw a diagram to represent this situation.

(b) Based on the information given and your diagram, how far is Airplane B from the airport (to the nearest tenth of a mile)?

Applications of Trigonometry (TE6)

Activity 8.6.10 Carlos, Jean, and Travis are camping in their tents. The distance between Carlos and Jean is 153 feet, the distance between Carlos and Travis is 201 feet, and the distance between Jean and Travis is 175 feet.

(a) Draw a diagram to represent this situation.

Hint. It might be helpful to label your triangle in terms of C, J, and T for the angles and c, j, and t for the sides of the triangle.

(b) What type of triangle represents this situation? How can you determine whether this triangle is a right triangle?

(c) Refer back to the previous section. Which trigonometric law (the Law of Sines or the Law of Cosines) would be the best one to use if we wanted to find the angle at which Carlos is from his friends?

(d) Find the angle at which Carlos is located from his friends to the nearest degree.

(e) Now that we know the angle at which Carlos is located from Jean and Travis, determine the angle (to the nearest degree) at which Travis is located from his friends by using the Law of Sines.

(f) Find the remaining angle left of the triangle.

Applications of Trigonometry (TE6)

Remark 8.6.11 Trigonometric functions can model relationships between different quantities that follow a periodic nature: height over time, distance over time, temperature over time and so on. Scientists observe this back-and-forth movement and collect data so they can model them using an equation or a graph. They then use this information to make predictions for the future.

Applications of Trigonometry (TE6)

Activity 8.6.12 The depth of the water in meters at a certain pier varies with the tide and is modeled by the equation $d(t) = 2 + \frac{1}{2}\sin\frac{\pi}{6}t$ where t is the number of hours after 10 a.m.

(a) How deep will the water be at high tide?

(b) How deep will it be at low tide?

(c) How many hours are there between two successive high tides?

(d) How many of these tide cycles are there in a 24 hour day?

Applications of Trigonometry (TE6)

Activity 8.6.13 A circular Ferris wheel is 120 meters in diameter and contains several carriages. Jesus and Allison enter a carriage at the bottom of the wheel and get off 24 minutes later after having gone around 8 times. When a carriage is at the bottom of the wheel, it is 1 meter off the ground.

(a) What is the maximum and minimum height of Jesus and Allison's carriage?

(b) Let $h(t)$ be a function that represents the height of the carriage t minutes after it has started moving. What is the period of $h(t)$?

(c) Which trigonometric function would be the best to use to model this situation?

Appendix A

Appendix

A.1 Graphs of Common Functions

Polynomial Functions.

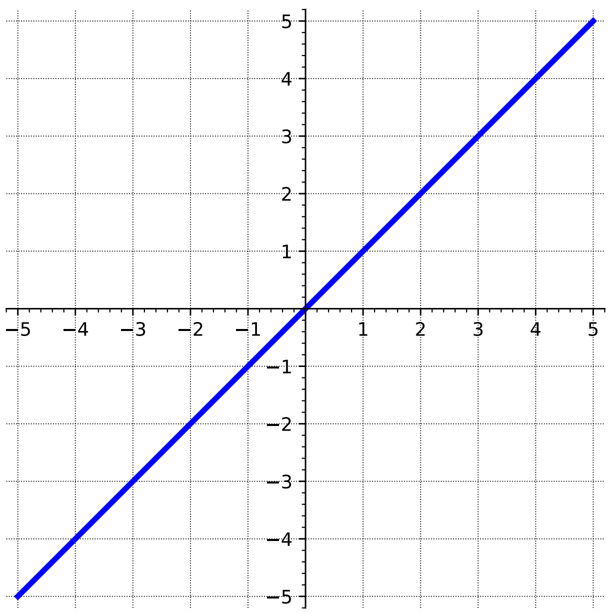


Figure A.1.1 Graph of $y = x$

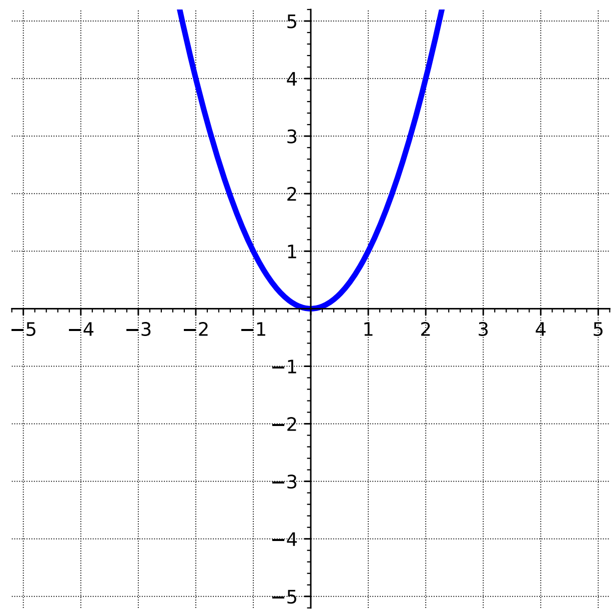


Figure A.1.2 Graph of $y = x^2$

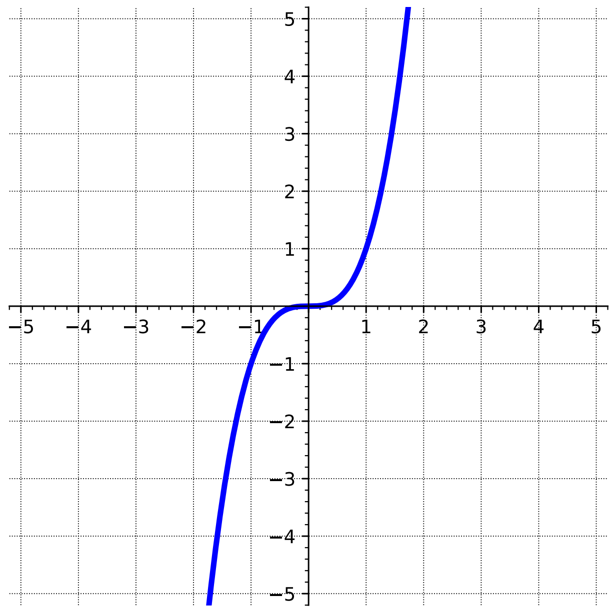


Figure A.1.3 Graph of $y = x^3$

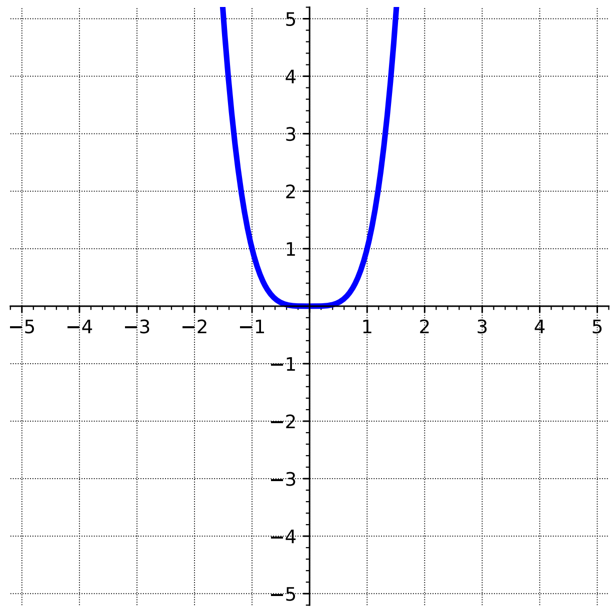


Figure A.1.4 Graph of $y = x^4$

Rational Functions.

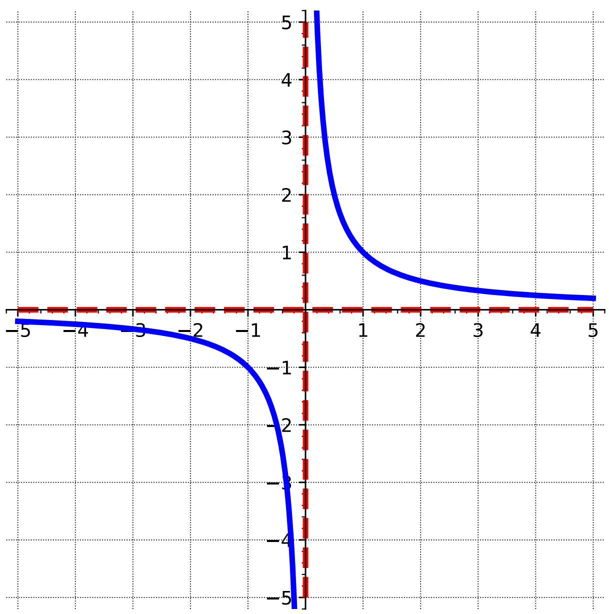


Figure A.1.5 Graph of $y = \dfrac{1}{x}$

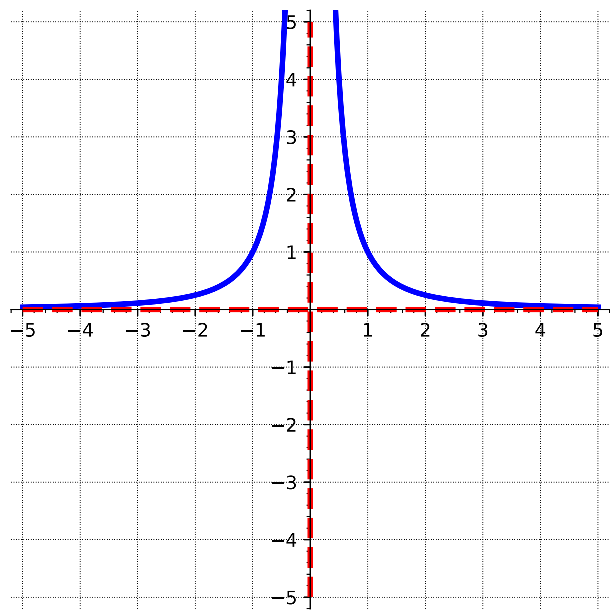


Figure A.1.6 Graph of $y = \dfrac{1}{x^2}$

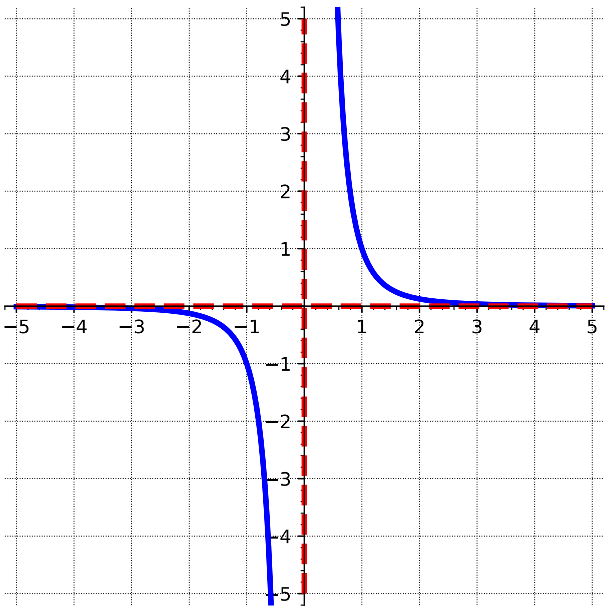


Figure A.1.7 Graph of $y = \dfrac{1}{x^3}$

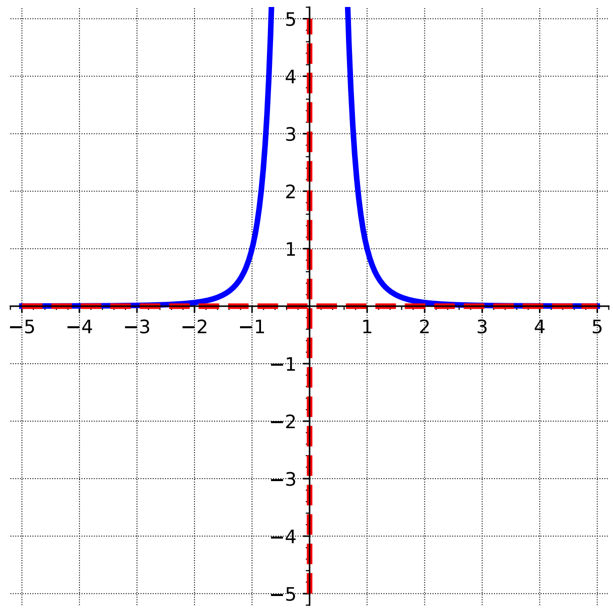


Figure A.1.8 Graph of $y = \dfrac{1}{x^4}$

Other Functions.

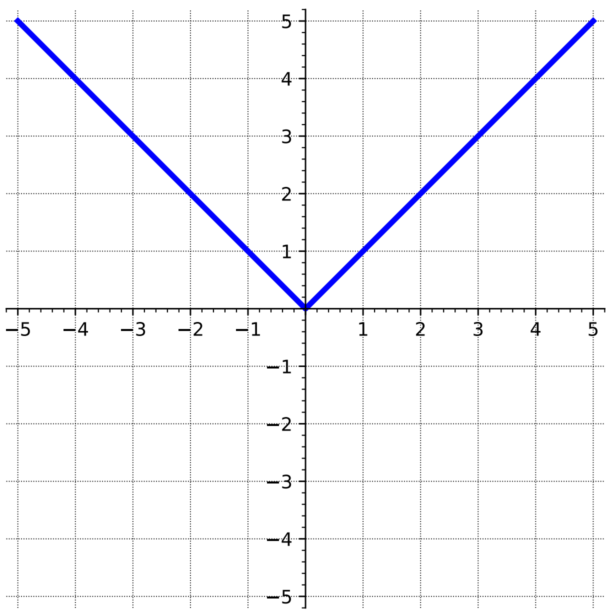


Figure A.1.9 Graph of $y = |x|$

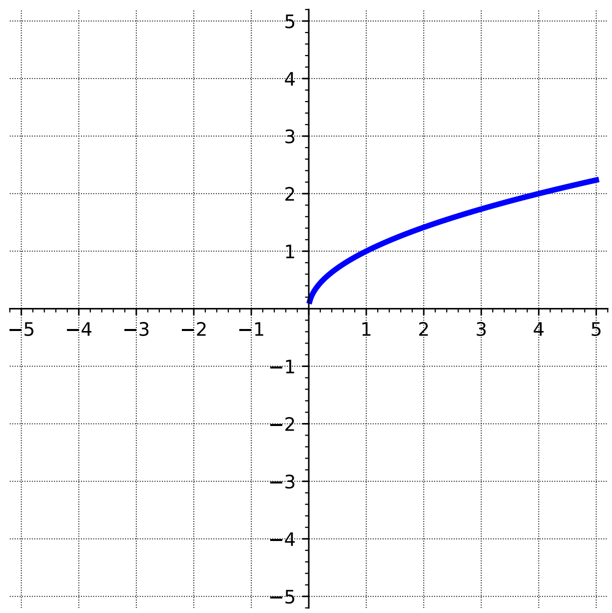


Figure A.1.10 Graph of $y = \sqrt{x}$

Exponential and Logarithmic Functions.

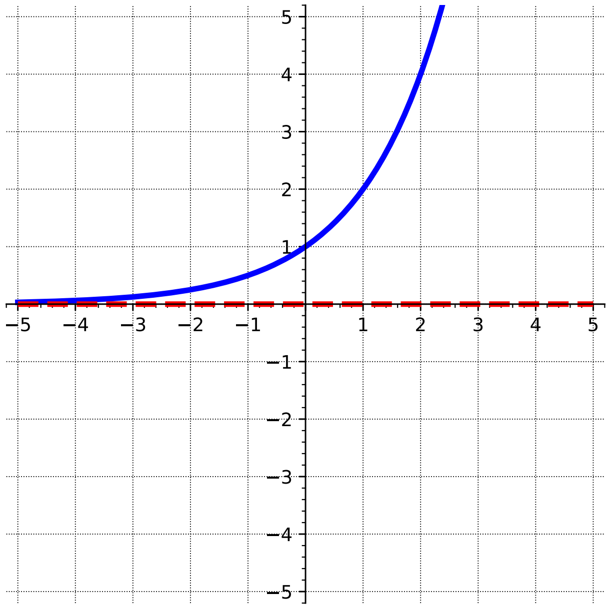


Figure A.1.11 Graph of $y = b^x$, with $b = 2$

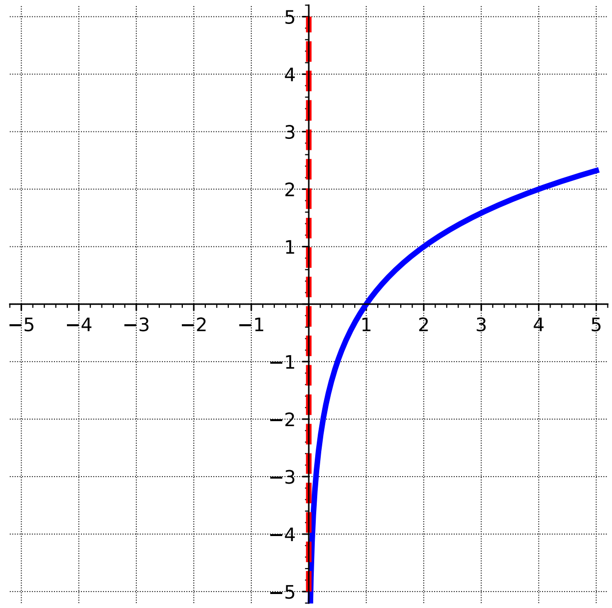


Figure A.1.12 Graph of $y = \log_b x$, with $b = 2$

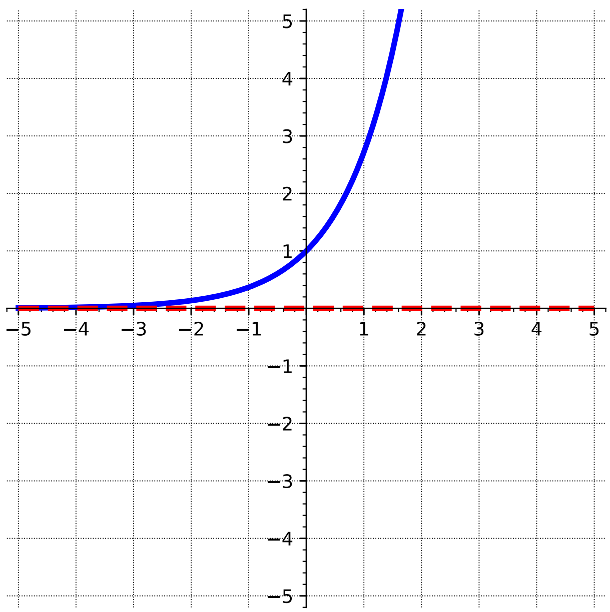


Figure A.1.13 Graph of $y = e^x$

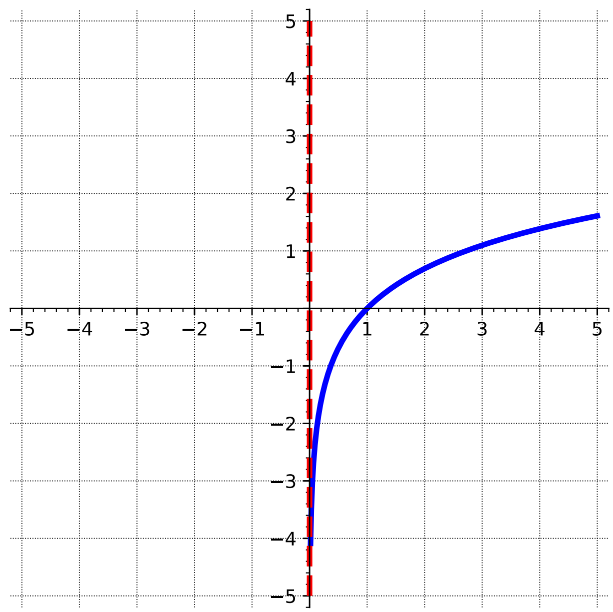


Figure A.1.14 Graph of $y = \ln x$

Trigonometric Functions.

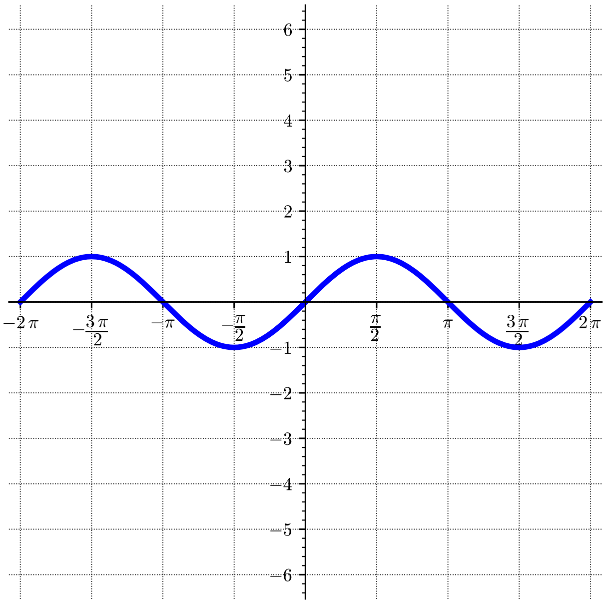


Figure A.1.15 Graph of $y = \sin(x)$

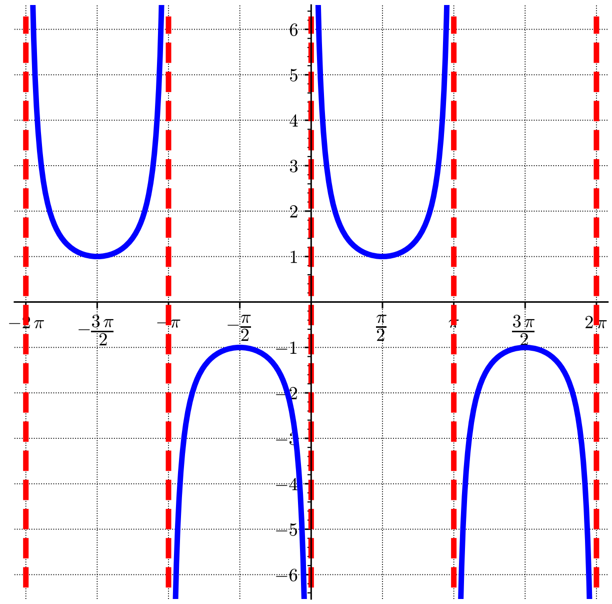


Figure A.1.16 Graph of $y = \csc x$

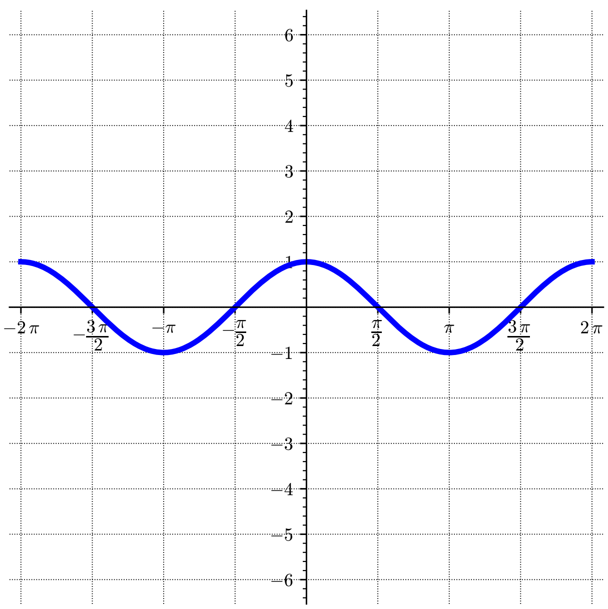


Figure A.1.17 Graph of $y = \cos x$

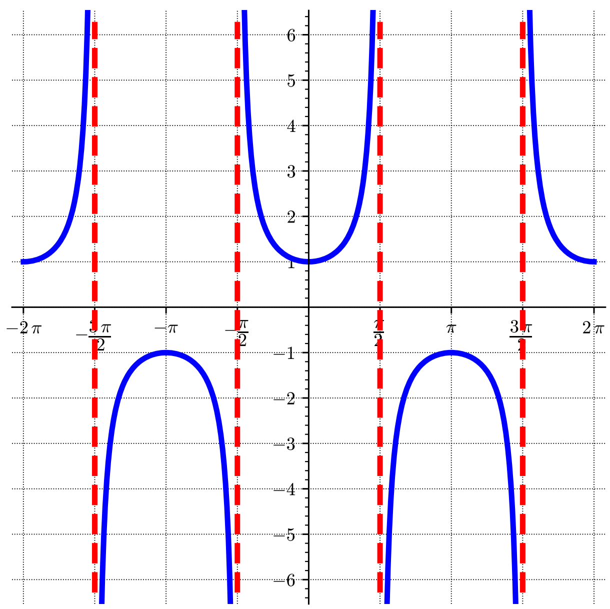


Figure A.1.18 Graph of $y = \sec x$

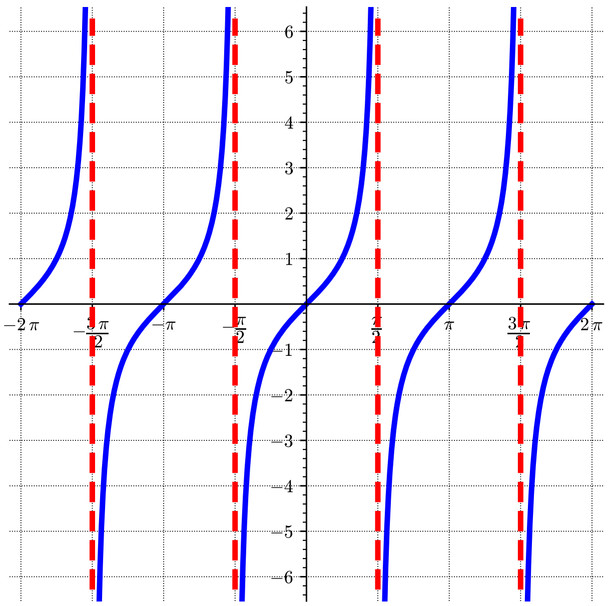


Figure A.1.19 Graph of $y = \tan x$

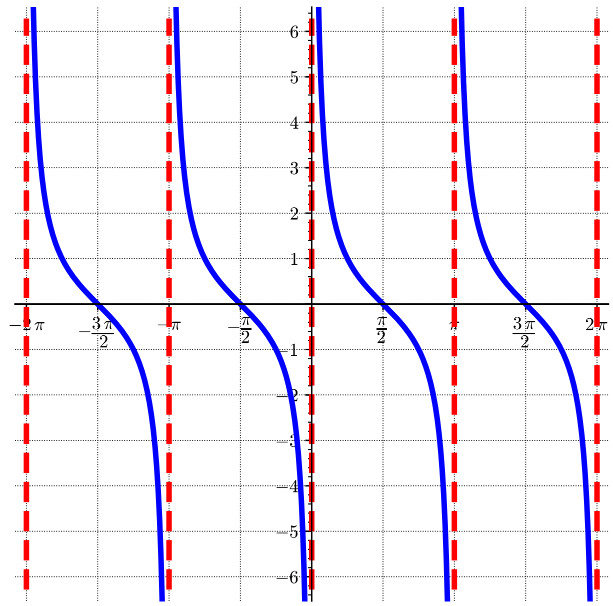


Figure A.1.20 Graph of $y = \cot x$

Inverse Trigonometric Functions.

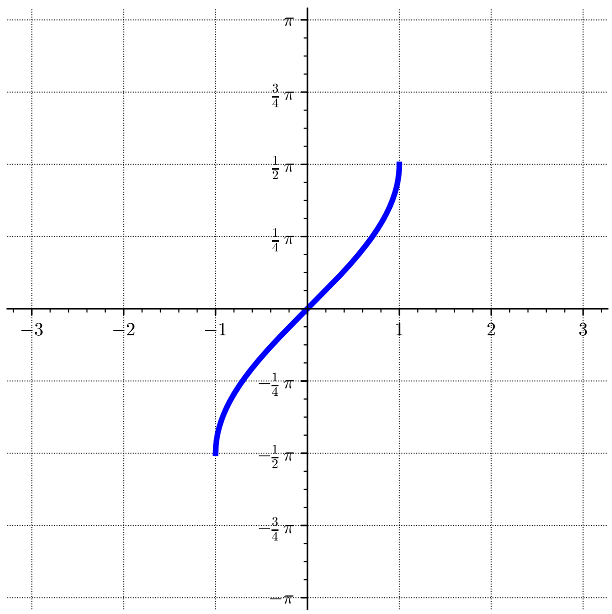


Figure A.1.21 Graph of $y = \sin^{-1}(x)$

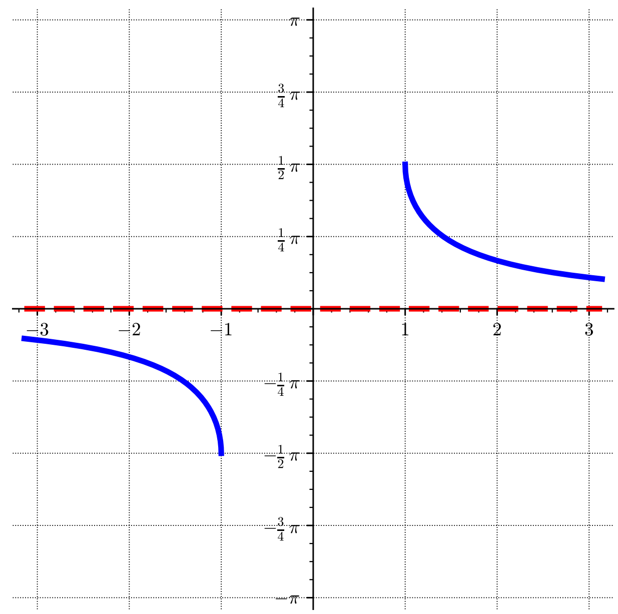


Figure A.1.22 Graph of $y = \csc^{-1}(x)$

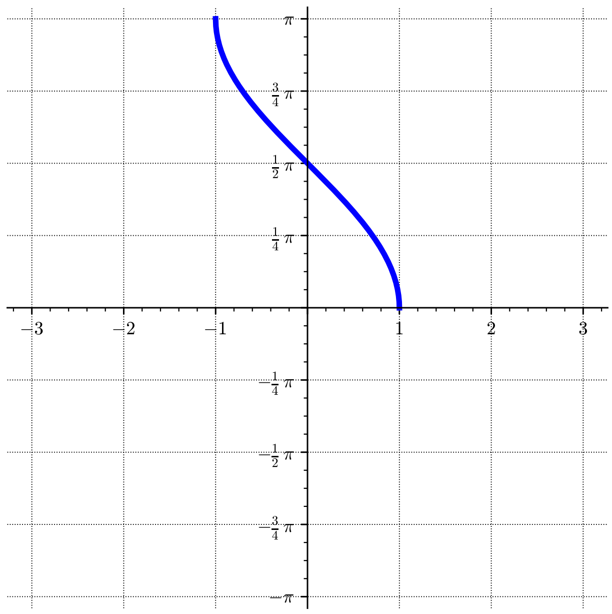


Figure A.1.23 Graph of $y = \cos^{-1}(x)$

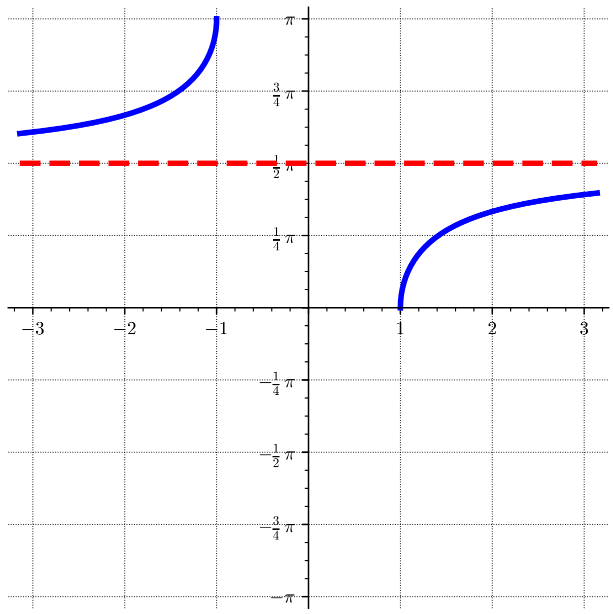


Figure A.1.24 Graph of $y = \sec^{-1}(x)$

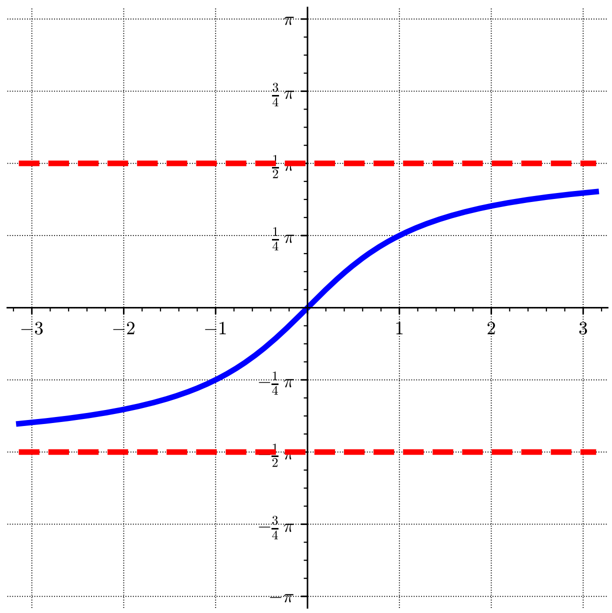


Figure A.1.25 Graph of $y = \tan^{-1}(x)$

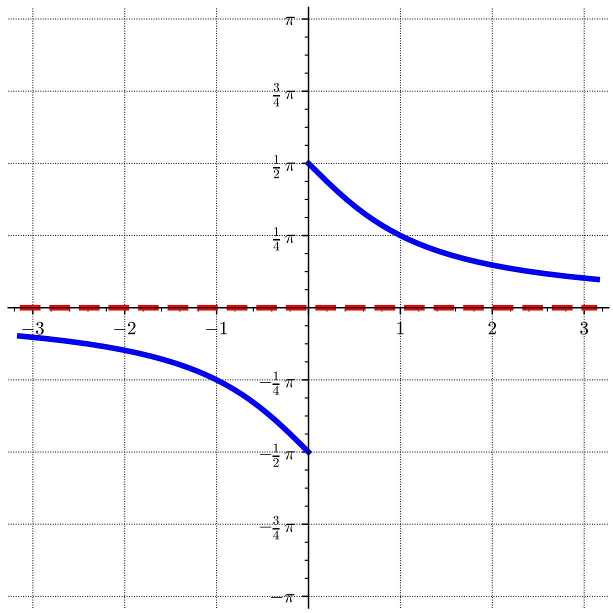


Figure A.1.26 Graph of $y = \cot^{-1}(x)$

A.2 Trigonometric Identities

Identity A.2.1 Pythagorean Identities.

- $\sin^2(\theta) + \cos^2(\theta) = 1$
- $\tan^2(\theta) + 1 = \sec^2(\theta)$

- $1 + \cot^2(\theta) = \csc^2(\theta)$

Identity A.2.2 Sum and Difference Formulas.

- $\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$
- $\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$
- $\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$
- $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$
- $\tan(\alpha + \beta) = \dfrac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$
- $\tan(\alpha - \beta) = \dfrac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$

Identity A.2.3 Double Angle Formulas.

- $\sin(2\theta) = 2\sin\theta\cos\theta$
- $\cos(2\theta) = \cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta = 2\cos^2\theta - 1$
- $\tan(2\theta) = \dfrac{2\tan\theta}{1 - \tan^2\theta}$

Identity A.2.4 Power Reduction Formulas.

- $\sin^2\theta = \dfrac{1 - \cos 2\theta}{2}$
- $\cos^2\theta = \dfrac{1 + \cos 2\theta}{2}$
- $\tan^2\theta = \dfrac{1 - \cos 2\theta}{1 + \cos 2\theta}$

Identity A.2.5 Half Angle Formulas.

- $\sin\frac{\theta}{2} = \sqrt{\frac{1-\cos\theta}{2}}$ *or* $\sin\frac{\theta}{2} = -\sqrt{\frac{1-\cos\theta}{2}}$, *depending on which quadrant* θ *is in.*
- $\cos\frac{\theta}{2} = \sqrt{\frac{1+\cos\theta}{2}}$ *or* $\cos\frac{\theta}{2} = -\sqrt{\frac{1+\cos\theta}{2}}$, *depending on which quadrant* θ *is in.*
- $\tan\frac{\theta}{2} = \sqrt{\frac{1-\cos\theta}{1+\cos\theta}} = \frac{\sin\theta}{1+\cos\theta} = \frac{1-\cos\theta}{\sin\theta}$ *or* $\tan\frac{\theta}{2} = -\sqrt{\frac{1-\cos\theta}{1+\cos\theta}} = \frac{\sin\theta}{1+\cos\theta} = \frac{1-\cos\theta}{\sin\theta}$, *depending on which quadrant* θ *is in.*